Selected general problems in chaos theory and nonlinear dynamics

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Abstract: - This paper is aimed to the problems which have been recently faced by the authors and still remains unsolved. These difficulties and dilemmas lead to the several hypotheses but without rigorous mathematical proofs. The problems mentioned in this brief paper should be solved by skilled mathematicians and physicists. Even though it is highly possible that similar mathematical models and associated tasks will be unfolded in many other scientific disciplines.

Key-Words: - Behavior quantification, chaos theory, nonlinear dynamical system

1 Introduction
It is well known that nonlinear dynamical systems with at least three (autonomous) or two (driven) degrees of freedom can exhibit irregular, noise-like behavior. This kind of motion is extremely sensitive to the changes of the initial conditions and is called chaos for its early discovery published in [2]. The most significant property of chaotic attractor is its fractal metric dimension, ergodicity and mixing. From the viewpoint of chaotic data sequence it’s the continuous and broad-band frequency spectrum [3].

Chaos has been reported from many distinct scientific disciplines, for example optics, chemistry, elementary particle interaction, population growth, economy, planetary motion, medicine, weather forecast, energy transmission, power electronics, analog circuits, digital networks and many other areas of common life. It is caused by the fact that very algebraically similar mathematical models describe the completely different real physical events. Once existed, chaos can be considered as universal phenomenon since the interpretation of the individual state variables or the internal system parameters is irrelevant.

This paper is organized as follows. Next chapter is aimed to the quantification of dynamical motion and possibility to search for some specific solution [4] in the parameter space of the general class of the dynamical systems [5]. Third chapter marks the problem of finding mathematical model associated with experimentally observed dynamical motion and the existing methods for attractor reconstruction [6]. Fourth chapter is focused to the so-called dynamical system of class C [7] with hypothetic configurations of the eigenvalues in each segment of the piecewise-linear vector field. In detail, geometry of the vector field is prescribed but parameter values leading to the chaotic behavior have not been uncovered. Fifth chapter questions routine for accurate and quickly-calculated quantification of the large state space attractors or the systems with very low dissipation. Sixth paragraph brings an idea involving nonlinear dynamics and probability of the chaotic motion in fully analog implementation of a neural network [8]. Seventh chapter asks a fundamental question about non-existence of any stochastic processes in nature. Finally, further progress and perspectives in chaos theory are revealed.

2 Searching for chaos
It turns out that any real dynamical system is basically nonlinear and has multiple state attractors depending on the choice of the initial conditions or system parameters. By setting these values such that it has physical meaning the complete information about every possible solution is no longer available. Although the chaotic attractor is often a global attracting set the eventuality of coexistence of the structurally different chaotic attractors is not totally impossible, especially if some complex nonlinearity is presented. To reveal such unfolding numerically advanced methods for system analysis should be invented and verified.

From the viewpoint of the lumped circuits these mentioned problems are even more visible. Assume the harmonic oscillator, i.e. second-order dynamical system. To overcome the natural losses in passive
RC network some sort of active element [9] must be employed. If the high frequency operation range is to be achieved the working capacitances become comparable to the parasitic ones and the order of such network rises unwillingly. Since each active building block has voltage or current saturation at the output port we encounter the higher-order nonlinear dynamical system. This feature also holds for the basically non-autonomous linear circuits like analog filters, especially biquads [10] and its higher-order modifications. Moreover, logical probability of chaos existence [11] grows proportionally with system dimension. To provide a concrete example the strange attractors have been recently reported in simple oscillators with diamond transistors [12].

To date, the possibility for specific dynamical motion localization if the describing mathematical model is known remains an unsolved problem. It is evident that such routine can be considered as the optimization task. Besides the maximal universality of the final procedure there are two other problems to deal with. These are marked in the text below.

### 2.1 Fitness Function

The proper choice of the objective function plays a crucial role in any optimization procedure. For the purpose of solution quantification the best one is some metric dimension of the state space attractor. The most precise approach is based on the counting of the real volume that can cover the entire attractor. The state space is separated by small cubes and the total number of cubes filled by a fiducial trajectory is estimated. The final dimension is given by a slope as three numbers which measure the average ratio of occupied by state space attractor the box-counting dimension guess and other parameters. On the other hand, attractor reconstruction theorem and its direct orthogonal base of vectors and avoid numerical errors Gram-Smith orthogonalization procedure [15] is added to the standard routine for calculation after the few iteration steps. The proper choice between upper mentioned motion quantifiers depends on the state attractor under inspection. For large volume occupied by state space attractor the box-counting method cannot be considered as well suited.

### 2.1 Searching Procedure

Due to the fact that the closed-form analytic solution of set of the nonlinear differential equations cannot be obtained we are restricted on the numerical ways of analysis. For the purpose of optimization we cannot utilize gradient methods and are restricted on stochastic approach. Few particular and successful results have been already published by the authors, both for piecewise-linear system [16] or polynomial vector field [17]. In these papers, genetic algorithm and particle swarm optimization has been used and compared, especially from the convergence speed and number of fitness function calculation point of view. It should be noted that the success in search is far from the mathematical proof [18] of chaos.

Another procedure for the chaotic solution quantification as well as improving search routine is the topic for our further research. Main drawbacks associated with this problem can be minimized by speed-up whole procedure [19] by using multi-core computers, parallel processing and grid calculations.

### 3 Data sequence recognition

Sometimes it is rather difficult or impossible to derive a set of the ordinary differential equations describing observed physical phenomena. In such a case state variables are measured by some proper device, e.g. digital oscilloscope and only discrete data sequence is available. Several methods capable to handle with general motion quantification and embedding dimension are known, mostly based on the time delays [20]. The results presented by the authors are subject to the errors as the routines are very sensitive to the choice of time gap, embedding dimension guess and other parameters. On the other hand, attractor reconstruction theorem and its direct application seems to be useful and of many practical applications [21].

The topic introduced in this chapter is waiting for the world-wide contributions of the other scientists and engineers.

### 4 Vector field geometry

Assume the following piecewise-linear dynamical systems expressed in the compact matrix form

\[
\dot{x} = Ax + b \frac{1}{2} \left( |w^T x + 1| - |w^T x - 1| \right),
\]

(1)
where dots represents time derivatives, \( x \in \mathbb{R}^3 \), \( A \) is 3×3 square matrix and \( b, w \) are 3×1 column vectors. This class of the systems can produce a large variety of the chaotic attractors [22] depending on the parameter values. Linear algebra and theory of the differential equations tell us that the dynamical motion is uniquely determined by the fixed points, eigenvalues and corresponding eigenspaces, shortly on the local vector field geometry. For the systems (1) this vector field is separated by two parallel boundary planes into three affine regions. There is just one unstable equilibria in each region. In the outer segments the geometry always becomes the saddle-spiral, i.e. \( \mathbb{R}^3 \oplus \mathbb{R}^3 u \oplus \mathbb{R}^3 s \) for the double-scroll attractor [23] or \( \mathbb{R}^3 \oplus \mathbb{R}^3 u \oplus \mathbb{R}^3 s \) for its dual equivalent [24]; upper index marks dimension of stable (s) or unstable (u) manifold. Internal geometry in inner segment of vector field is of opposite configuration of the manifolds. To this end, there is another one typical strange attractor generated by (1) and named double-hook [25] with internal geometry composed of three eigenvectors \( \mathbb{R}^3 \oplus \mathbb{R}^3 u \oplus \mathbb{R}^3 s \oplus \mathbb{R}^3 s \). Interesting question has been recently asked in [26]: for which set of system (1) parameters there are some chaotic attractors with outer geometry \( \mathbb{R}^3 \oplus \mathbb{R}^3 u \oplus \mathbb{R}^3 s \)? If so, can be derived theorems about restrictions and demands for the chaos evolution familiar to one of the Shilnikov’s [27], [28]?

Let suppose the subclass of (1) written as single third-order differential equation, namely
\[
\ddot{x} - q_1 \dot{x} + q_2 x + \frac{1}{\lambda} (p_3 - q_3) (|x+1|-|x-1|),
\]
where \( q_1, q_2, q_3 \) and \( p_3 \) are system parameters as well as coefficients of the characteristic polynomial. Clearly, behavior of this system is determined by four parameters instead of six as it holds for (1). Thus reachable area of the values of six eigenvalues is restricted and not every configuration of vector field can be modeled by this equation. The upper mentioned question can be asked also in this case and is much more challenging than it is for (1). The authors believe that there is a positive answer to this problem and can be reached for example by brute force method described in the second chapter. If this approach is used, the fitness function should be penalized in the case of incorrect eigenvalues.

5 Large attractor classification
The problem with rapid quantification of the large state attractors using capacity dimension has been already mentioned and is quite understandable. This obstacle can be removed by using dimension based on the Lyapunov exponents as long as dynamical flow is smooth and has continuous derivatives. But there is a pack of the systems where matrix of linearization can not be calculated or its utilization leads to the completely false results. For example assume the following mathematical model of the system with cyclically symmetrical vector field and jump nonlinearity [29]
\[
\begin{align*}
\dot{x} &= ax + \text{sign} [\sin(by)] \\
\dot{y} &= ay + \text{sign} [\sin(bz)] \\
\dot{z} &= az + \text{sign} [\sin(bx)]
\end{align*}
\]
where \( a \) and \( b \) are system parameters. It is worth nothing that partial derivative of the sign function is zero, positive or negative infinity. By considering the individual clusters of the state space (with size defined by \( b \) the Jacobi matrix in the Lyapunov’s exponent calculation routine cannot be constant but should contain variable elements, the most probably functions.

Figure 1: 3D view on the typical attractor generated by Thomas (upper picture) and Gotthans-Petrzela (lower picture) system with \( a=0 \) and \( b=10 \).
The most straightforward way is to calculate the numerical value of the derivative in each iteration step. Of course, such value will strongly depend on the actual iteration step and the final results can be very different. On the first look it seems that both Gotthans-Petrzela system (3) and Thomas system [30] are closely related; on the second thought the same analysis method cannot be utilized. Also the typical labyrinth chaos attractor can be observed in the case of both systems as shown in Fig. 1. Both figures were integrated using build-in fourth-order Runge-Kutta method in Mathcad.

The importance of large attractor complexity and shape quantification goes from the dynamical motion of elemental particle without interactions, i.e. Brownian motion [31].

6 Chaos in analog neural network
Assume the analog implementation of single cell in the neural network described by Hindmarch-Rose model [32] which is

\[
\begin{align*}
\dot{x} &= y + \phi(x) - z + I \\
\dot{y} &= \psi(y) - y \\
\dot{z} &= r[x(s - x) - z]
\end{align*}
\]  

(4)

In the context of such neuronal activity nonlinear functions are

\[
\phi(x) = ax^3 - x^3 \\
\psi(x) = 1 - bx^2
\]  

(5)

and \(a, b, r, s, x_R, I\) are system parameters. This mathematical model is widely used and can generate a variety of bursting and peaking signals. Some set of parameters leads to the long-term unpredictable behavior and chaos, see Fig. 2. To make a network the neurons are connected with each other via single stimulus \(I\). This control parameter should be subject for nonlinear weight function analogically to the feed forward artificial neural network often used for solving technical problems by discrete optimization.

What is of great importance is the fact that axon is forced by synapse and this is described in terms of (4) and (5) by single state variable \(x(t)\). Now let’s think about the probability of chaotic motion in the analog neural network composed of one hundred neurons. Such network can be easily mathematically described [33], numerically integrated and analyzed as well as modeled by the electronic circuit [34]. The sophisticated guesses presented by trustworthy scientists suggest that the probability of chaos (more likely hyperchaos in this case) in this large analog neural network is almost a reality. Such thesis is in accordance with recent experimentally measured brain processes which resembles chaos in many fundamental aspects.

Figure 2: Typical bursting and peaking waveform visualized as 3D attractor, parameters \(a=2.5, b=5, r=0.001, s=4, x_R=-1.6\) and \(I=5\).

The practical implementation of large analog neural network belongs also to the partially unanswered question; VLSI on-chip realization will probably be the correct solution.

7 Determinism vs stochastic processes
Probably the most interesting open problem for further discussion is the postulate that there are no stochastic processes in nature. The authors believe that random-like behavior can be observed only if studied dynamical system is somehow idealized or not completely described. Roughly speaking, if each state variable is observable and each functional system parameter value is numerically known there exists mathematical model with finite-order and associated numerical solution. There is also a basic level of predictability. The reason for this is simple: even chaotic system with many degrees of freedom and strongly nonlinear vector field has finite largest Lyapunov exponent making a short-term future motion prediction possible. This proposition goes to the arbitrary dimension of system, no matter if it describes microscopical movement of the elemental particles or we scale-up to the evolution of universe. This can be confusing, brave and rather utopian idea causing immediate headaches. One can also argue that such models can not be constructed and many real and important variables and parameters must be omitted. Yes, this is true without questioning. Moreover, there is no enough mathematical power for analysis of such complex mathematical models. And if we finally do it and obtain accurate model of the being itself what about backward numerical integration to the very birth of our galaxy, time approaching big bang?
8 Conclusion
In this review paper some fundamental questions from the chaos theory and nonlinear dynamics have been pointed. These problems represent only a very small fraction of the theoretical and practical difficulties which are faced in the dynamical system modeling and analysis. The list of more concrete but less general problems can be found in [34].

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