On Asymptotic Behavior and Numerical Resolution of One Nonlinear Maxwell’s Model

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Abstract: Large time behavior of solutions to one nonlinear Maxwell’s model is studied. Initial-boundary value problems with mixed boundary conditions is considered. The rates of convergence are given too. Numerical resolution of problem is given as well.

Key–Words: Nonlinear integro-differential equations, large time asymptotics, numerical resolution

1 Introduction

A great variety of applied problems are modeled by nonlinear integro-differential equations. Such equations arise, for example, for mathematical modeling of the process of penetrating of magnetic field in the substance. In a quasistationary case the corresponding system of Maxwell’s equations has the form [5]:

\[ \frac{\partial H}{\partial t} = -\text{rot}(\nu_m \text{rot}H), \]

\[ c_v \frac{\partial \theta}{\partial t} = \nu_m (\text{rot}H)^2, \]

where \( H = (H_1, H_2, H_3) \) is a vector of the magnetic field, \( \theta \) is temperature, \( c_v \) and \( \nu_m \) characterize the thermal capacity and electroconductivity of the substance. System (1) defines the process of diffusion of the magnetic field and equation (2) – change of the temperature at the expense of Joule’s heating without taking into account the heat conductivity.

If \( c_v \) and \( \nu_m \) depend on temperature \( \theta \), i.e., \( c_v = c_v(\theta) \), \( \nu_m = \nu_m(\theta) \), then the system (1),(2) can be rewritten in the following form [3]:

\[ \frac{\partial H}{\partial t} = -\text{rot}\left[a \left( \int_0^t |\text{rot}H|^2 \, d\tau \right) \text{rot}H \right], \]

where function \( a = a(S) \) is defined for \( S \in [0, \infty) \).

Note that (3) integro-differential model is complex. Equations and systems of (3) type still yield to the investigation only for special cases [1]-[3],[6],[7],[9]-[19]. The existence and uniqueness of the global solutions of such models have been proved in [1]-[3],[12] using of Galerkin and compactness methods [8],[20]. Materials for solvability and uniqueness properties of the (3) type models can be found also in [6],[7],[9].

If the magnetic field has the form \( H = (0, 0, U) \) and \( U = U(x, t) \), where \( U = U(x, t) \) is a scalar function of time and of one spatial variables, then \( \text{rot}H = (0, -\frac{\partial U}{\partial x}, 0) \) and system (3) will take the form:

\[ \frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left[a(S) \frac{\partial U}{\partial x} \right], \]

where \( S(x, t) = \int_0^t \left( \frac{\partial U}{\partial x} \right)^2 \, d\tau. \)

In the work [6] some generalization of (3) type equation is proposed. In particular, assuming the magnetic field has the form \( H = (0, 0, U) \) and again \( U = U(x, t) \) then following integro-differential equation is obtained [12]:

\[ \frac{\partial U}{\partial t} = a(S) \frac{\partial^2 U}{\partial x^2}, \]

where \( S(t) = \int_0^t \int_0^1 \left( \frac{\partial U}{\partial x} \right)^2 \, dx \, d\tau. \)

The asymptotic behavior of the solutions and numerical approximation of (4) and (5) type equations and corresponding systems for two component magnetic field have been object of intensive research in recent years [10]-[19].
The purpose of this note is to study the asymptotic behavior of solutions of the equation (4). Our object is to give large time asymptotics of the solutions of the initial-boundary value problem with Dirichlet boundary condition on one side of lateral boundary and Neumann boundary condition on other side. The attention is paid on the case \( a(S) = (1 + S)^p \), \( 0 < p \leq 1 \).

The rest of the paper is organized as follows. In the second section we discuss asymptotic behavior of solution of initial-boundary value problem. Section three is devoted to the numerical resolution. In the last forth section some conclusions are given.

## 2 Asymptotic behavior of solution

In the domain \( Q = (0, 1) \times (0, \infty) \) let us consider the following initial-boundary value problem:

\[
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left[ a(S) \frac{\partial U}{\partial x} \right], \quad (x, t) \in Q, \quad (6)
\]

\[
U(0, t) = 0, \quad \left. \frac{\partial U(x, t)}{\partial x} \right|_{x=1} = 0, \quad t \geq 0, \quad (7)
\]

\[
U(x, 0) = U_0(x), \quad x \in [0, 1], \quad (8)
\]

where

\[
S(x, t) = \int_0^t \left( \frac{\partial U}{\partial x} \right)^2 dt, \quad (9)
\]

or

\[
S(t) = \int_0^t \int_0^1 \left( \frac{\partial U}{\partial x} \right)^2 dx dt, \quad (10)
\]

\( U_0 = U_0(x) \) is a given function.

In [10] asymptotic behavior of solution and in [4] numerical resolution of problem (6)-(8),(10) was considered.

The main purpose of this work is to prove the following statement.

**Theorem 1** If \( a(S) = (1 + S)^p \), \( 0 < p \leq 1 \); \( U_0 \in H^2(0, 1) \cap H_0^1(0, 1) \), then for the solution of problem (6)-(9) the following estimate holds:

\[
\left| \frac{\partial U(x, t)}{\partial x} \right| + \left| \frac{\partial U(x, t)}{\partial t} \right| \leq C \exp \left( -\frac{t}{2} \right).
\]

Note that we use usual Sobolev spaces \( H^k(0, 1) \) and \( H_0^k(0, 1) \). Here and below \( C, C_1 \) and \( c \) denote positive constants independent of \( t \).

One must note that in similar fashion as in [15] the following statement can be proven for problem (6)-(9):

**Theorem 2** If \( a(S) = (1 + S)^p \), \( 0 < p \leq 1 \); \( U_0 \in H^2(0, 1) \cap H_0^1(0, 1) \), then for the solution of problem (6)-(9) the following estimate holds:

\[
\left| \frac{\partial U(x, t)}{\partial x} \right|_{L_2(0, 1)} + \left| \frac{\partial U(x, t)}{\partial t} \right|_{L_2(0, 1)} \leq C \exp \left( -\frac{t}{2} \right).
\]

Now we are going to prove Theorem 1. To prove this theorem we need some auxiliary statements.

**Lemma 1** The solution of problem (6)-(9) satisfies the estimate

\[
\int_0^t \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 dx d\tau \leq C.
\]

**Proof:** Let us differentiate (6) with respect to \( t \)

\[
\frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial \tau} \left( (1 + S)^p \frac{\partial U}{\partial \tau} \right) \right] = 0,
\]

multiplying (11) by \( \partial U/\partial \tau \), and by integrating the resulting relation over the domain \((0, 1)\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 dx + \int_0^1 (1 + S)^p \left( \frac{\partial^2 U}{\partial \tau \partial x} \right)^2 dx
\]

\[
+ p \int_0^1 (1 + S)^{p-1} \left( \frac{\partial U}{\partial \tau} \right)^3 \frac{\partial^2 U}{\partial \tau \partial x} dx = 0.
\]

This, together with the Poincare-Friedrichs inequality, implies that

\[
\frac{d}{dt} \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 dx + \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 dx
\]

\[
+ \frac{p}{2} \int_0^1 (1 + S)^{p-1} \left( \frac{\partial U}{\partial \tau} \right)^4 dx \leq 0.
\]

Integrating relation (13) over the domain \((0, t)\) and by using the formula for integration by parts we obtain the inequality

\[
\int_0^t \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 dx d\tau + 2 \int_0^t \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 dx d\tau \leq C,
\]

which completes the proof of Lemma 1.

**Lemma 2** For the function \( S \) the following estimate holds

\[
c\varphi^{1/(1+2p)}(t) \leq 1 + S(x, t) \leq C\varphi^{1/(1+2p)}(t),
\]
where

\[ \varphi(t) = 1 + \int_0^t \int_0^1 (1 + S)^{2p} \left( \frac{\partial U}{\partial x} \right)^2 \, dx \, d\tau. \quad (14) \]

**Proof:** From (9) it follows that

\[ \frac{\partial S}{\partial t} = \left( \frac{\partial U}{\partial x} \right)^2, \quad S(x, 0) = 0. \]

Multiplying the last equation by \((1 + S)^{2p}\), we obtain

\[ \frac{1}{1 + 2p} \frac{\partial (1 + S)^{1+2p}}{\partial t} = \sigma^2, \quad (15) \]

where

\[ \sigma(x, t) = (1 + S)^p \frac{\partial U}{\partial x}. \quad (16) \]

According to this notation, (6) can be represented in the form

\[ \frac{\partial U}{\partial t} = \frac{\partial \sigma}{\partial x}. \]

We have

\[ \sigma^2(x, t) = \int_0^1 \sigma^2(y, t) \, dy + 2 \int_0^x \sigma(\xi, t) \frac{\partial \sigma(\xi, t)}{\partial \xi} \, d\xi \, dy \quad (17) \]

Taking into account Lemma 1 and relations (14), (15), (17), we arrive at

\[ \frac{1}{1 + 2p} (1 + S)^{1+2p} \leq \int_0^t \int_0^1 \sigma^2(y, \tau) \, d\tau \, dy + \int_0^t \int_0^1 \left( \frac{\partial U(x, \tau)}{\partial \tau} \right)^2 \, dx \, d\tau \]

\[ \leq \frac{1}{1 + 2p} \leq 2 \int_0^t \int_0^1 \sigma^2(y, \tau) \, d\tau \, dy + C_1 \leq C_2 \varphi(t). \]

Therefore,

\[ 1 + S(x, t) \leq C \varphi^{1/(1+2p)}(t). \quad (18) \]

Analogously,

\[ \frac{1}{1 + 2p} (1 + S)^{1+2p} \geq \frac{1}{2} \int_0^t \int_0^1 \sigma^2(y, \tau) \, d\tau \, dy - C_1 = \frac{1}{2} \varphi(t) - C_2. \quad (19) \]

Since

\[ C_2(1 + S)^{1+2p} \geq C_2 \]

it follows from (19) and (20) that

\[ 1 + S(x, t) \geq c \varphi^{1/(1+2p)}(t). \quad (21) \]

Finally, the estimates (18) and (21) imply the assertion of Lemma 2.

Using Lemma 2, relation (14), and Theorem 2, we obtain

\[ \frac{d \varphi(t)}{dt} = \int_0^1 (1 + S)^{2p} \left( \frac{\partial U}{\partial x} \right)^2 \, dx \]

\[ \leq C \varphi^{2p/(1+2p)}(t) \exp(-t). \]

Integrating this inequality and by using (14), we obtain the inequality

\[ 1 \leq \varphi(t) \leq C, \]

and consequently, by Lemma 2, we have

\[ 1 \leq 1 + S(x, t) \leq C. \quad (22) \]

Relations (16) and (17), together with (22) and Theorem 2, imply the inequality

\[ \sigma^2(x, t) \leq 2 \int_0^t \int_0^1 (1 + S)^{2p} \left( \frac{\partial U}{\partial x} \right)^2 \, dx \]

\[ + \int_0^t \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 \, dx \leq C \exp(-t). \]

Finally, from (16) and (22) we obtain

\[ \left| \frac{\partial U}{\partial \tau} \right| = |\sigma(x, t)(1 + S)^{-p}| \leq C \exp \left( -\frac{t}{2} \right). \quad (23) \]

Using Cauchy-Schwarz inequality in (12), multiplying scalarly by \(\exp(2t)\) and integrate the resulting equation over the interval \((0, t)\) we get

\[ \int_0^t \exp(2\tau) \frac{d}{d\tau} \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 \, dx \, d\tau \]

\[ \int_0^t \exp(2\tau) \frac{d}{d\tau} \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 \, dx \, d\tau \]
+ \int_0^t \exp(2\tau) \int_0^1 (1 + S)^p \left( \frac{\partial^2 U}{\partial x \partial \tau} \right)^2 \, dx \, d\tau \\
\leq p^2 \int_0^t \exp(2\tau) \int_0^1 (1 + S)^{p-2} \left( \frac{\partial U}{\partial x} \right)^6 \, dx \, d\tau.

By taking into account inequality (22) and Theorems 1 and 2 and by performing simple transformations, we obtain the inequalities

\int_0^t \exp(2\tau) \int_0^1 \left( \frac{\partial^2 U}{\partial x \partial \tau} \right)^2 \, dx \, d\tau
\leq - \exp(2t) \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 \, dx + \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 \, dx \bigg|_{t=0}
+ 2 \int_0^t \exp(2\tau) \int_0^1 \left( \frac{\partial U}{\partial \tau} \right)^2 \, dx \, d\tau + C \int_0^t \exp(-\tau) \, d\tau
\leq \frac{C_1 + C_2 \exp(t) + C_3}{4} \int_0^t \left( \frac{\partial^2 U}{\partial \tau^2} \right)^2 \, dx \, d\tau + C_4 \exp(-t)
+ C_5 \int_0^t \exp(2\tau) \int_0^1 \left( \frac{\partial^2 U}{\partial x \partial \tau} \right)^2 \, dx \, d\tau + C_6 \int_0^t \exp(-\tau) \, d\tau
+ C_7 \int_0^t \exp(2\tau) \exp(-\tau) \int_0^1 \left( \frac{\partial^2 U}{\partial \tau \partial x} \right)^2 \, dx \, d\tau,

i.e.

\int_0^t \left( \frac{\partial^2 U}{\partial \tau^2} \right)^2 \, dx \leq C \exp(-t).

This a priori estimate, together with the relation

\frac{\partial U(x, t)}{\partial t} = \frac{1}{y} \frac{\partial U(y, t)}{\partial y} \frac{1}{\partial \tau} + \frac{1}{\partial \tau} \frac{\partial U \xi(t)}{\partial \tau} \frac{d\xi}{dy}

and Theorem 2 proves Theorem 1.

3 Numerical resolution

In the cylinder (0, 1) x (0, T), where T is a positive constant, let us consider the following initial-boundary value problem:

\frac{\partial U}{\partial t} - \frac{\partial}{\partial x} \left\{ \left[ 1 + \int_0^1 \left( \frac{\partial U}{\partial x} \right)^2 \, dx \right] \frac{\partial U}{\partial x} \right\} = f(x, t),
\[ U(0, t) = 0, \quad \frac{\partial U(x, t)}{\partial x} \bigg|_{x=1} = 0, \quad t \geq 0, \quad (25) \]

\[ U(x, 0) = U_0(x), \quad x \in [0, 1]. \]

Here \( f = f(x, t) \) and \( U_0 = U_0(x) \) are given functions of their arguments.

In order to describe the finite difference method for problem (25), we introduce a net whose mesh point are denoted by \( (x_i, t_j) = (ih, j\tau) \), where \( i = 0, 1, ..., M; \quad j = 0, 1, ..., N \) with \( h = \frac{1}{M}, \quad \tau = \frac{T}{N}. \)

The initial line is denoted by \( j = 0 \). The discrete approximation at \( (x_i, t_j) \) is designed by \( u_i^j \) and the exact solution to the problem (25) by \( U_i^j \).

We will use the following known notations:

\[
\begin{align*}
&u_{i+1}^{j+1} - u_i^{j+1} = \frac{h}{h}, \quad u_{i-1}^{j+1} = \frac{u_{i+1}^{j+1} - u_i^{j+1}}{h}, \\
u_i^{j+1} - u_i^{j} = \tau, \quad u_i^{j} - u_i^{j-1} = \frac{\tau}{\tau}.
\end{align*}
\]

Let us correspond to the problem (25) the following difference scheme:

\[ u_i^{j+1} - \left\{ 1 + \tau \sum_{k=1}^{j+1} \left( u_{i+k}^j \right)^2 u_{i-k}^j \right\} = f_i^j, \]

\[ i = 1, 2, ..., M - 1, \quad j = 0, 1, ..., N - 1, \quad (26) \]

\[ u_i^0 = 0, \quad u_i^j = 0, \quad j = 0, 1, ..., N, \quad u_i^j = U_0, \quad i = 0, 1, ..., M. \]

**Theorem 3** If the problem (25) has a sufficiently smooth solution \( U = U(x, t) \), then the solution \( u_i^j = (u_1^j, u_2^j, ..., u_M^j), \quad j = 1, 2, ..., N \) of the finite difference scheme (26) tends to the \( U_i^j = (U_1^j, U_2^j, ..., U_M^j) \) for \( j = 1, 2, ..., N \) as \( \tau \to 0, \quad h \to 0 \) and the following estimate is true

\[ \|u_i^j - U_i^j\|_{L^2(0, 1)} \leq C(\tau + h), \quad j = 1, 2, ..., N. \]

Now let us consider some results of numerical experiments.

In test experiment we have chosen the right hand side of equation (25) so that the exact solution is given by

\[ u(x, t) = x(1-x)^2 \cos(4\pi + t), \]

which satisfy boundary conditions in (25).

In the figure 1 we plotted the numerical solution and the exact solutions at \( t = 0.2 \) and \( t = 1.0 \) (Fig. 1). As it is visible from these pictures, the numerical and exact solutions are almost identical.

In the second experiment we have chosen zero right hand side and initial data given by

\[ u(x, 0) = x(1-x)^2 \cos(24\pi x). \]

In this case, we know (Theorem 1) that the solution will decay in time. In following pictures (Figs. 2, 3) are plotted numerical solution for three different time value.

**Figure 1:** The solutions at \( t = 0.2 \) and \( t = 1.0 \). The exact solution is solid line and the numerical solution is marked by +.

**Figure 2:** The numerical solutions at \( t = 0.04, 0.06 \).

**Figure 3:** The numerical solutions at \( t = 0.08, 0.1 \).

We have experimented with several other initial data for problem (25). In all cases we noticed that numerical solutions are approaching to zero as it is shown in theoretical researches.

**4 Conclusion**

Theoretical results show that solution of problem (6)-(9) tends to zero. Carried out numerical experiments show that in all cases numerical solution fully agree with the theoretical results.

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