Linear stability of a convective flow in a vertical pipe

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Abstract: - The linear stability of a convective flow in a vertical pipe generated by internal heat sources of constant volume density is analyzed in the present paper. The linear stability problem is solved for different values of the parameters of the problem. It is shown that for low Prandtl numbers instability is governed by hydrodynamic modes due to the interaction of counter-flowing fluid streams. The second instability mode (thermal mode) appears for large Prandtl numbers and becomes dominant since lower critical Grashof numbers correspond to the thermal mode. Reasonable agreement with experimental data is found.

Key-Words: - stability of fluid flows, internal heat sources, collocation method

1 Introduction

Non-isothermal flows often occur in engineering applications. Heat transfer characteristics of flows in vertical pipes and channels are quite different for laminar and turbulent flows. As a result, much attention is devoted in the literature to the stability analysis of non-isothermal flows in vertical ducts.

Experimental investigation of a heated flow in a vertical pipe, if a constant heat flux is used at the wall, is conducted in [1]. It is shown in [1] that the first azimuthal mode is always the most unstable. Theoretical analysis of the linear stability of flows in vertical channels and pipes is presented in [2]-[6]. One of the major conclusions from the analysis in [2]-[6] is that there are two instability modes: hydrodynamic mode (which is the most unstable for small Prandtl numbers) and thermal mode (which is the most unstable for high Prandtl numbers). In addition, it is found that the critical Grashof numbers are quite low in both cases.

An important application of non-isothermal flows is related to convective flows generated by internal heat sources. Examples include the theory of thermal ignition [7], convection in the Earth's mantle [8] and miniaturization of electronic components [9].

Theoretical investigation of the stability of a convective flow generated by internal heat sources in a vertical plane channel is conducted in [10], [11]. It is shown in [10] that the role of thermal factors is relatively small at low Prandtl numbers. However, at large Prandtl numbers hydrodynamic instability at the boundary between counter-flows moving in opposite directions is replaced by thermal instability in the form of thermal running waves. Stability of internally generated convection in a tall vertical annulus is analyzed in [12] and [13] for the cases of homogeneous and inhomogeneous internal heat sources, respectively.

Stability of a convective flow in a vertical circular cylinder with uniform heat sources is studied in the present paper. Both hydrodynamic mode and thermal mode are analyzed. The critical Grashof numbers for the onset of instability are calculated for different Prandtl numbers. Reasonable agreement is found between the critical Grashof numbers for the most unstable azimuthal mode and experimental data.

2 Mathematical formulation of the problem

The Navier-Stokes equations under the Boussinesq approximation can be written in the following dimensionless form

$$\frac{\partial v_r}{\partial t} + \mathrm{Gr}\left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} + \Delta v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r^2}, \quad (1)$$

$$\frac{\partial v_\theta}{\partial t} + \mathrm{Gr}\left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \Delta v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}, \quad (2)$$

$$\frac{\partial v_z}{\partial t} + \mathrm{Gr}\left(v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} - \frac{v_\theta{}^2}{r} \right) = -\frac{\partial p}{\partial z} + \Delta v_z + T, \quad (3)$$

$$\frac{\partial T}{\partial t} + \mathrm{Gr}\left(v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = \frac{1}{\mathrm{Pr}}\Delta T + \frac{1}{\mathrm{Pr}}, \quad (4)$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0, \quad (5)$$

where (r, θ, z) are the cylindrical polar coordinates, v_r, v_θ and v_z are the velocity components, p is the pressure and T is the temperature. Thermal convection in a vertical cylinder is produced by homogeneous internal heat sources of volume density Q. The measures of length, time, velocity, temperature and pressure are R, R^2/ν, $g\beta QR^4/(\nu\kappa\rho c_p)$, $QR^2/(\rho\kappa c_p)$, and $g\beta QR^3/(\kappa c_p)$, respectively. Here R is the radius of the cylinder, ρ is the density, ν the kinematic viscosity, β the coefficient of thermal expansion, κ the thermal conductivity, g the acceleration due to gravity, and c_p is the heat capacity. The Grashof number, Gr, and the Prandtl number, Pr, are defined as follows: $\mathrm{Gr} = g\beta QR^5/(\nu^2\kappa\rho c_p)$ and $\mathrm{Pr} = \nu/\kappa$.

Equations (1)–(5) have the following steady solution:

$$v_r = 0,\ v_\theta = 0,\ v_z = v_0(r),$$

$$T = T_0(r),\ p = p_0(z). \quad (6)$$

Substituting (6) into (1)–(5) leads to the system

$$-\frac{dp_0}{dz} + \frac{d^2 v_0}{dr^2} + \frac{1}{r}\frac{dv_0}{dr} + T_0 = 0, \quad (7)$$

$$\frac{d^2 T_0}{dr^2} + \frac{1}{r}\frac{dT_0}{dr} + 1 = 0 \quad (8)$$

with the boundary conditions

$$v_0(1) = 0,\ T_0(1) = 0. \quad (9)$$

The functions $v_0(r)$ and $T_0(r)$ are assumed to be bounded at $r = 0$. In addition, we assume that the total fluid flux through the cross-section of the pipe is equal to zero (the pipe is closed):

$$\int_0^1 r v_0(r) dr = 0. \quad (10)$$

The solution of (7)–(10) has the form

$$v_0(r) = \frac{1}{192}(1 - 4r^2 + 3r^4),$$

$$T_0(r) = \frac{1}{4}(1 - r^2),\ p_0(z) = \frac{1}{6}z + \mathrm{const}. \quad (11)$$

We consider a perturbed flow of the form

$$v_r = v_r',\ v_\theta = v_\theta',\ v_z = v_0 + v_z',$$

$$T = T_0 + T',\ p = p_0 + p', \quad (12)$$

where v_r', v_θ', v_z', T' and p' are small perturbations. In accordance with the method of normal modes the perturbations are assumed to be of the form

$$v_r'(r,\theta,z,t) = u(r)\exp(-\lambda t + in\theta + ikz),$$

$$v_\theta'(r,\theta,z,t) = v(r)\exp(-\lambda t + in\theta + ikz),$$

$$v_z'(r,\theta,z,t) = w(r)\exp(-\lambda t + in\theta + ikz), \quad (13)$$

$$T'(r,\theta,z,t) = \varphi(r)\exp(-\lambda t + in\theta + ikz),$$

$$p'(r,\theta,z,t) = q(r)\exp(-\lambda t + in\theta + ikz),$$

where $u(r)$, $v(r)$, $w(r)$, $\varphi(r)$, $q(r)$ are the amplitudes of the normal perturbations, k and n are the axial and azimuthal wavenumbers, respectively.

Substituting (11)–(13) into (1)–(5) and linearizing the equations in a neighborhood of the base flow (11) we obtain the following linear system of ordinary differential equations:

$$-\lambda u + ik\mathrm{Gr} v_0 u = -\frac{dq}{dr} + Lu - \frac{u}{r^2} - \frac{2inv}{r^2}, \quad (14)$$

$$-\lambda v + ik\mathrm{Gr} v_0 v = -\frac{inq}{r} + Lv - \frac{v}{r^2} + \frac{2inu}{r^2}, \quad (15)$$

$$-\lambda w + \mathrm{Gr}\left(u\frac{dv_0}{dr} + ikwv_0\right) = -ikq + Lv + \varphi, \quad (16)$$

$$-\lambda\varphi + \mathrm{Gr}\left(u\frac{dT_0}{dr} + ik\varphi v_0\right) = \frac{1}{\mathrm{Pr}}L\varphi, \quad (17)$$

$$\frac{du}{dr} + \frac{u}{r} + \frac{inv}{r} + ikw = 0. \quad (18)$$

The boundary conditions at $r = 1$ are

$$u(1) = 0,\ v(1) = 0,\ w(1) = 0,\ \varphi(1) = 0,\ q(1) = 0. \quad (19)$$

The boundary conditions at $r = 0$ are derived in [14] and depend on the value of the azimuthal wavenumber n.

If $n = 0$, the conditions have the form

$$u(0) = 0,\ v(0) = 0,\ \frac{dw}{dr}(0) = 0,$$

$$\frac{d\varphi}{dr}(0) = 0,\ q(0) = 0. \quad (20)$$

For $n = 1$, the conditions are

$$u(0) + iv(0) = 0,\ 2\frac{du}{dr}(0) + i\frac{dv}{dr}(0) = 0, \quad (21)$$

$$w(0) = 0,\ \varphi(0) = 0,\ q(0) = 0.$$

For the case $n \geq 2$, the boundary conditions have the form

$$u(0) = 0,\ v(0) = 0,\ w(0) = 0,\ \varphi(0) = 0,\ q(0) = 0. \quad (22)$$

Problem (14)–(22) is an eigenvalue problem. The stability of base flow (11) is determined by the sign of the real parts of the eigenvalues λ: flow (11) is said to be stable if all the real parts of λ are positive, and unstable if at least one of the real parts of λ is negative.

3 Numerical results

The eigenvalue problem (14)–(22) is solved by means of the pseudospectral collocation method based on Chebyshev polynomials (preliminary results of the current investigation were reported in [15]). The functions $u(r), v(r), w(r), \varphi(r)$ and $q(r)$ are sought in the form

$$u(x) = \sum_{m-0}^{N-1} a_m T_m(x),\ v(x) = \sum_{m-0}^{N-1} b_m T_m(x),$$

$$w(x) = \sum_{m-0}^{N-1} c_m T_m(x),\ \varphi(x) = \sum_{m-0}^{N-1} d_m T_m(x), \quad (23)$$

$$u(x) = \sum_{m-0}^{N-1} a_m T_m(x),$$

where $x = 2r - 1$, $T_m(x)$ is the Chebyshev polynomial of degree m of the first kind, a_m, b_m, c_m, d_m and e_m are unknown coefficients. The collocation points are

$$x_j = \cos\frac{\pi j}{N-1},\quad j = 0,1,2,...,N-1. \quad (24)$$

Substituting (23) into (14)–(22) and evaluating all the functions and their derivatives at the collocation points (24) we obtain a generalized eigenvalue problem of the form

$$(A - \lambda B)y = 0, \quad (25)$$

where y is the vector that contains all unknowns a_m, b_m, c_m, d_m and e_m.

Calculations are performed for different values of the parameters of the problem. Both asymmetric modes ($n = 0$) and asymmetric modes ($n \geq 1$) are analyzed. The least stable mode in all cases considered is the first azimuthal mode with $n = 1$. The dependence of the Grashof number on the longitudinal wavenumber k is shown in Fig. 1 for the case $\mathrm{Pr} = 0.79,\ n = 1$.

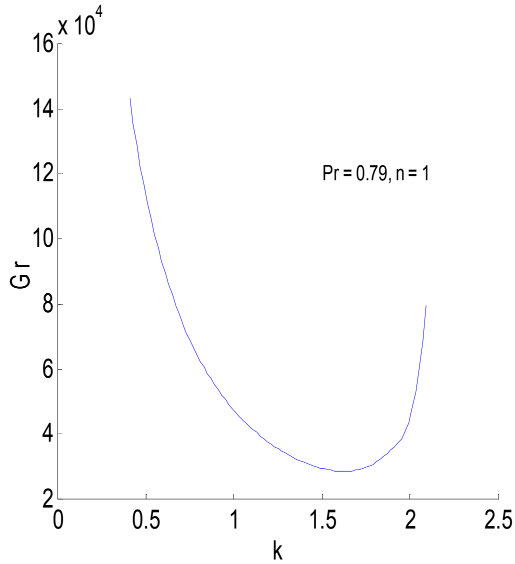

Fig.1. The Grashof number versus the wavenumber k number for the case $\mathrm{Pr} = 0.79,\ n = 1$.

It is seen from Fig. 1 that the stability curve consists of one branch and has a well-defined minimum (the critical Grashof number). The region of instability is above the curve.

As the Prandtl number increases, a second branch of the stability curve appears in the region of small wavenumbers. The second branch has a lower minimum. Hence, the two

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branches of the neutral stability curves can be identified as the hydrodynamic mode and the thermal mode [10], [11]. The hydrodynamic mode exists for all Prandtl numbers and it leads to instability for low Prandtl numbers. On the other hand, the thermal mode appears only at relatively large Prandtl numbers and it is the most unstable mode. The appearence of the second instability mode is demonstrated in Fig. 2.

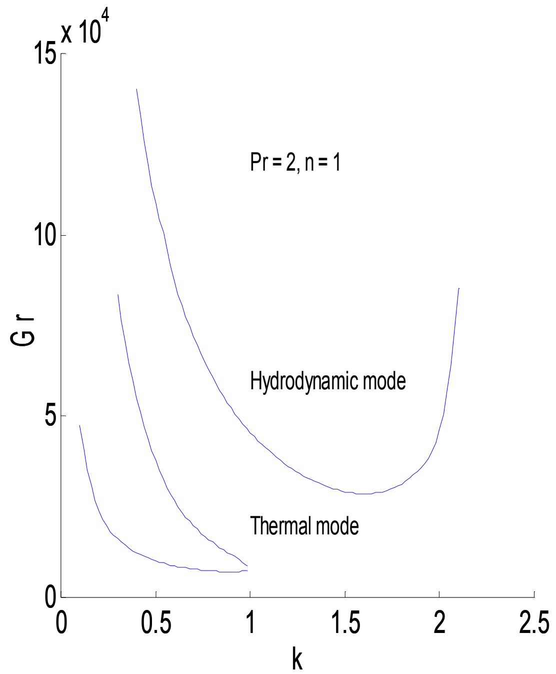

Fig.2. The Grashof number versus the Prandtl number for the case $\mathrm{Pr} = 2,\ n = 1$.

As can be seen from Fig. 2, the thermal mode has a lower critical Grashof number. Calculations show that the phase velocity of perturbations increases as the Prandtl number increases. This type of instability is termed in [11] as instability in the form of thermal running waves.

4 Comparison with experiments

In the present section the results of numerical calculations (in terms of the critical Grashof numbers) are compared with experimental data [16]. It is shown in [16] that a convective flow is generated in a vertical cylinder due to homogeneous internal heat sources. The flow consists of two counter-flowing streams: ascending stream near the axis of the cylinder and descending stream near the wall. The difference between the experimentally measured velocity and theoretical velocity distribution given by (11) was of order 5%.

Experimental data show that, starting from some Grashof number flow, (11) loses stability since a spiral vortex appears near the boundary of counter-flowing streams. The first azimuthal mode ($n = 1$) is found to be the most unstable mode in all experiments.

The calculated values of the critical Grashof numbers for different Prandtl numbers are shown in Fig. 3 (solid curve).

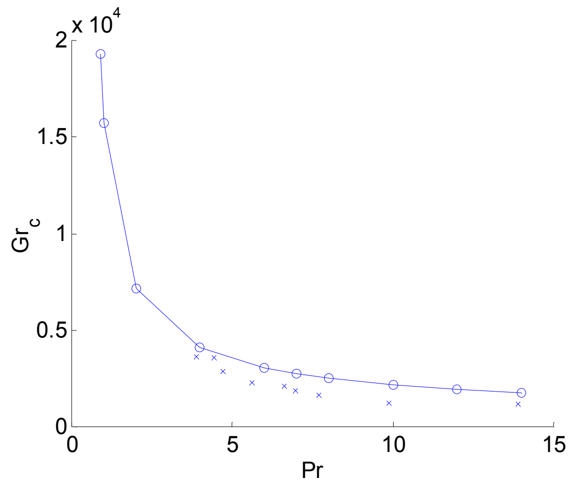

Fig.3. The dependence of the critical Grashof number on the Prandtl number (solid curve – theory, x – experimental points).

It is seen from Fig. 3 that the critical Grashof numbers decreases as the Prandtl number increases. For $\mathrm{Pr} \geq 2$, the minimum of the neutral stability curve corresponds to the thermal mode (see Fig. 2). Experimental data are in a qualitative agreement with theoretical calculations in terms of instability mode (both experimental data and theory suggest that the first azimuthal mode with $n = 1$ is the most unstable). It is seen from Fig. 3 that experimentally observed critical Grashof numbers decrease as the Prandtl number increases. This fact is also supported by theory. In addition, both experimental data and theoretical calculations indicate a relatively large speed of perturbations for large Prandtl numbers. However, there are quantitative differences between experimental data and theory – experimentally observed critical Grasshof numbers are smaller than

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the values predicted by theory. One possible source of discrepancy is the relatively large (up to 5%) error in the base flow velocity.

5 Conclusion

Linear stability of a convective flow generated by homogeneous internal heat sources is analyzed in the present paper. It is found that two modes of instability exist for large Prandtl numbers: hydrodynamic mode and thermal mode. In all cases instability occurs in the form of a spiral vortex that corresponds to the first azimuthal mode with $n = 1$. The thermal mode becomes dominant for large Prandtl numbers. Thermal instability occurs in the form of thermal running waves.

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