A note on Derivations of Commutative Rings

MICHAEL GR. VOSKOGLOU
School of Technological Applications
Graduate Technological Educational Institute (T. E. I.)
Meg. Alexandrou 1, 26334 Patras
GREECE

e-mail: voskoglou@teipat.gr, mvosk@hol.gr
URL: http://eclass.teipat.gr/RESEmSTE101/document

Abstract: We study properties of a differentially simple commutative ring $R$ with respect to a set $D$ of derivations of $R$. Among the others we investigate the relation between the $D$-simplicity of $R$ and that of the local ring $R_P$ with respect to a prime ideal $P$ of $R$ and we prove a criterion about the $D$-simplicity of $R$ in case where $R$ is a 1-dimensional (Krull dimension) finitely generated algebra over a field of characteristic zero and $D$ is a singleton set. The above criterion was quoted without proof in an earlier paper of the author.

Key-Words: Derivations, Commutative rings, Local rings, Finitely generated algebras, Differential simplicity, Skew polynomial rings

1 Introduction

All the rings considered in this paper are commutative with identity and all the fields are of characteristic zero, unless it is otherwise stated. A local ring is understood to be a Noetherian ring with a unique maximal ideal; if $R$ is not Noetherian, then we call it a quasi-local ring. A Noetherian ring with finitely many maximal ideals is called a semi-local ring. For special facts on commutative algebra we refer freely to [1],[7], and [11], while for the concepts of Algebraic Geometry used in the paper we refer freely to [3].

Let $R$ be a ring (not necessarily commutative) and let $D$ be a set of derivations of $R$. Then an ideal $I$ of $R$ is called a $D$-ideal, if $d(I) \subseteq I$ for all derivations $d$ in $D$, and $R$ is called a $D$-simple ring if it has no non trivial $D$-ideals. When $D$ is a singleton set, say $D=\{d\}$, then, for reasons of simplicity, $I$ is called a $d$-ideal and $R$ is called a $d$-simple ring respectively. In general $R$ is called a differentially simple ring, if there exists at least one set $D$ of derivations of $R$, such that $R$ is a $D$-simple ring.

Every $D$-simple ring $R$ contains the field $F=\text{C}(R) \cap \bigcap_{d \in D} \ker d$, where $\text{C}(R)$ denotes the center of $R$, and therefore, if $R$ is of characteristic zero, then it contains the field of rational numbers.

Non commutative differentially simple rings exist in abundance, e.g. every simple ring $R$ is $D$-simple for any set $D$ of derivations of $R$, and that is why our interest is turned to commutative rings only. Some characteristic examples of commutative $D$-simple rings, where $D$ is a non singleton set of derivations, are given below.

**Example 1.1:** The polynomial ring $R=k[x_1,x_2,\ldots,x_n]$ over a field $k$ is a $D$-simple ring, where $D=\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\}$.  

**Proof:** If $I$ is a non zero $D$-ideal of $R$ and $f$ is in $I$, we can write $f=\sum_{i=0}^{k} f_i x_1^k$, with $f_i$ in $k[x_2,\ldots,x_n]$ for each i. Then $\frac{\partial^k f}{\partial x_1^k}=kf_0$ is a non zero element of $I \cap k[x_2,\ldots,x_n]$. Repeating the same argument for $f_0$ and keep going in the same way one finds, after $n$ at most steps, that $I \cap k \neq \{0\}$. Thus $I=R$.

**Example 1.2:** The power series ring $R=k[[x_1,x_2,\ldots,x_n]]$ over a field $k$ is a $D$-simple ring, where $D$ is as in example 1.1.

**Example 1.3:** Let $R=\frac{IR[x,y,z]}{(x^2+y^2+z^2-1)}$ be the coordinate ring of the real sphere (IR denotes the field of the real numbers). Consider the IR-derivations $d_1$ and $d_2$ of the polynomial ring $IR[x,y,z]$ defined by $d_1: x \rightarrow y+z, y \rightarrow z-x, z \rightarrow -y, \text{ and } d_2: x \rightarrow y+2z, y \rightarrow xyz-x, z \rightarrow -xy^2-2x$. Then, since $d_1(x^2+y^2+z^2-1)=0$, for $i=1,2$, $d_i$
induces an IR-derivation of R, denoted also by \( d_i \). 
Set \( D = \{ d_1, d_2 \} \), then R is a D-simple ring (cf. [22], Lemma 3.1). 
The following example concerns a non trivial case of a differentially simple ring of prime characteristic:

**Example 1.4:** Let \( k \) be a field of prime characteristic, say \( p \), and let \( R \) and \( D \) be as in example 1.1. Then \( I=(x_1^p, x_2^p, \ldots, x_n^p) \) is obviously a D-ideal of \( R \) and therefore each \( \frac{\partial}{\partial x_i} \), \( i=1,2,\ldots,n \), induces a derivation, say \( d_i \), of the factor ring \( R/I \) by \( d_i(f+I)=\frac{\partial f}{\partial x_i}+I \), for all \( f \) in \( R \). Set \( D'=\{ d_1,d_2,\ldots,d_n \} \), then \( R/I \) is a \( D' \)-simple ring.

**Proof:** If \( A \) is a non zero \( D' \)-ideal of \( R/I \), then \( A'=\{ f+I \in A : f+I \in A \} \) is obviously a D-ideal of \( k[x_1, x_2, \ldots, x_n] \) containing properly \( I \). It becomes evident that there exists \( f \) in \( A' \) with all its terms of the form \( cx_1^{m_1}x_2^{m_2}\ldots x_n^{m_n} \), with \( c \) in \( k \) and \( 0 \leq m \leq p \) , for each \( i=1, 2,\ldots,n \). Let \( m \) be the greatest integer which appears as exponent of \( x_i \) in the terms of \( f \). Without loss of the generality we may assume that \( m \neq 0 \). Then \( 0 \neq \partial^m f = m!f_i(x_2,\ldots,x_n) \) is in \( A' \). If \( f_i \) is not in \( k \), we repeat the same argument for \( f_i \) and we keep going in the same way until we find, after a finite number of steps, that \( A' \cap k \neq \{0\} \). Thus \( A=R/I \) and the result follows.-

2. Characteristic properties of d-simple rings.

There is no general criterion known to decide whether or not a given ring \( R \) is differentially simple. However it seems that there is a connection between the differential simplicity and the Krull dimension of \( R \) (denoted by \( \dim R \)). The following result demonstrates this connection, when \( R \) is of prime characteristic:

**Theorem 2.1:** Let \( R \) be a ring of prime characteristic, say \( p \), and let \( D \) be a set of derivations of \( R \), such that \( R \) is D-simple. Then \( R \) is a 0-dimensional, quasi-local ring.

**Proof:** Let \( M \) be a maximal ideal of \( R \) and let \( I \) be the ideal of \( R \) generated by the set \( \{ m^p : m \in M \} \). Then, since \( R \) is of characteristic \( p \), \( I \) is a proper D-ideal of \( R \), therefore the D-simplicity of \( R \) implies that \( I=\{0\} \). Thus \( M \) is contained in the nil radical, say \( N \), of \( R \) and therefore \( M=N \). Let now \( P \) be a prime ideal of \( R \) contained in \( M \). Then, since \( N \) is equal to the intersection of all prime ideals of \( R \) and \( M=N \), we get that \( M=P \). Thus \( N \) is the unique prime ideal of \( R \) and this proves the theorem.-

As an immediate consequence of the above theorem, if \( R \) is a domain, then \( R \) is a field (since \( M=N=\{0\} \) and therefore the interest is turned mainly to rings of characteristic zero. In this case it is well known that a differentially simple ring is always a domain, while, if a ring \( R \) contains the rational numbers and has no non-zero prime D-ideals for a set \( D \) of derivations of \( R \), then \( R \) is a D-simple ring (cf. [8], Corollary 1.5). Seidenberg proved that, if \( R \) is a domain which is a finitely generated algebra over a field, then \( R \) is a Der R-simple ring (where Der \( R \) denotes the set of all derivations of \( R \)), if, and only if, \( R \) is a regular ring (cf. [14], Theorems 3 and 5). Thus, if \( R \) is D-simple for any set \( D \) of derivations of \( R \), then \( R \) is a regular ring. Hart [4] proved that this is actually true for the wider class of G-rings, which contains all finitely generated algebras and all complete local rings over fields and is closed under localization ([111], pp. 249-257). The converse of this statement is not true in general; e.g. the coordinate ring \( R \) of the real sphere (example 3.1 above), although it is regular, admits no derivation \( d \) such that \( R \) is a d-simple ring (cf [5], example iii). However, if \( R \) is a regular local ring of finitely generated type over a field \( k \) (i.e. a localization of a finitely generated k-algebra at a prime ideal of \( R \)), Hart [5] constructed a derivation \( d \) of \( R \), such that \( R \) is a d-simple ring. The above results show that for a wide class of rings of characteristic zero the d-simplicity is connected with the regularity, a property which requires from a ring \( R \) to have a "special" kind of dimension. In fact, in this case every maximal ideal \( M \) of \( R \) can be generated by \( \dim R_M \) elements, where \( R_M \) denotes the localization of \( R \) at \( M \). In particular, if \( R \) is a finitely generated algebra, then \( R \) can be generated by \( \dim R \) elements.

Consider now the localization \( R_P \) of a ring \( R \) at a proper prime ideal \( P \) of \( R \). Then every derivation \( d \) of \( R \) extends to a derivation of \( R_P \) by the usual rule of differentials for rational functions. We prove the following result:

**Theorem 2.2:** Let \( R \) and \( R_P \) be as above. Then, if \( R \) is a D-simple ring with respect to a set \( D \) of derivations of \( R \), \( R_P \) is also D-simple. Conversely, if \( R_P \) is D-simple with respect to a set \( D \) of derivations of \( R_P \) whose restrictions to \( R \) are derivations of \( R \), then \( R \) is also a D-simple ring.
Proof: Assume first that R is a D-simple ring, while $R_p$ is not. Let q be a prime ideal of $R_p$, such that $d(q) \subseteq q$ for all $d$ in D. Then there exists a prime ideal Q of R not meeting P, such that $q = Q^e = \{ \frac{r}{s} : r \in Q$, $s \in R-P \}$ (cf. [1], Proposition 3.11). Since R is a D-simple ring, we can find r in Q, such that $d(r) \notin Q$ for all $d$ in D. But $d(\frac{r}{s}) = d(r)$ is in $Q^e = q$, therefore $d(r)$ is in Q, which is absurd.

For the converse, assume that there exists a non zero prime ideal, say Q, of R, such that $d(Q) \subseteq Q$, for all derivations d of D. Then, for all r in Q, $d(r) \in Q \iff sd(r) \in Q$, for all s in P-R $\iff \frac{sd(r) - rd(s)}{s^2} = d(\frac{r}{s}) \in Q^e$. Thus $R_p$ is not a D-simple ring, which is a contradiction. –

Corollary 2.3: Let R, $R_p$ and D be as in the converse statement of the above theorem. Assume further that R is a semi-local finitely generated algebra over a field k. Then R is a field.

Proof: By Noether’s normalization Lemma R is an integral extension of a polynomial ring, say $k[x_1,x_2,...,x_m]$, and dim R=m. Further, by the correspondence of maximal ideals in integral extensions, the number of maximal ideals of $k[x_1,x_2,...,x_m]$ should be finite. But $(x_1-a_1, x_2-a_2,...,x_m-a_m)$ is a maximal ideal of $k[x_1,x_2,...,x_m]$, for all choices of the $a_i$'s in k, $i=1,2,...,m$. Since k is of characteristic zero, k is an infinite field, therefore, if m ≠ 0, then $k[x_1,x_2,...,x_m]$ has an infinite number of maximal ideals, which is a contradiction. Thus m=0 and the Krull dimension of R is zero. But, by Theorem 2.2, R is a D-simple ring, therefore R is a domain and the result follows.

The following example illustrates Theorem 2.2:

Example 2.4: Let $R=k[x_1,x_2]$ be a polynomial ring over a field k; then it is easy to check that P=$(x_1)$ is a prime ideal of R. By example 1.1 R is a D-simple ring, with D=$\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \}$. Therefore, by Theorem 2.2, $R_p$ is also D-simple, where D in this case denotes the set of the extensions of $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ to $R_p$.

3. Finitely generated d-simple algebras of dimension 1

Let D be a non singleton set of derivations of a ring R, and assume that R is d-simple for some d in D, then obviously R is also a D-simple ring.

The converse is not true; e.g., although there exists a set D of 2 derivations of the coordinate ring R of the real sphere such that R is D-simple (cf. Example 1.3), there is no derivation d of R such that R is d-simple.(cf. [5], example iii).

For a second counter example notice that, if a complete local ring R is d-simple for some derivation d of R, then the Krull dimension of R is 1 (cf. [21], Theorem 2.3). Thus, although the power series ring $R=k[[x_1,x_2,...,x_n]]$ over a field k is $\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2},...,\frac{\partial}{\partial x_n} \}$-simple (cf. Example 1.2), there is no derivation d of R such that R is a d-simple ring. Thus the “strongest” property of a ring R concerning its differential simplicity is to admit a derivation d, such that R is d-simple.

Notice now that, if R is a differentially simple ring (and therefore a domain) of dimension 0, then R is a field. On the contrary, they are known several non trivial cases of d-simple rings of dimension 1, such as the polynomial and the power series rings in one variable over a field (see examples 1.1 and 1.2 above), as well as non trivial examples of 1-dimensional rings which are not d-simple for any derivation d (e.g. see example of [6]). Therefore it looks interesting to a search for necessary and sufficient conditions for the d-simplicity of 1-dimensional rings. We shall do this below for finitely generated algebras over a field k. In this case it is easy to prove:

Proposition 3.1: Let R be a finitely generated algebra over a field k, say $R=k[y_1,y_2,...,y_n]$. Assume further that there exists a derivation d of R such that R is a d-simple ring. Then $R=(d(y_1),d(y_2),...,d(y_n))$.

Proof: Theorem 2.4 of [21].

In Theorem 2.4 of [21] was quoted without proof that the converse of Proposition 3.1 is also true. In order to prove this we need the following two lemmas:

Lemma 3.2: Let R and S be algebras over a field k such that S is integral over R, and let d be a derivation of R which extends to a derivation of S. Then, if P is a prime d-ideal of R and T is a prime ideal of S such that $T \cap R = P$, T is a d-ideal of S.
Proof: Let t be an element of T such that d(t) is not in T. Since S is integral over R, there exists a positive integer n such that
\[ f(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_1t + a_0, \]
with a_i in R for each i=0,1,…,n-1. Thus
\[ a_0 = -t^{n-1}a_{n-1}t^{n-2} + \ldots + a_0 = 0, \]
in T \cap R = P, therefore d(a_0) is also in P. But
\[ d(f(t)) = nf(t) + da_{n-1}t^{n-1} + \ldots + a_0, \]
so d(a_0) = 0, which contradicts our hypothesis. The converse is also true by Proposition 3.1. -

The following example illustrates the above theorem:

**Example 3.5:** The coordinate ring
\[ R = \frac{k[x_1,x_2]}{(x_1^2 - 2)} \]
of the circle defined over a field k admits k-derivations d, such that R is a d-simple ring.

**Proof:** Theorem 2.2 of [23].

4. Remarks and examples

(i) As it becomes evident from the last part of the proof of Lemma 3.2, if k is an algebraically closed field, then Theorem 3.3 is a straightforward consequence of the Hilbert’s Nullstellensatz. Also, if we take k to be the field of rational numbers, then, since every derivation of R is a k-derivation, Theorem 3.3 holds for any derivation of R.

(ii) A derivation d of a ring R is called a simple derivation, if R is a d-simple ring. They are certainly known examples of finitely generated algebras of dimension greater than 1, and even of infinite dimension, admitting simple derivations, typified by polynomial rings in finitely and infinitely many variables over a field k, and by Laurent polynomial rings in finitely many variables, say n, over k provided that the dimension of k (as a vector space) over the field of the rational numbers is greater or equal to n (cf. [21], section 3). In particular, for the polynomial ring k[x,y] most of the published examples of its simple derivations with d(x)=1 are of the form d = \frac{\partial}{\partial x} + F(x,y) \frac{\partial}{\partial y}, where F(x,y) is a polynomial of k[x,y] with deg F(x,y) ≤ 2 (e.g. [2], [6], [10], etc). Nowicki [13] proved recently that
\[ \frac{\partial}{\partial x} + (y + px) \frac{\partial}{\partial y}, \]
where s is an arbitrary positive integer and 0 ≠ p in k, is a simple derivation of
k[x,y]. His proof is based on the well known fact that a derivation d of k[x,y] such that d(x)=1 is simple, if, and only if, d has no Darboux polynomials (cf. [12]).

(iii) It is well known that, if the coordinate ring of a variety, say Y, over a field k is regular, then Y is a smooth variety (i.e. Y has no singular points). This result, combined to the fact that a d-simple finitely generated algebra is a regular ring (cf. [14]), shows that, if R is the coordinate ring of a singular variety, then R admits no simple derivations. In [23] we have presented some characteristic examples of smooth varieties over k (e.g. cylinder, real torus considered as a 2-dimensional surface in 4 dimensions, etc), which admit at least one simple derivation. We emphasize that this is not true in general, the typical counter example being the coordinate ring of the real sphere (see example 1.3 above). But what happens if k is an algebraically closed field? Consider for example the coordinate ring

\[ R = \frac{C[x,y,z]}{(x^2 + y^2 + z^2 - 1)} \]

of the complex sphere (with C we denote the field of complex numbers). We have studied the unpublished work of a colleague, who claims that he has constructed a simple derivation of R. Although we have some doubts about the correctness of the given proof, this possibility raises the following question: If R is the coordinate ring of a smooth variety over an algebraically closed field, does R admit simple derivations? The answer to this question is not known (at least to me).

(iv) The D-simplicity of a ring R (not necessarily commutative) is connected to the simplicity of the corresponding skew polynomial ring R[x,D] of derivation type over R (cf. [15], [20] and [22]), while the prime (semiprime) ideals of S=R[x,D] are connected with the D-prime (semiprime) ideals of R (cf. [16],[17] and [18]). More explicitly, if R is a D-simple ring of characteristic zero and all the elements of D are outer derivations of R, then S is a simple ring. Conversely if S is a simple ring, then R is D-simple and no d in D is an inner derivation induced by an element of \( \bigcap_{d \in D, \ker d} \). An analogous result holds for the case of prime characteristic. Also, if P is a prime ideal of S, then P \( \cap \) S is a D-prime ideal of R, while, if I is a D-prime ideal of R, then IS is a prime ideal of S. We recall that an ideal I of R is called a D-prime ideal, if, given any two D-ideals A, B of R such that AB \( \subseteq \) I, we have that either A \( \subseteq \) I , or B \( \subseteq \) I .

(v) For other interesting properties of the differential ideals of a ring the reader may look [19].

References


