A General Nonlinear Transformation For The evaluation of nearly singular integrals

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Abstract: - In this paper, a general nonlinear transformation is adopted and applied to calculating the potential and its derivative at the interior points very close to the boundary.

Key-Words: - BEM; potential problems; nearly singular integrals; boundary layer effect; transformation; Numerical method

1 Introduction

The BEM is power and efficient computational methods if integrals are evaluated accurately, and the main advantages of the BEM resulting from the reduction of the dimension of the boundary value problem are well-known. However, it is popular as well that the standard BEM formulations include singular and nearly singular integrals, and thus the integrations should be performed very carefully. Other than the nearly singular integral, many direct and indirect algorithms for singular integral have been developed and used successfully [1-11]. Therefore, the key point in achieving the required accuracy and efficiency of the BEM is not the singular integral but the nearly singular integral.

In this paper, a general nonlinear transformation is adopted and applied to calculating the potential and its derivative at the interior points very close to the boundary in 2D potential problems. The proposed transformation is constructed based on the idea of diminishing the difference of the orders of magnitude or the scale of change of operational factors. After the BIEs are discretized on the boundary, the nearly weakly singular, strongly singular and hyper-singular integrals can be calculated accurately by using the present method. The nonlinear transformation is available for linear and quadratic elements. Both temperatures and its derivative at the interior points very close to the boundary are accurately computed. The algorithm derived in this paper substantially simplifies the programming and provided a general computational method for solving thin coating problems.

2 Non-singular boundary integral equations (BIEs)

It is well known that the domain variables would be computed by using integral equations only after all the boundary quantities have been obtained, and the accuracy of boundary quantities directly affects the validity of the interior quantities. However, when calculating the boundary quantities, we have to deal with the singular boundary integrals, and a good choice is to use the regularized BIEs. In this paper, we always assume that $\Omega$ is a bounded domain in $\mathbb{R}^2$, $\Omega^c$ is its open complement; $\Gamma = \partial \Omega$ denotes their common boundary; $t(x)$ and $n(x)$ are the unit tangent and outward normal vectors of $\Gamma$ to domain $\Omega$ at point $x$, respectively. For two dimensional potential problems, the equivalent non-singular BIEs with direct variables are given in [8].

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\[
\int_{\Gamma} u(x,y)q(x)d\Gamma = \int_{\Gamma} [u(x) - u(y)] q'(x,y)d\Gamma, \quad y \in \Gamma
\]

(1)

where we can take the fundamental solution \( u^*(x,y) \) for Eq. (1) as

\[
u^*(x,y) = -\frac{1}{2\pi} \ln r
\]

(2)

in which \( r = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \); \( y_1 \) and \( y_2 \) are the coordinates of source point \( y \); \( x_1 \) and \( x_2 \) are the coordinates of field point \( x \).

After Eq. (1) is discretized to numerically evaluate the boundary unknown variables, the potential at interior points can be obtained by using the following integral equation

\[
u(y) = \int_{\Gamma} u^*(x,y)q(x)d\Gamma - \int_{\Gamma} q'(x,y)u(x)d\Gamma
\]

(3)

In order to determine the flux at interior point \( y \), taking derivative of Eq. (3) with respect to the coordinates of the source point \( y \) there is

\[
\nabla_j \nu(y) = \int_{\Gamma} \nabla u^*(x,y)q(x)d\Gamma - \int_{\Gamma} u(x)\nabla q'(x,y)d\Gamma
\]

(4)

The Gaussian quadrature is directly used to calculate the integrals in discretized equations (3) and (4) in the conventional boundary element method. However, when the field point \( y \) is very close to the integral element \( \Gamma_e \), the distance \( r \) between the field point \( y \) and the source point \( x \) tends to zero. This causes the integrals in the discretized equations (3) and (4) nearly singular. Therefore, the physical quantities at interior points cannot be calculated accurately by using the conventional Gaussian quadrature.

The above mentioned nearly singular integrals can be expressed as the following generalized integrals:

\[
\begin{align*}
I_1 &= \int_{\Gamma} \psi(x) \ln r^2 d\Gamma \\
I_2 &= \int_{\Gamma} \psi(x) \frac{1}{r^{2\alpha}} d\Gamma
\end{align*}
\]

(5)

where \( \alpha > 0 \), \( \psi(x) \) is a well-behaved function including the Jacobian, the shape functions and ones which arise from taking the derivative of the integral kernels. Under such a circumstance, either a very fine mesh with massive integration points or a special integration technique needs to be adopted. In the last two decades, numerous research works have been published on this subject in the BEM literature. Most of the work has been focused on the numerical approaches, such as subdivisions of the element of integration, adaptive integration schemes, exact integration methods and so on. However, most of these earlier methods are either inefficient or can not provide accurate results when the interior points are very close to the boundary. In this paper, a very efficient transformation method is employed to calculate the nearly singular integrals in the discretized equations (3) and (4). Consequently, the accurate results of the physical quantities at interior points very close to the boundary are obtained.

3 Nearly singular integrals under linear elements

The quintessence of the BEM is to discretize the boundary into a finite number of segments, not necessarily equal, which are called boundary elements. Two approximations are made over each of these elements. One is about the geometry of the boundary, while the other has to do with the variation of the unknown boundary quantity over the element. In this section, the geometry segment is modeled by a continuous linear element.

Assuming \( x^1 = (x_{11}, x_{12}) \), \( x^2 = (x_{21}, x_{22}) \) are the two extreme points of the linear element \( \Gamma_j \), then the element \( \Gamma_j \) can be expressed as

\[
x_k(\xi) = N_i(\xi)x_{ik} + N_j(\xi)x_{jk}, \quad \xi \in [-1,1], k = 1, 2
\]

(6)

where \( N_i(\xi) = (1 - \xi)/2, N_j(\xi) = (1 + \xi)/2 \).

Letting \( s_i = x_{i2} - x_{i1}, w_i = y_i - (x_{i2} + x_{i1})/2 \), one has

\[
r_j = \frac{r_{ij}}{r} = \frac{y_i - x_{ij}}{r} = \frac{s_i \xi/2 + w_i}{r}
\]

(7)

\[
r^2 = |x - y|^2 = r^2 = A\xi^2 + B\xi + E = L^2[(\xi - \eta)^2 + d^2]
\]

(8)

Where
\[ A = s_1 s_2 / 4, B = s_1 w_1, E = w_1 w_2, \eta = -B / 2A, \]
\[ L = \sqrt{A}, d = \sqrt{4AE - B^2} / 2A. \]

With the aid of the Eq. (8), the nearly singular integrals in Eq. (5) can be rewritten as
\[
\begin{align*}
I_1 &= \int_{\gamma_1} \int_{\gamma_2} g(\xi) \ln[(\xi - \eta)^2 + d^2] d\xi + \ln E \int_{\gamma_1} g(\xi) d\xi \\
I_2 &= \int_{\gamma_1} \int_{\gamma_2} \frac{g(\xi)}{L^{2a}[((\xi - \eta)^2 + d^2)^a]} d\xi
\end{align*}
\]

(9)

where \( g(\cdot) \) is a regular function that consists of shape function and Jacobian.

### 4 The transformation for nearly singular integrals

In Eqs. (9), if \( d \) is very small, the above integrals would present various orders of near singularity. The key to achieving high accuracy is to find an method to calculate these integrals accurately for a small value of \( d \).

The integrals \( I_1 \) and \( I_2 \) in Eqs. (9) can be reduced to the following integrals by simple deduction
\[
\begin{align*}
I_1 &= \int_0^d g(x) \ln(x^2 + d^2) dx \\
I_2 &= \int_0^d \frac{g(x)}{A(x^2 + d^2)^a} dx
\end{align*}
\]

(10)

where \( A \) is a constant which is possibly with different values in different element integrals; \( g(\cdot) \) is a regular function that consists of shape function, Jacobian and ones which arise from taking the derivative of the integral kernels.

Introducing the following nonlinear transformation
\[
x = d(e^{\theta t} - 1), t \in [-1, 1]
\]

where \( k = \ln \sqrt{1 + A/d} \).

Substituting (11) into (10), then the integrals \( I_1 \) and \( I_2 \) can be rewritten as follows
\[
\begin{align*}
I_1 &= dk \int_0^t g(t) \ln d^2 e^{\frac{1}{2t}} dt + dk \int_0^t g(t) \ln(d(e^{\frac{1}{2t}} - 1)^2 + 1) e^{\frac{1}{2t}} dt \\
I_2 &= d^{2-2a} k \int_0^t \frac{g(t) e^{\frac{1}{2t}}}{(e^{\frac{1}{2t}} - 1)^2 + 1} dt
\end{align*}
\]

(12)

We can observe that \((e^{\frac{1}{2t}} - 1)^2 + 1 \geq 1\). Thus, the integrand is fully regular even if the value of \( d \) is very small.

By following the procedures described above, the near singularity of the boundary integrals has been fully regularized even if the interior point very close to the boundary need to be calculated. The final integral formulations are obtained as shown in Eqs. (12), which can now be computed straightforward by using the standard Gaussian quadrature.

### 5 Conclusions

In this paper, a general nonlinear transformation, based on the direct regularized boundary element method, is adopted and applied to calculating the potential and its derivative at the interior points very close to the boundary in 2D potential problems. The proposed transformation is available for linear elements.

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