Finite element method for Burgers equation using cubic B-spline approximation

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Abstract: In this paper we present numerical solution of Burgers equation using cubic B-spline functions. In the past, difficulties have been experienced while getting its numerical solution. So it is worth to find the schemes that can solve it efficiently giving accurate and stable solutions. The current work aims to develop an algorithm that is easy to understand and implement. Numerical experiments shows that the scheme is capable in achieving results of high accuracy.

B-spline, Finite element method, Burger equation, Space, Time.

1 Introduction

Numerical solution of the unsteady Navier-Stokes (NS) equation is always a challenging problem. From the mid 1980’s, main focus is being given to the simulation of Navier-Stokes equations using mathematical and computer modeling. As a simplified model of the Navier-Stokes equation, the one-dimensional viscous Burgers equation represents a first step in the hierarchy of approximation of Navier-Stokes equations. Burgers equation shows a similar features with Navier-Stokes equation due to the form of the nonlinear convection term and the occurrence of the viscosity term. Thus, before concentrating on the numerical solution of the Navier-Stokes equation, it seems reasonable to first study a simple model of the Burgers equation. Shock flows, traffic flows, nonlinear wave propagation/shock waves, gas dynamics, turbulence [2, 3] and so forth are some important fields where Burgers model is of great use and is very helpful in capturing a variety of phenomena associated with fluid flows. In 1951 it was first solved by Cole and Hopf [5, 4]. Later on, many researchers focused on its solution (see, e.g., [6, 7, 11, 12]) and thus, it is a very active area of research till the date. In this paper, we consider the one dimensional Burgers equation

\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = \nu \frac{\partial^2 U}{\partial x^2}, \]  

(1)

which may be considered as a model equation for decay of turbulence within a box of length \( L \) and \( \nu \) measures kinematic viscosity. Physical boundary conditions require \( U \) to be zero at the ends such that \( U \to 0 \) as \( x \to 0, L \).

2 Solution scheme

Let us consider \( 0 = x_0 < x_1 < \ldots < x_N = L \) as a partition of \( [0, L] \) by the knots \( x_i, \ i = 0, 1, \ldots, N \) and let \( B_i \) be the cubic B-splines with knots at points \( x_i \) defined as [17]. The least square formulation of equation (1) is described as

\[ \delta \int_0^t \int_0^L \left( \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} - \nu \frac{\partial^2 U}{\partial x^2} \right)^2 \ dx dt = 0. \]  

(2)

Now, each spline covers four elements so that each element \( [x_m, x_{m+1}] \) is covered by four splines. In each element, using the transformation \( x = x_m + \eta \Delta x \), \( 0 \leq \eta \leq 1 \). Cubic B-spline functions in terms of local coordinates are expressed as

\[ \phi_{m-1} = (1-\eta)^3; \]
\[ \phi_m = 4 - 3\eta + 3(1-\eta)^2 - 3(1-\eta)^3; \]
\[ \phi_{m+1} = 1 + 3\eta + 3\eta^2 - 3\eta^3; \] \[ \phi_{m+2} = \eta^3. \]  

(3)

The splines \( B_m(x) \) and its two principle derivatives vanish outside the interval \( [x_{m-2}, x_{m+2}] \). On each interval \( [t_n, t_{n+1}], \Delta t = t_{n+1} - t_n \) is mapped by local coordinate \( \tau \) such that \( t = t_n + \tau \Delta t, \ 0 \leq \tau \leq 1 \). We identify the finite elements with the each interval \( [x_m, x_{m+1}] \) and using the above transformation, equation (2) over each typical element can be written as

\[ \delta \int_0^1 \int_0^1 \left( \frac{\partial U}{\partial \tau} + \alpha \frac{\partial U}{\partial \eta} - \beta \frac{\partial^2 U}{\partial \eta^2} \right)^2 \ d\eta d\tau = 0, \]  

(4)

where \( \alpha = U \frac{\Delta t}{\Delta x} \) and \( \beta = \nu \frac{\Delta t}{\Delta x^2} \). The integral equation (4) takes its minimum value with the variation in
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Over each element \([x_m, x_{m+1}]\) when

\[
\int_0^1 \int_0^1 (\frac{\partial U}{\partial \tau} + \alpha \frac{\partial U}{\partial \eta} - \beta \frac{\partial^2 U}{\partial \eta^2}) \times \\
\delta \left( \frac{\partial U}{\partial \tau} + \alpha \frac{\partial U}{\partial \eta} - \beta \frac{\partial^2 U}{\partial \eta^2} \right) dx dy = 0. \tag{5}
\]

The term \(\delta \left( \frac{\partial U}{\partial \tau} + \alpha \frac{\partial U}{\partial \eta} - \beta \frac{\partial^2 U}{\partial \eta^2} \right)\) if taken as weight function, our formulation (5) turns to Petrov-Galerkin method. Taking variation of \(U\) over each element \([x_m, x_{m+1}]\), we seek the approximation of the form \(U_N(\eta, \tau) = \sum_{j=m-1}^{m+2} \psi_j(\eta)(\sigma_j + \tau \Delta \sigma_j)\). Where \(\psi_{m-1}, \psi_m, \psi_{m+1}, \psi_{m+2}\) are B-splines acting as shape functions for each element and \(\sigma_{m-1}, \sigma_m, \sigma_{m-1}, \sigma_{m+2}\) are nodal parameters. When cubic B-spline functions are employed in the given approximation, our weight function takes the form \(\left( \psi_m + \alpha \tau \psi_m' - \beta \psi_m'' \right)\). Inserting in (5) gives us matrix form of the equations as

\[
\sum_{j=m-1}^{m+2} \left\{ A^e + \frac{\alpha}{2} D^e + (D^e)^T \right\} + \beta B^e - \\
\frac{\alpha \beta}{3} (E^e + (E^e)^T) + \frac{\alpha^2}{3} B^e + \frac{\beta^2}{3} C^e \\
- \frac{\beta}{2} (\psi_i \psi_j' + \psi_j \psi_i') \left\{ \Delta \sigma_j^n \right\} \\
+ \sum_{j=m-1}^{m+2} \left\{ \alpha D^e + \frac{\alpha^2}{2} B^e + \beta B^n + \frac{\beta^2}{2} C^e \\
- \frac{\alpha \beta}{2} (E^e + (E^e)^T) + \beta (\psi_i \psi_j') \left\{ \sigma_j^n \right\} \right\}, \tag{6}
\]

where \(\sigma = (\sigma_{m-1}, \sigma_m, \sigma_{m+1}, \sigma_{m+2})^T\). The matrices \(A^e, B^e, C^e, D^e, E^e\) for the individual elements are given by

\[
A_{i,j}^e = \int_0^1 B_i B_j d\eta, \\
B_{i,j}^e = \int_0^1 B_i' B_j' d\eta, \\
C_{i,j}^e = \int_0^1 B_i'' B_j' d\eta, \\
D_{i,j}^e = \int_0^1 B_i B_j' d\eta, \\
E_{i,j}^e = \int_0^1 B_i' B_j'' d\eta \tag{7}
\]

with \(i, j\) taking values \(m - 1, m, m + 1, m + 2\) and \(m = 0, 1, \ldots, N - 1\). Assembling contributions from all the elements gives a global system of equations as with \(A, B, C, D, E\) as assembled matrices derived from the element matrices \(A^e, B^e, C^e, D^e, E^e\) which can be expressed as

\[
N \Delta \sigma^n + M \sigma^n = 0. \tag{8}
\]

Defining \(\sigma = \sigma^n\) and \(\Delta \sigma = \sigma^{n+1} - \sigma^n\), we get the matrix system

\[
\begin{bmatrix}
A + \frac{\alpha}{2} (D + DT) + \left( \frac{\alpha^2}{3} + \beta \right) B - \frac{\alpha \beta}{3} (E + ET) \\
+ \frac{\beta^2}{3} E - \frac{\beta}{2} (\psi_i \psi_j' + \psi_j \psi_i') \\
\end{bmatrix} \sigma^{n+1} = \begin{bmatrix}
A + \frac{\alpha}{2} (D + DT) - \frac{\alpha^2}{6} B + \frac{\alpha \beta}{6} (D + DT) - \\
\frac{\beta^2}{6} E + \beta (\psi_i \psi_j') \\
\end{bmatrix} \sigma^n, \tag{9}
\]

which gives recurrence relationship for \(\sigma^n\) and (8) can be written as

\[
N \sigma^{n+1} = P \sigma^n. \tag{10}
\]

Where \(P_i = N_i - M_i\) and elements of matrices \(N, M\). Iteration process can be started from the above recurrence relation after getting the initial vector.

### 3 Test problems

In this section, we have made a careful analysis of the given Burgers equation (1) whose solutions are studied by [9, 13, 14, 16] etc. To demonstrate the capability of present algorithm, computations are made for different values of viscosity and time as well. For the first test problem, we have boundary conditions \(U(0, t) = 0, U(1, t) = 0, t > 0\) and initial condition \(U(x, 0) = \sin(\pi x), 0 \leq x \leq 1\), for the Burgers equation (1). Numerical results for the first test example are presented and compared with those obtained by [9, 13] using Hopf-Cole(HC), Restrictive Hopf-Cole(RHC) and exact solution in Table 1. A comparison between the numerical solutions and exact solution can also be visualized through Figures 1 for \(\nu = 0.1, 0.7\) at different times. For the second example, we have initial condition \(U_0 = 4x(1 - x)\) and the given homogeneous boundary conditions \(U(0, t) = 0, U(1, t) = 0, t > 0\). Table 2 shows the results obtained for this problem where, a comparison is made with the published data (Hopf-Cole(HC), Restrictive Hopf-Cole(RHC) and exact solution). A comparison between the exact solution and the given method of solution is clearly noticed. Graphical results can be seen in Figures 2 for different values of \(\nu\) as \(0.2, 0.7, 0.05, 0.09\) at different times. From the

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results obtained, we observe that the numerical solutions coincided with the exact solution and the agreement between numerical results and the exact solution appeared very satisfactory.

References:

Table 1: Comparison of results for the first test problem at different times taking $\nu = 0.1, \Delta t = 0.0001, h = 0.0125$.

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Table 2: Comparison of results for the second test problem at different times taking $\nu = 1.0, \Delta t = 0.0001, h = 0.0125$.

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Figure 1: Results obtained at \( \nu = 0.1, 0.7 \) for first test problem.

Figure 2: Results obtained at \( \nu = 0.2, 0.7 \) for second test problem.