An Application of the Ni(x) Integral Function to Nonhomogeneous Airy’s Equation

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Abstract: - In this article, we discuss a recently introduced function, Ni(x), and its use in the formulation and solution of non-homogeneous Airy’s differential equation with constant forcing function.

Key-words: Airy’s Functions; Scorer Functions; Non-homogeneous Airy’s ODE; Ni(x) Function.

1 Introduction
Many problems in mathematical physics that are governed by ordinary differential equations (ODE) are apt to reduction to Airy’s ODE by appropriate transformation or change of variables (cf. [1], [3], [6], [7], and the references therein). A well-known form of Airy’s ODE in the unknown function $y(x)$ is given by:

$$y'' - xy = f(x)$$

...(1)

Solving equation (1) has been the subject of a large number of studies that lead to the introduction of Airy’s functions and Scorer’s functions, [7]. In particular, solution to the homogeneous part of ODE (1), namely,

$$y'' - xy = 0$$

...(2)

is given by, [7]:

$$y = c_1 Ai(x) + c_2 Bi(x)$$

...(3)

where $Ai(x)$ and $Bi(x)$ are two linearly independent functions, defined by the following integrals [7]:

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos \left( xt + \frac{1}{3} t^3 \right) dt$$

...(4)

$$Bi(x) = \frac{1}{\pi} \int_0^\infty \sin \left( xt + \frac{1}{3} t^3 \right) dt + \frac{1}{\pi} \int_0^\infty \exp \left( xt - \frac{1}{3} t^3 \right) dt$$

...(5)

The Wronskian of $Ai(x)$ and $Bi(x)$ is given by, [7]:

$$W(Ai(x), Bi(x)) = \frac{dAi}{dx} Bi(x) - \frac{dBi}{dx} Ai(x) = \frac{1}{\pi}$$

...(6)

The non-homogeneous ODE (1) has a particular solution given by the Scorer functions, [7], as given in what follows:

i) When $f(x) = -\frac{1}{\pi}$, a particular solution is given by

$$Gi(x) = \frac{1}{\pi} \int_0^\infty \sin \left( xt + \frac{1}{3} t^3 \right) dt$$

...(7)

ii) When $f(x) = \frac{1}{\pi}$, a particular solution is given by

$$Hi(x) = \frac{1}{\pi} \int_0^\infty \exp \left( xt - \frac{1}{3} t^3 \right) dt$$

...(8)

The functions $Gi(x)$ and $Hi(x)$ are known as Scorer’s functions. It can be seen from (7), (8) and (5) that

$$Gi(x) + Hi(x) = Bi(x)$$

...(9)
For real values of $x$, Airy’s and Scorer’s functions are real-valued. Extensive analysis has been carried out when the argument of these functions is complex, and methods of evaluating Airy’s and Scorer’s functions have been developed and extensively studied (cf. [2], [3], [4], [6], [7]. And the references therein).

In order to solve initial value problems involving ODE (1) and (2), one requires the following values of Airy’s and Scorer’s functions at the initial point $x = 0$, [7]:

\[
\begin{align*}
G_i(0) &= \frac{1}{2} Hi(0) = \frac{1}{3} Bi(0) = \frac{1}{\sqrt{3}}Ai(0) \\
&= \frac{1}{3^{7/6}I(\frac{2}{3})} \quad \text{(10)}
\end{align*}
\]

\[
\begin{align*}
G_i(0) &= \frac{1}{2} \dot{H}(0) = \frac{1}{3} \dot{B}(0) = -\frac{1}{\sqrt{3}}\dot{A}(0) \\
&= \frac{1}{3^{5/6}I(\frac{1}{3})} \quad \text{(11)}
\end{align*}
\]

The aim of the current work is find solutions to some initial value problems involving ODE (1) using Airy’s and Scorer’s functions, and to discuss the $Ni(x)$ integral function that was recently introduced by Nield and Kuznetsov [5], and how it is used in solving initial value problems.

2 The $Ni(x)$ Function

In a recent paper, Nield and Kuznetsov [5] introduced an integral function, $Ni(x)$, that arose in their modeling and solution of flow over a porous layer, where the governing equation is similar in form to ODE (1). To accomplish their solution, they conveniently introduced the following integral function, defined in terms of Airy’s functions:

\[
Ni(x) = Ai(x) \int_0^x Bi(t) \, dt - Bi(x) \int_0^x Ai(t) \, dt.
\]  
\[\ldots(12)\]

First derivative of $Ni(x)$ is given by, [5]:

\[
\dot{Ni}(x) = \dot{Ai}(x) \int_0^x Bi(t) \, dt - \dot{Bi}(x) \int_0^x Ai(t) \, dt.
\]  
\[\ldots(13)\]

An interesting and useful feature of this function is realized as follows: the second derivative of this function involves the Wronskian of Airy’s functions, namely:

\[
Ni''(x) = Ai'' \int_0^x Bi(t) \, dt - Bi'' \int_0^x Ai(t) \, dt - W(Ai(x), Bi(x)).
\]  
\[\ldots(14)\]

where, as given in equation (6), $W(Ai(x), Bi(x)) = \frac{1}{\pi}$. We will use this feature in formulating and solving initial value problems associated with ODE (1).

The $Ni(x)$ function has a number of other interesting and useful properties, some of which have already been analyzed (cf. [5]), while others are yet to be discovered.

The following values of $Ni(x)$ and its first and second derivatives at $x = 0$ can easily be obtained from (12), (13) and (14):

\[
Ni(0) = \dot{Ni}(0) = 0. \quad \ldots(15)
\]

\[
Ni''(0) = -W(Ai(0), Bi(0)) = -\frac{1}{\pi} \quad \ldots(16)
\]

In what follows, we will use the $Ni(x)$ function in the formulation and solution of initial value problems.

3 Formulation of Initial Value Problems in Terms of the $Ni(x)$ Function

Consider the initial value problem composed of solving ODE (1) subject to the initial conditions:

\[
y(0) = \alpha \quad \ldots(17)
\]

\[
\dot{y}(0) = \beta \quad \ldots(18)
\]

where $\alpha$ and $\beta$ are known constants.

General solution to ODE (1) is the sum of the complementary function, given by equation (3) as the
solution to the homogeneous Airy’s ODE, and the particular solution, $y_p$, which we will formulate here in terms of the $Ni(x)$ function.

Based on the method of variation of parameters, we assume the particular solution of the form:

$$y_p = u_1Ai(x) + u_2Bi(x).$$  

...(19)

The functions $u_1$ and $u_2$ are given, respectively, by the following form:

$$u_1 = \frac{\int_0^x f(t)Bi(t)dt}{Ni'(0)},$$  

...(20)

$$u_2 = -\frac{\int_0^x f(t)Ai(t)dt}{Ni'(0)}.$$  

...(21)

Particular solution of ODE (1) can then be expressed, for any forcing function, $f(x)$, as:

$$y_p = Ai(x)\int_0^x f(t)Bi(t)dt - Bi(x)\int_0^x f(t)Ai(t)dt$$

$$- \frac{\int_0^x f(t)Ai(t)dt}{Ni'(0)}.$$  

...(22)

General solution to ODE (1) thus takes the form:

$$y = c_1Ai(x) + c_2Bi(x) + Ai(x)\int_0^x f(t)Bi(t)dt$$

$$- Bi(x)\int_0^x f(t)Ai(t)dt - \frac{\int_0^x f(t)Ai(t)dt}{Ni'(0)}.$$  

...(23)

Now, using conditions (17) and (18) in (23), we obtain the following system of linear equations that we can solve to obtain the arbitrary constants, $c_1$ and $c_2$:

$$c_1Ai(0) + c_2Bi(0) = \alpha$$  

...(24)

$$c_1Ai'(0) + c_2Bi'(0) = \beta.$$  

...(25)

Solution of (24) and (25) can be expressed in terms of the Gamma function as:

$$c_1 = \pi \left\{ \frac{1/6\alpha}{\Gamma(\frac{1}{3})} - \frac{\beta}{\Gamma(\frac{2}{3})} \right\}$$  

...(26)

$$c_2 = \pi \left\{ \frac{\alpha}{3^{1/3}\Gamma(\frac{1}{3})} + \frac{\beta}{3^{2/3}\Gamma(\frac{2}{3})} \right\}.$$  

...(27)

It is clear from (22) that finding $y_p$ depends on evaluation of the involved integrals. Once the integrals are evaluated, we substitute (26) and (27) in (23) to render the initial value problem completely determined.

It is worth noting that the values of the arbitrary constants in (26) and (27) are valid for all initial value problems involving ODE (1), subject to conditions (17) and (18), when the general solution is expressed in the form (23).

Now, let us consider the following cases of the forcing function $f(x)$.

**Case 1: $f(x) = \delta = constant.$**

Using $f(x) = \delta$ in (22), we obtain the following particular solution that we express here in terms of $Ni(x)$:

$$y_p = \frac{\delta}{N\Gamma'(0)} \left\{ Ai(x)\int_0^x Bi(t)dt - Bi(x)\int_0^x Ai(t)dt \right\}$$

$$ + \frac{\delta}{Ni'(0)} Ni(x).$$  

...(28)

General solution to ODE (1) thus takes the form:

$$y = c_1Ai(x) + c_2Bi(x) + \frac{\delta}{Ni'(0)} Ni(x)$$  

...(29)

where $Ni''(0)$ is given by (16).

A solution satisfying the initial conditions (17) and (18) is obtained by substituting (26) and (27) in (29), namely:

$$y = \pi \left\{ \frac{1/6\alpha}{\Gamma(\frac{1}{3})} - \frac{\beta}{\Gamma(\frac{2}{3})} \right\} Ai(x) +$$

$$\pi \left\{ \frac{\alpha}{3^{1/3}\Gamma(\frac{1}{3})} + \frac{\beta}{3^{2/3}\Gamma(\frac{2}{3})} \right\} Bi(x) + \frac{\delta}{Ni'(0)} Ni(x).$$  

...(30)

For the sake of comparison, we will consider the two cases of: $f(x) = -\frac{1}{\pi}$, and $f(x) = \frac{1}{\pi}$ in the nonhomogeneous ODE (1) whose particular solutions has been given in terms of Scorer’s functions.
When \( f(x) = -\frac{1}{\pi} \), equation (29) yields the general solution:

\[
y = c_1 Ai(x) + c_2 Bi(x) + Ni(x)
\]  
\[\text{...(31)}\]

Solution satisfying the initial conditions (17) and (18) is obtained by using (26) and (27) in (31), namely:

\[
y = \pi \left( \frac{3^{1/6} \alpha}{r^{2/3}(\xi)} - \frac{\beta}{3^{2/3}r^{2/3}(\xi)} \right) Ai(x) + \pi \left( \frac{\alpha}{3^{1/3}r^{2/3}(\xi)} + \frac{\beta}{3^{2/3}r^{2/3}(\xi)} \right) Bi(x) + Ni(x)
\]  
\[\text{...(32)}\]

In terms of Scorer's function, the general solution to ODE (1) for \( f(x) = -\frac{1}{\pi} \), takes the form

\[
y = c_1 Ai(x) + c_2 Bi(x) + Gi(x)
\]  
\[\text{...(33)}\]

where \( Gi(x) \) is as defined in equation (7).

A solution that satisfies initial conditions (17) and (18) takes the form:

\[
y = \pi \left( \frac{3^{1/6} \alpha}{r^{2/3}(\xi)} - \frac{\beta}{3^{2/3}r^{2/3}(\xi)} \right) Ai(x) + \pi \left( \frac{\alpha}{3^{1/3}r^{2/3}(\xi)} + \frac{\beta}{3^{2/3}r^{2/3}(\xi)} \right) Bi(x) + Gi(x)
\]  
\[\text{...(34)}\]

Solution (32) is expressed in terms of \( Ni(x) \) while solution (34) is expressed in terms of \( Gi(x) \). The two solutions are equivalent, as can be seen if one uses the relationship between \( Gi(x) \) and \( Ni(x) \), [5], and the arbitrary constants are re-evaluated:

\[
Ni(x) = Gi(x) - \frac{Bi(x)}{3}
\]  
\[\text{...(35)}\]

When \( f(x) = \frac{1}{\pi} \), equation (29) easily yields the general solution:

\[
y = c_1 Ai(x) + c_2 Bi(x) - Ni(x)
\]  
\[\text{...(36)}\]

Solution satisfying the initial conditions (17) and (18) is obtained by using (26) and (27) in (34), namely:

\[
y = \pi \left( \frac{3^{1/6} \alpha}{r^{2/3}(\xi)} - \frac{\beta}{3^{2/3}r^{2/3}(\xi)} \right) Ai(x) + \pi \left( \frac{\alpha}{3^{1/3}r^{2/3}(\xi)} + \frac{\beta}{3^{2/3}r^{2/3}(\xi)} \right) Bi(x) - Ni(x)
\]  
\[\text{...(37)}\]

In terms of Scorer’s function, the general solution to ODE (1) for \( f(x) = \frac{1}{\pi} \), takes the form:

\[
y = c_1 Ai(x) + c_2 Bi(x) + Hi(x)
\]  
\[\text{...(38)}\]

where \( Hi(x) \) is defined in equation (8).

A solution that satisfies initial conditions (17) and (18) takes the form:

\[
y = \pi \left( \frac{3^{1/6} \alpha}{r^{2/3}(\xi)} - \frac{\beta}{3^{2/3}r^{2/3}(\xi)} \right) Ai(x) + \pi \left( \frac{\alpha}{3^{1/3}r^{2/3}(\xi)} + \frac{\beta}{3^{2/3}r^{2/3}(\xi)} \right) Bi(x) + Hi(x)
\]  
\[\text{...(39)}\]

Again, solutions (37) and (39) are equivalent in light of (9) and (35), which yield the following relationship between \( Ni(x) \) and \( Hi(x) \), and the arbitrary constants are re-evaluated:

\[
Ni(x) = \frac{2}{3} Bi(x) - Hi(x)
\]  
\[\text{...(40)}\]

4 Conclusion

In this work we offered formulation of the non-homogeneous Airy’s ODE and its solution in terms of the newly introduced integral function, \( Ni(x) \). This function offers an alternative to the use of Scorer’s functions in the solution of non-homogeneous Airy’s ODE, and possesses some interesting features. While we only discussed the case where the forcing function is constant, this work sets the stage for formulating initial value problems involving more general forcing functions.

References


