on one nonlinear two-dimensional diffusion system

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Abstract: The two-dimensional system of nonlinear partial differential equations is considered. This system arises in process of vein formation of young leaves. Decomposition and variable directions type finite difference schemes are studied. Convergence of these schemes are given.

Key–Words: System of nonlinear partial differential equations, decomposition and variable directions type finite difference schemes.

1 Introduction

Nonlinear systems of partial differential equations describing various processes of diffusion are the subject of investigating for many scientists.

The main features of such systems often are expressed in fact that they contain equations of different kinds, which are strongly connected to each other. Mentioned condition for each concrete system determines the usage of respective methods of research, because general theory is incompletely developed for such systems even in linear case. Naturally arises the questions of approximate solution of these problems which also are connected with serious complexities as well.

The considered model is connected with process of vein formation in meristematic tissues of young leaves. Mentioned model has the following form [14]:

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \frac{\partial}{\partial x} \left( V \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y} \left( W \frac{\partial U}{\partial y} \right) , \\
\frac{\partial V}{\partial t} &= -V + f \left( V \frac{\partial U}{\partial x} \right) , \\
\frac{\partial W}{\partial t} &= -W + g \left( W \frac{\partial U}{\partial y} \right) .
\end{align*}
\]

(1)

Here \( U \) is the signal concentration and \( V, W \) are diffusion coefficients for flux parallel to \( Ox \) and \( Oy \) axes, respectively, \( f \) and \( g \) are given sufficiently smooth functions of their arguments, which satisfy the following conditions: \( 0 < d \leq f(r) \leq D, \ 0 < d \leq g(s) \leq D, \ |f'(r)| < D, \ |g'(s)| < D \), where \( d \) and \( D \) are constants.

The essential difficulties arise in the processes of constructing, investigating and realization of the numerical algorithms for model (1). Besides nonlinearity the complexity of studying such problems are conditioned also by its two-dimensionality. Therefore, naturally arises the question of reduction this problem to easier ones. In particular, it is very important to reduce the two-dimensional problem to the set of one-dimensional problems.

In [2], [14] some qualitative and structural properties of solutions of the boundary-value problems for the system (1) are established. In [2] investigations are carried out for one-dimensional case. The large theoretical and practical importance of the investigation and numerical solution of the initial-boundary value problems for the systems of type (1) is pointed out in [2] and [14].

Investigation and numerical solution systems of (1) type are done in numerous works (see, for example, [2], [4], [5], [7]-[12], [14], [15]).

Beginning from the basic works [3], [16] in which the scheme of variable directions were suggested, the methods of constructing of effective algorithms for the numerical solution of the multi-dimensional problems of mathematical physics and the sphere of problems solvable with the help of these algorithms were essentially extended. At present there are some effective algorithms for solving the multi-dimensional problems (see, for example, [6], [13] and [17] references there in). These algorithms mainly belong to the methods of splitting-up or sum-approximation according to their approximate properties.

In the work [1] the difference schemes belonging to the class of algorithms of variable directions are given. In [8] the construction and investigation of
the such kind scheme for multidimensional case of (1) type system is discussed for the initial-boundary value problem with Dirichlet boundary conditions.

In the present note one kind of such a scheme for the system (1) is given. We should note that some questions of the convergence of such type scheme as well as average model of sum approximation for the system (1) are discussed in the papers [4], [7]-[10]. The convergence of the difference scheme for one-dimensional analogue of the system (1) is studied in [7].

In the parallelepiped \( Q = [0, 1] \times [0, 1] \times [0, T] \), where \( T \) is a given positive constant, consider the system (1) with following boundary and initial conditions:

\[
U(x, 0, t) = U(x, 1, t) = 0, \\
U(0, y, t) = U(1, y, t) = 0, \\
U(x, y, 0) = U_0(x, y), \\
V(x, y, 0) = V_0(x, y), \\
W_0(x, y, 0) = W_0(x, y).
\]

Let us assume that \( U_0, V_0 \) and \( W_0 \) are given sufficiently smooth functions, such that \( U_0(x, y) \geq c, V_0(x, y) \geq c, W_0(x, y) \geq c \), where \( c \) is positive constant. Suppose that all necessary consistence conditions are satisfied and there exists the sufficiently smooth solution of the problem (1)-(3). It should be noted that the uniqueness of the solution of the problem (1)-(3) is studied in [4].

Under the conditions on functions \( f, g \) and \( V_0, W_0 \) it is not difficult to obtain the following estimates:

\[
c < V(x, y, t) < C, \\
c < W(x, y, t) < C,
\]

where \( c \) and \( C \) are positive constants.

Later we shall follow notations from [17]. Introduce on the domain \( Q \) the grids:

\[
\tilde{\omega}_{hr} = \tilde{\omega}_h \times \tilde{\omega}_h \times \omega_r, \\
\tilde{\omega}_{1hr} = \tilde{\omega}_{1h} \times \omega_r, \\
\tilde{\omega}_{2hr} = \tilde{\omega}_{2h} \times \omega_r, \\
\omega_r = \{ t_k = k\tau, k = 0, 1, ..., K, K\tau = T \},
\]

where

\[
\omega_h = \{ (x, y) = (ih, jh), i, j = 0, ..., M, Mh = 1 \}, \\
\omega_{1h} = \{ (x, y) = ((i - 1/2)h, jh), i, j = 1, ..., M \}, \\
\omega_{2h} = \{ (x, y) = (ih, (j - 1/2)h), i, j = 1, ..., M \},
\]

\[
u^k = u^k - u^{k-1}, \quad u^k = \frac{u^k - u^{k-1}}{\tau}, \quad u^k = \frac{u^{k+1} - u^k}{\tau}. 
\]

Let us correspond to problem (1)-(3) following decomposition finite difference scheme:

\[
\frac{u^{k+1} - u^k}{\tau} = (v^{k+1} u_{1x}^{k+1})_x, \\
\frac{v^{k+1} - v^k}{\tau} = -v^{k+1} + f(v^k u_{1x}^k), \\
u_1(x_i, y_j, t_k) = u_2(x_i, y_j, t_{k+1}), \\
u_1(x_i, y_j, 0) = U_0(x, y), \quad (x_i, y_j) \in \omega_h, \\
v(x_i, y_j, 0) = V_0(x, y), \quad (x_i, y_j) \in \omega_{1h}, \\
u_1(0, y_j, t_{k+1}) = 0, \quad u_1(1, y_j, t_{k+1}) = 0, \\
j = 0, 1, ..., M, \quad k = 0, 1, ..., K - 1;
\]

\[
u_2(x_i, y_j, t_k) = u_1(x_i, y_j, t_{k+1}), \quad (x_i, y_j) \in \omega_h, \\
w(x_i, y_j, 0) = W_0(x, y), \quad (x_i, y_j) \in \omega_{2h}, \\
u_2(x_i, 0, t_{k+1}) = 0, \quad u_1(x_i, 1, t_{k+1}) = 0, \\
i = 0, 1, ..., M, \quad k = 0, 1, ..., K - 1.
\]

Here functions \( u_1, u_2 \) are defined on \( \omega_{hr} \); \( v, w \) - on \( \omega_{1hr} \) and \( \omega_{2hr} \) respectively.

Let us correspond to the problem (1)-(3) following scheme of variable directions:

\[
\frac{u_1^{k+1} - u_1^k}{\tau} = (v^{k+1} u_{1x}^{k+1})_x + (v^k u_{1y}^k) y, \\
\frac{v^{k+1} - v^k}{\tau} = -v^{k+1} + f(v^k u_{1x}^k), \\
u_1(x_i, y_j, t_k) = u_2(x_i, y_j, t_{k+1}), \\
u_1(x_i, y_j, 0) = U_0(x, y), \quad (x_i, y_j) \in \omega_h, \\
v(x_i, y_j, 0) = V_0(x, y), \quad (x_i, y_j) \in \omega_{1h}, \\
u_1(0, y_j, t_{k+1}) = 0, \quad u_1(1, y_j, t_{k+1}) = 0,
\]
j = 0, 1, ..., M, k = 0, 1, ..., K - 1;

\[
\frac{u_2^{k+1} - u_2^k}{\tau} = (u_2^{k+1})_{x} + (u_2^{k+1})_{y}, \\
\frac{w^{k+1} - w^k}{\tau} = -w^{k+1} + g(w_2^k),
\]

(7)

Under the sufficiently smoothness of exact solution of the problem (1)-(3) the difference schemes (4),(5) and (6),(7) approximate the problem (1)-(3) with the rate \( O(\tau + h^2) \).

Let us introduce following notations for the errors: \( Z_1 = u_1 - U, \ Z_2 = u_2 - U, \ S_1 = v - V, \ S_2 = w - W \).

The following statement takes place.

**Theorem** If the problem (1)-(3) has the sufficiently smooth solution then finite difference schemes (4), (5) and (6), (7) converge to the exact solution of the problem (1)-(3) when \( \tau \rightarrow 0, \ h \rightarrow 0 \) and the following inequality holds

\[
\|Z_1\|_{\omega} + \|Z_2\|_{\omega} + \|S_1\|_{\omega} + \|S_2\|_{\omega} \leq C(\tau + h^2).
\]

Here \( C \) is a positive constant independent of \( \tau \) and \( h \), norms are discrete analogous of the norm of space \( L_2 \).

The problem similar to (1)-(3) with Dirichlet boundary conditions on part of boundary and Neumann boundary conditions on other side is also studied. In this case instead of (2) the following boundary conditions are considered:

\[
U(0, 0, t) = 0, \quad U(x, 0, t) = 0,
\]

\[
V(x, y, t) = \frac{\partial U(x, y, t)}{\partial x} \bigg|_{x=1} = \eta_1(y, t), \tag{8}
\]

\[
W(x, y, t) = \frac{\partial U(x, y, t)}{\partial y} \bigg|_{y=1} = \eta_2(x, t),
\]

where \( \eta_1 \) and \( \eta_2 \) are given functions.

Let us note that boundary conditions here are dictated by biological viewpoint [14].

The results analogical that are stated in above mentioned Theorem are also valid for the problem (1),(3),(8).

2 Conclusion

Numerous numerical experiments are done for the problem (1)-(3) using studied (4),(5) and (6),(7) schemes and their modifications for the problem (1),(3),(8). Carried out numerical experiments show that in all cases numerical solutions fully agree with the theoretical results.

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**References:**


