Double Conductor Line above a Half-space with Varying Electric and Magnetic Properties

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Abstract: - The change in impedance per unit length of a double conductor line located above a conducting half-space with varying electric conductivity and magnetic permeability is calculated for the case where both electric conductivity and magnetic permeability are exponentially varying functions of the vertical coordinate. Closed-form solution of the problem is found by means of the Hankel transform. The solution is expressed in terms of improper integrals containing modified Bessel functions of complex argument. Calculations are performed for different values of the parameters of the problem using software package Mathematica.

Key-Words: - eddy current testing, magnetic permeability, electric conductivity, Hankel transform

1 Introduction

In many cases theoretical models of eddy current non-destructive testing are based on the assumption that the properties of a conducting medium (electric conductivity and magnetic permeability) are constants [1]. In applications such as surface hardening and decarbonization [2], [3] the electric conductivity and magnetic permeability of the conducting medium can vary with respect to one geometrical coordinate. There are two basic approaches to eddy current modeling of conducting media with varying properties. One approach is based on the assumption that the conducting medium with varying properties can be replaced by a multilayer medium where each layer has constant electric conductivity and magnetic permeability. Following this approach, up to 50 layers of a multilayer medium are used in [4] to represent the variation of the electric and magnetic properties in the vertical direction.

The second approach is based on closed-form solutions of the problem. It is known that in some cases analytical solutions can be found under the additional assumption that the electric conductivity and magnetic permeability are approximated by relatively simple continuously varying functions of one geometrical coordinate. Examples can be found in [5]-[8] where the change in impedance of the coil is found by means of known special functions. Analytical solution for the case where a single-turn circular coil is located above a conducting half-space is found in [5] under the assumption that either the electric conductivity or magnetic permeability are exponential functions of the vertical coordinate. The results reported in [5] are generalized in [8] for the case where both parameters of the medium, that is, the electric conductivity and magnetic permeability are exponential functions of the vertical coordinate.

In the present paper a problem similar to [8] is considered for the case where the source of the external current in a double conductor line formed by two infinitely long wires located parallel to the interface. The change in impedance of the double line is found in terms of improper integrals containing Bessel functions of complex argument.

2 Mathematical Formulation of the Problem

Different configurations of eddy current probes are used in engineering applications. One example is a probe in the form of a rectangular frame carrying an alternating current. It is known [9] that if the ratio of the sides of the frame is 1:4 or smaller, then the rectangular probe can be approximated by the double conductor line.

Suppose that two horizontal infinitely long wires are located above a conducting half-space. The alternating current in the wires at the points (y_0, h) and (y_1, h) is equal to $\pm I\exp(j\omega t)$, respectively, where I is the amplitude of the current. Regions $R_0 = \{z > 0\}$ and $R_1 = \{z < 0\}$ represent the upper and lower half-space, respectively. The electric conductivity σ and magnetic permeability μ of region R_1 are exponentially varying functions of the vertical coordinate z of the form

$$\sigma = \sigma_m e^{\alpha z}, \ \mu = \mu_0 \mu_m e^{\beta z}, \tag{1}$$

where σ_m, μ_m, α, β are constants and μ_0 is the magnetic constant. Exponential dependence of μ on the vertical coordinate is confirmed by experimental data [2] for the case of a coating on ferromagnetic metal.

The amplitudes of the vector potentials in regions R_0 and R_1, respectively, have only one non-zero component in the x-direction [7] and are functions of y and z only, that is,

$$A_0 = A_0(y,z),\ A_1 = A_1(y,z). \qquad (2)$$

The systems of equations for the vector potentials A_0 and A_1 has the form

$$\frac{\partial^2 A_0}{\partial y^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I\delta(y-y_0)\delta(z-h) + \mu_0 I\delta(y-y_1)\delta(z-h), \qquad (3)$$

$$\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} - \beta\frac{\partial A_1}{\partial z} - j\omega\sigma_m\mu_0\mu_m e^{(\alpha+\beta)z}A_1 = 0, \qquad (4)$$

where $\delta(y)$ is the Dirac delta-function.

The boundary conditions are

$$A_0\,|_{z=0} = A_1\,|_{z=0}, \quad \frac{\partial A_0}{\partial z}\,|_{z=0} = \frac{1}{\mu_m}\frac{\partial A_1}{\partial z}\,|_{z=0}. \qquad (5)$$

The following conditions are assumed to be satisfied at infinity:

$$A_0,\ A_1,\ \frac{\partial A_0}{\partial y},\ \frac{\partial A_1}{\partial y} \to 0, \quad \text{as} \quad \sqrt{y^2+z^2} \to \infty. \qquad (6)$$

3 Problem Solution

The solution to (3)-(6) can be represented as the sum of even and odd solutions (see [7]):

$$A_i(y,z) = A_{i\,\mathrm{even}}(y,z) + A_{i\,\mathrm{odd}}(y,z), \quad i = 0,1. \qquad (7)$$

The right-hand side of (3) can also be written as the sum of even and odd solutions. Suppose that the right-hand side of (3) is denoted by $f(y,z)$. It can easily be shown that

$$f(y,z) = f_{\mathrm{even}}(y,z) + f_{\mathrm{odd}}(y,z), \qquad (8)$$

where

$$f_{\mathrm{even}}(y,z) = f_{\mathrm{odd}}(y,z) = \frac{\mu_0 I}{2}[\delta(y-y_1) - \delta(y-y_0)]\delta(z-h). \qquad (9)$$

The Fourier cosine transform of the form

$$\tilde{A}_i^{(c)}(\lambda,z) = \int_0^\infty A_{i\,\mathrm{even}}(y,z)\cos\lambda y\,dy, \quad i = 0,1 \qquad (10)$$

is applied to (3)-(6) (where the right-hand side of (3) is replaced by $f_{\mathrm{even}}(y,z)$) in order to find the even component, $A_{i\,\mathrm{even}}(y,z)$, $i = 0,1$. In the transformed space the functions $A_0^{(c)}(\lambda,z)$ and $A_1^{(c)}(\lambda,z)$ satisfy the following system of equations

$$\frac{d^2\tilde{A}_0^{(c)}}{dz^2} - \lambda^2\tilde{A}_0^{(c)} = \frac{\mu_0 I}{2}(\cos\lambda y_1 - \cos\lambda y_0)\delta(z-h), \qquad (11)$$

$$\frac{d^2\tilde{A}_1^{(c)}}{dz^2} - \beta\frac{d\tilde{A}_1^{(c)}}{dz} - \lambda^2\tilde{A}_1^{(c)} - j\omega\sigma_m\mu_0\mu_m e^{(\alpha+\beta)z}\tilde{A}_1^{(c)} = 0 \qquad (12)$$

with the boundary conditions

$$\tilde{A}_0^{(c)}\,|_{z=0} = \tilde{A}_1^{(c)}\,|_{z=0}, \quad \frac{d\tilde{A}_0^{(c)}}{dz}\,|_{z=0} = \frac{1}{\mu_m}\frac{d\tilde{A}_1^{(c)}}{dz}\,|_{z=0}. \qquad (13)$$

Consider two subregions of R_0, namely, $R_{00} = \{0 < z < h\}$ and $R_{01} = \{z > h\}$. We denote the solutions in these regions by $\tilde{A}_{00}^{(c)}$ and $\tilde{A}_{01}^{(c)}$, respectively.

The general solution to (11) in R_{00} has the form

$$\tilde{A}_{00}^{(c)}(\lambda,z) = C_1 e^{\lambda z} + C_2 e^{-\lambda z}. \qquad (14)$$

In subregion R_{01} we select the general solution that is bounded as $z \to \infty$:

$$\tilde{A}_{01}^{(c)}(\lambda,z) = C_3 e^{-\lambda z}. \qquad (15)$$

Similarly, the general solution to (12) which remains bounded as $z \to -\infty$ has the form (see[10]):

$$\tilde{A}_1^{(c)}(\lambda,z) = C_4 e^{\beta z/2} I_\nu\left(c e^{(\alpha+\beta)z/2}\right), \qquad (16)$$

where $I_\nu(s)$ is the modified Bessel's function of the first kind of order ν, and the parameters c and ν are given by

$$c = \frac{2\sqrt{j\omega\mu_0\mu_m\sigma_m}}{\alpha+\beta}, \quad \nu = \frac{\sqrt{\beta^2 + 4\lambda^2}}{\alpha+\beta}.$$

There are four unknown constants in (14)-(16) and only two boundary conditions (13). The third condition is obtained in the form

$$\tilde{A}_{01}^{(c)}\,|_{z=h} = \tilde{A}_{00}^{(c)}\,|_{z=h}, \qquad (17)$$

since the vector potential is continuous at $z = h$. The last condition can be derived from equation (11). Integrating (11) with respect to z from $h-\varepsilon$ to $h+\varepsilon$ ($\varepsilon > 0$) and taking the limit as $\varepsilon \to +0$, we obtain

$$\frac{d\tilde{A}_{01}^{(c)}}{dz}\,|_{z=h} - \frac{d\tilde{A}_{00}^{(c)}}{dz}\,|_{z=h} = \frac{\mu_0 I}{2}(\cos\lambda y_1 - \cos\lambda y_0). \qquad (18)$$

Using solutions (14)-(16) and boundary conditions (13), (17) and (18) the following system of linear equations is obtained:

$$
\begin{cases}
C_1 e^{\lambda h} + C_2 e^{-\lambda h} - C_3 e^{-\lambda h} = 0 \\
-\lambda C_1 e^{\lambda h} + \lambda C_2 e^{-\lambda h} - \lambda C_3 e^{-\lambda h} \\
\quad = \dfrac{\mu_0 I}{2}(\cos\lambda y_1 - \cos\lambda y_0) \\
C_1 + C_2 - C_4 I_\nu(c) = 0 \\
\lambda C_1 - \lambda C_2 - C_4\left[\dfrac{\beta}{2} I_\nu(c) + c\dfrac{\alpha+\beta}{2} I_\nu^{'}(c)\right] = 0
\end{cases} \tag{19}
$$

The constants C_1, C_2, C_3 and C_4 are determined from (19) and have the form

$$
\begin{aligned}
C_1 &= \frac{\mu_0 I}{4\lambda}(\cos\lambda y_0 - \cos\lambda y_1)e^{-\lambda h}, \\
C_2 &= \frac{\mu_0 I(\cos\lambda y_0 - \cos\lambda y_1)}{4\lambda[(2\lambda\mu_m + \beta)I_\nu(c) + c(\alpha+\beta)I_\nu^{'}(c)]} \\
&\quad \times e^{-\lambda h}[(2\lambda\mu_m - \beta)I_\nu(c) - c(\alpha+\beta)I_\nu^{'}(c)], \\
C_3 &= C_1 e^{\lambda h} + C_2, \\
C_4 &= \frac{\mu_0\mu_m I(\cos\lambda y_0 - \cos\lambda y_1)e^{-\lambda h}}{[(2\lambda\mu_m + \beta)I_\nu(c) + c(\alpha+\beta)I_\nu^{'}(c)]}.
\end{aligned} \tag{20}
$$

Hence, the solution in the transformed space is defined by (14)-(16) where the coefficients C_1, C_2, C_3 and C_4 are obtained from (20). Applying the inverse Fourier cosine transform

$$
A_{i\,\text{even}}(y,z) = \frac{2}{\pi}\int_0^\infty \tilde{A}_i^{(c)}(\lambda,z)\cos\lambda y d\lambda, \;\; i = 0,1 \tag{21}
$$

to (14)-(16), we obtain the solution to (3)-(6) for the even component, $A_{i\,\text{even}}(y,z), i = 0,1$, of the vector potential in regions R_0 and R_1. In particular, the solution in free space (region R_0) has the form

$$
A_{0\,\text{even}}(y,z) = A_{0\,\text{even}}^{\text{free}}(\lambda,z) + A_{0\,\text{even}}^{\text{ind}}(\lambda,z), \tag{22}
$$

where

$$
\begin{aligned}
A_{0\,\text{even}}^{\text{free}}(y,z) &= \frac{\mu_0 I}{2\pi}\int_0^\infty (\cos\lambda y_0 - \cos\lambda y_1) \\
&\quad \times e^{-\lambda|z-h|}\cos\lambda y \frac{d\lambda}{\lambda}
\end{aligned} \tag{23}
$$

is the vector potential due to double conductor line in an unbounded free space (region R_1 is absent).

The second term on the right-hand side of (22) is equal to

$$
\begin{aligned}
A_{0\,\text{even}}^{\text{ind}}(y,z) &= \frac{\mu_0 I}{2\pi}\int_0^\infty \frac{(2\lambda\mu_m - \beta)I_\nu(c) - c(\alpha+\beta)I_\nu^{'}(c)}{(2\lambda\mu_m + \beta)I_\nu(c) + c(\alpha+\beta)I_\nu^{'}(c)} \\
&\quad \times (\cos\lambda y_0 - \cos\lambda y_1)e^{-\lambda(z+h)}\cos\lambda y \frac{d\lambda}{\lambda}
\end{aligned} \tag{24}
$$

and represents the induced component of the vector potential in free space due to the conducting half-space.

The Fourier sine transform

$$
\tilde{A}_i^{(s)}(\lambda,z) = \int_0^\infty A_{i\,\text{odd}}(y,z)\sin\lambda y dy, \; i = 0,1 \tag{25}
$$

and the inverse Fourier sine transform

$$
A_{i\,\text{odd}}(y,z) = \frac{2}{\pi}\int_0^\infty \tilde{A}_i^{(s)}(\lambda,z)\sin\lambda y d\lambda, \; i = 0,1 \tag{26}
$$

are used to determine the odd components, $A_{i\,\text{odd}}(y,z),\ i = 0,1$, of the vector potential in regions R_0 and R_1. It can be shown that the odd component of the solution is given by formulas (22)-(24) where cosine is replaced by sine. In particular, the induced (odd) component of the vector potential in region R_0 can be written as follows

$$
\begin{aligned}
A_{0\,\text{odd}}^{\text{ind}}(y,z) &= \frac{\mu_0 I}{2\pi}\int_0^\infty \frac{(2\lambda\mu_m - \beta)I_\nu(c) - c(\alpha+\beta)I_\nu^{'}(c)}{(2\lambda\mu_m + \beta)I_\nu(c) + c(\alpha+\beta)I_\nu^{'}(c)} \\
&\quad \times (\sin\lambda y_0 - \sin\lambda y_1)e^{-\lambda(z+h)}\sin\lambda y \frac{d\lambda}{\lambda}
\end{aligned} \tag{27}
$$

Using (7), (24) and (27) we obtain the solution to problem (3)-(6) in the form

$$
\begin{aligned}
A_0^{\text{ind}}(y,z) &= A_{0\,\text{even}}^{\text{ind}}(y,z) + A_{0\,\text{odd}}^{\text{ind}}(y,z) \\
&= \frac{\mu_0 I}{2\pi}\int_0^\infty \frac{(2\lambda\mu_m - \beta)I_\nu(c) - c(\alpha+\beta)I_\nu^{'}(c)}{(2\lambda\mu_m + \beta)I_\nu(c) + c(\alpha+\beta)I_\nu^{'}(c)} \\
&\quad \times [\cos\lambda(y-y_0) - \cos\lambda(y-y_1)]e^{-\lambda(z+h)}\frac{d\lambda}{\lambda}.
\end{aligned} \tag{28}
$$

Finally, we compute the change in impedance per unit length of the double conductor line using the formula

$$
Z_{\text{per unit length}}^{\text{ind}} = \frac{j\omega}{I}\oint_C A_0^{\text{ind}}(y,z)dx. \tag{29}
$$

Substituting (28) into (29) we obtain

$$
Z_{\text{per unit length}}^{\text{ind}} = \frac{\mu_0\omega}{\pi}Z,
$$

where

$$Z = j\int_0^\infty \frac{(2s\mu_m - \tilde{\beta})I_\nu(\tilde{c}) - \tilde{c}(\tilde{\alpha}+\tilde{\beta})I_\nu'(\tilde{c})}{(2s\mu_m + \tilde{\beta})I_\nu(\tilde{c}) + \tilde{c}(\tilde{\alpha}+\tilde{\beta})I_\nu'(\tilde{c})} \quad (30)$$
$$\times[1-\cos s]e^{-2\gamma s}\frac{ds}{s},$$

where

$$\tilde{c} = \frac{2\eta\sqrt{j}}{\tilde{\alpha}+\tilde{\beta}},\ \nu = \frac{\sqrt{\tilde{\beta}^2+4s^2}}{\tilde{\alpha}+\tilde{\beta}},\ \eta = r_c\sqrt{\omega\sigma_m\mu_0\mu_m},$$

$\tilde{\alpha} = \alpha r_c,\ \tilde{\beta} = \beta r_c,\ \gamma = \frac{h}{r_c}$ and $r_c = y_0 - y_1$ is the distance between the wires.

4 Numerical Results

The results of numerical computations are presented in this section. Integral (30) is evaluated numerically using package Mathematica since it allows one to compute modified Bessel function of variable order and complex argument.

The change in impedance is computed for different values of the parameters $\tilde{\alpha}$, $\tilde{\beta}$ and η. The parameter γ was fixed at 0.05 for all calculations.

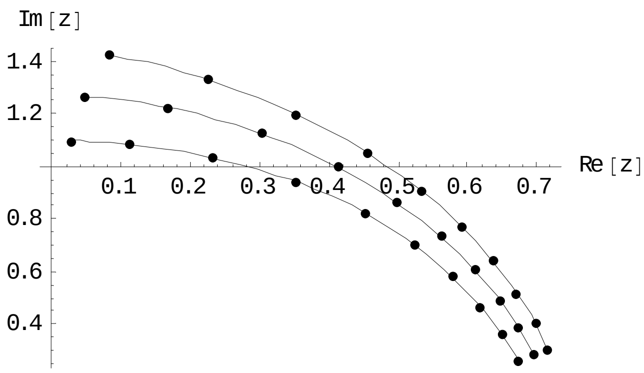

Fig.1. The change in impedance given by (30) for $\tilde{\alpha} = 0$ and three values of $\tilde{\beta}$, namely, $\tilde{\beta} = 0.5;\ 1.5;\ 2.5$ (from top to bottom). The points shown on the graph correspond to $\eta = 1,\ 2,\ ,3,...,10$ (from left to right).

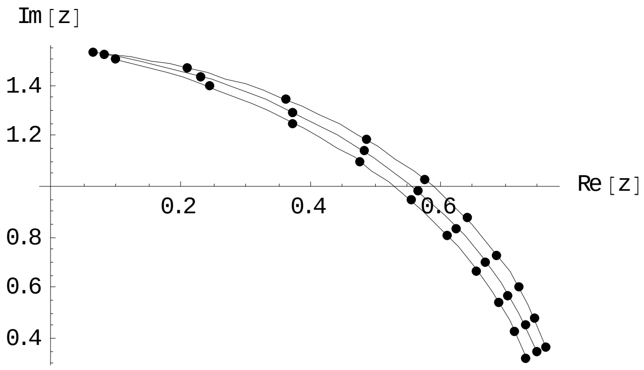

Fig.2. The change in impedance given by (30) for $\tilde{\beta} = 0$ and three values of $\tilde{\alpha}$, namely, $\tilde{\alpha} = 0.5;\ 1.5;\ 2.5$ (from top to bottom). The points shown on the graph correspond to $\eta = 1,\ 2,\ ,3,...,10$ (from left to right).

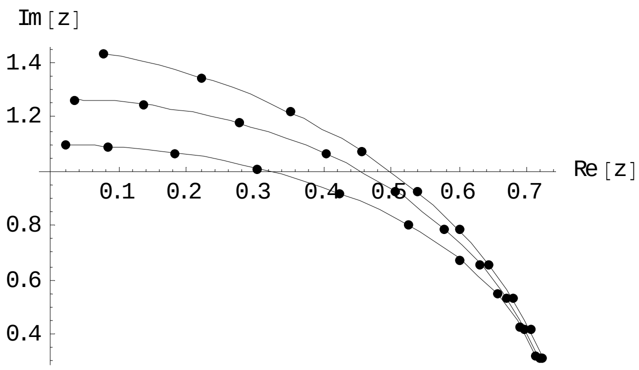

Fig.3. The change in impedance given by (30) for $\tilde{\alpha} = 0.5,\ \tilde{\beta} = 0.5;\ \tilde{\alpha} = 1.5,\ \tilde{\beta} = 1.5;\ \tilde{\alpha} = 2.5,\ \tilde{\beta} = 2.5;$ (from top to bottom). The points shown on the graph correspond to $\eta = 1,\ 2,\ ,3,...,10$ (from left to right).

It is interesting to note that for large frequencies (large η values) the change in impedance seem to be independent on the values of $\tilde{\alpha}$ and $\tilde{\beta}$ for the case $\tilde{\alpha} = \tilde{\beta}$.

4 Conclusion

Closed-form solution for the change in impedance of a double conductor line located above a conducting half-space with varying electric conductivity and magnetic permeability is obtained in the present paper. The electric conductivity and magnetic permeability are assumed to vary exponentially with depth. The solution is obtained in terms of improper integral containing

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modified Bessel's functions of the first kind of complex argument. It is shown in experiments [1], [2] that in some applications the magnetic permeability vary exponentially with respect to the vertical coordinate. Thus, the obtained closed-form solution can be applied for the analysis of the inverse problem in order to estimate the unknown parameters (σ and/or μ) of the conducting medium in the case where these parameters are not constants.

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