Identities and Inequalities Derived from Euclid’s Algorithm with Applications in Cutting-Covering Receipts

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Abstract: Starting from the original demonstration of the Euclid's Algorithm (Elements, Book VII,2) we deduce one using rectangles. From this proof we deduce after some calculus some identities and inequalities that we use in Cutting-Covering Receipts. Given two integers \(a\) and \(x\) then in Euclid’s Algorithm last remainder is 0. So, we can deduce a useful relation \(a \cdot x = q_1 x^2 + q_2 r_1^2 + \ldots + q_n d^2\). Starting from here we can produce some inequalities as \(\sum q_i \leq a \cdot x + 1 - a^2\).

Key-words: Euclid’s algorithm, cutting-covering receipts.

1 Introduction
We prefer in this introduction to reproduce, in original, pieces of Euclid’s algorithm:
Book VII (original topic)

Definition 1: Unity is that with in respect to every thing is called one.
Definition 2: And number is a set made of units.
Definition 3: A prime number is that measured only by unity.
Definition 4: Two prime numbers each to other are those which have common measure only the unity.

Problem 2
Find the highest common measure of two given numbers which are not prime each to other.

Proof: Let \(AB\) and \(CD\) the given numbers prime each to other. We have to find the biggest common measure of them. Then, if \(CD\) is a measure of \(AB\) and \(CD\) measures himself then \(CD\) is a common measure of \(CD\) and \(AB\). And obviously is the biggest because no greater number as \(CD\) will measure \(CD\).

But if \(CD\) does not measure \(AB\), then subtracting successively the smallest of the numbers \(AB, CD\) from the biggest, it will remain a certain number which measures the previous. It will not remain the unity, because then \(AB, CD\) will be prime numbers each to other, contrary to the hypothesis. So it will remain a certain number which measures the previous. And \(CD\) measuring \(BE\) remaining \(EA\) smaller and \(EA\) measuring \(DF\) remaining \(CF\) smaller and \(CF\) measuring \(AE\). Then, because \(CF\) measures \(AE\) and \(AE\) measures \(DF\) results \(CF\) measures \(DF\). But it is a measure for himself, so it will measure the entire \(CD\). But \(CD\) measures \(BE\) so \(CF\) measures \(BE\); \(CF\) measures also \(EA\) so it will measure also the entire \(BA\). But it measures also \(CD\) so \(CF\) measures \(AB\) and \(CD\). Consequently \(CF\) is a common measure of \(AB\) and \(CD\). I say also that it is the greatest. If \(CF\) is not the greatest measure of \(AB, CD\) then a certain number greater than \(CF\) will measure \(AB\) and \(CD\). Let it be \(G\) and because \(G\) measures \(CD\)
and CD measures BE then G measures BE. But it measures the entire BA so it will measure the reminder AE. But AE measures DE, so G will measure DF. But it measures the entire DC so it will measure the reminder CF. so the bigger will measure the smallest which is a contradiction. Consequently no greater number than CF will measure AB and CD; so CF is the biggest common measure of AB and CD.

A

E

C

F

G

B

D

2 Problem Formulation

The practical problem we are trying to model is: we have to cover a rectangular room with a cover (linoleum or carpet). This material is in a roll of fixed width and by cutting it with a guillotine we get another rectangle. We want to cover the room with a minimum number of pieces and to loose a minimum amount of material (the lost represents the difference between area of the bought material and the area of the room).

3 Solution

The algorithm we propose is based on the following two properties:

1. If we put the cover on the floor overlapping two adjacent sides of the cover over two adjacent sides of the floor and we cut the remainder of material (that does not cover the floor), we obtain a new piece of cover and a new piece of floor that have the same properties described by condition c1.

2. The cover may be put on the floor in two directions (see the example fig 1 and 2)

3. After we laid out the material in one of the directions we face the same issue regarding dimensions smaller than a or b or x or y.

Algorithm 1.

The algorithm (proposed in [2]) is recursively generating a binary tree: if the initial problem is to cover the rectangle of dimensions a and b with a cover of dimensions x and y then it is the root $T(a, b, x, y)$; putting x on a we obtain the right sub-tree with the root $T(a-x, b, x, y-b)$ and putting x on b we obtain the left sub-tree with the root $T(a, b-x, x, y-a)$.

$$T(a, b, x, y)$$

$$T(a', b', x', y') \quad T(a'', b'', x'', y'')$$

where $a'=a$, $b'=b-x$, $x'=x$ and $y'=y-a$, if $b>x$ or $a'=a-x$, $b'=b$, $x'=x-b$ and $y'=y$ if $x>b$ and $a''=a-x$, $b''=b$, $x''=x$ and $y'=y-b$ if $a>x$ or $a''=a$, $b''=b-y$, $x''=x-a$, $y''=y$ if $a<x$

A solution would be the construction of a binary tree only on the right side:

$$T(a, b, x)$$

$$T(a, b-y, x-a)$$

$$T(a, b-2y, x-2a)$$

$$\ldots$$

$$T(a, b-q_1y, r_1)$$

where $x=a*q_1+r_1$
\[ T(a-r_1, b', r_1) \]
\[ T(a-2r_1, b', r_1) \]
\[ \ldots \]
\[ T(r_2, b', r_1) \]
\[ \text{where } a = q_2 r_1 + r_2 \]
\[ T(r_2, b'-r_2, r_1- r_2) \]
\[ \ldots \]
\[ T(r_{n-1}, 0', r_n) \]

There is also a practical problem if the material has some geometrical model. The successive dimensions of the covering material and the surface to be covered are: \( x, a, r_1, r_2, \ldots, r_n, 0 \), the remainders of the divisions of Euclid’s algorithm.

The number of pieces \( S = \sum_{i=1}^{n} q_i \) will be the sum of the successive quotients from the same Euclid’s algorithm.

We extended this algorithm to a L-shape room in [1]. In [5] we presented an algorithm with polynomial complexity based on Euclid’s algorithm.

4 Some identities and inequalities

Let us consider a rectangular planar surface of sides \( a \) and \( b \) (suppose \( a < b \)) to be covered with pieces from a rectangular planar surface of sides \( x \) an \( y \) (\( x > y \)) satisfying the relations \( ab = xy \).

Keeping the orientation of the pieces in the covering process we have to make a convention: we shall choose fitting units of lengths on \( a, b, x, \) and \( y \) so:
- the same unit of length on \( a \) and \( x \) and the same unit of length on \( b \) and \( y \);
- the number of units of length on \( a \) coincide with the number of units of length on \( y \) and the same condition for \( b \) and \( x \);

We call unit rectangle a rectangle with the sides the unit length on \( a \) and \( b \).

The covering-cutting algorithm now take some ideas from the original Euclid’s algorithm presented in the introduction but the algorithm we present here is more natural.

The initial rectangle will be covered in these conditions with this receipt:
\[ x = q_1 a + r_1 \]
\[ a = q_2 r_1 + r_2 \]
\[ \ldots \]
\[ r_{n-2} = q_{n-1} r_{n-1} \]

Now a numerical example:
let \( a = 8, b = 19, x = 19, y = 8 \).
Euclid’s algorithm gives:
\[ x = q_1 a + r_1 = 2a + 3 \]
\[ a = q_2 r_1 + r_2 = 2r_1 + 2 \]
\[ r_1 = q_3 r_2 + r_3 = 1 \cdot r_2 + 1 \]
\[ r_2 = q_4 r_3 = 2 r_3 \]

And then we have the cutting receipt:

\[ \text{Fig. 3} \]

Now we can emphasize (starting from Euclid’s algorithm) an identity with areas. Counting the unit rectangles in Fig. 3 we can see that we have:
\[ ab = 2a^2 + 2r_1^2 + 1r_2^2 + 2r_3^2 \]

We can prove this identity in general. Let us multiply every relation in the covering receipt by its divisor and then add all the relations obtaining:
\[ xa = q_1 a^2 + q_2 r_1^2 + \ldots + q_{n} r_{n-1}^2 \]
or
with \( r_0 = a \) and where \( q_i, i = 1, n \) are the quotients in and \( r_i, i = 1, n - 1 \) are the correspondent reminders in the Euclid’s algorithm.

With respect to the last identity obtained, the biggest rectangular pattern we can use in the covering material has \( r_{n-1}^2 \) unit rectangles.

The number of pieces covering the surface is

\[
S = \sum_{i=1}^{n} q_i .
\]

Let us observe that we can write, using (1),

\[
\sum_{i=1}^{n} q_i = ax + \sum_{i=1}^{n} q_i r_{i-1}^2 = \]

\[
= ax + 1 - a^2 + (q_1 - 1)(1 - a^2) + \sum_{i=2}^{n} q_i (1 - r_{i-1}^2).
\]

1 - a^2 < 0 and 1 - r_{i-1}^2 < 0 so we obtain

\[
(2) \quad \sum_{i=1}^{n} q_i \leq ax + 1 - a^2 .
\]

Inequality (2) was deduced also in (6) by geometrical meanings.

References


