Recursively Defined Sequences and CAS

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Abstract: In mathematics curricula of secondary schools and also in many basic courses of mathematics at universities it is possible to identify notions based on infinite sequences. Students are usually acquainted with the fact that sequences are defined either by formulas for the *n*-th terms or recursively. Coursework is mainly concentrated on arithmetic and geometric sequences and their recurrence definitions. Other less traditional applications of recursively defined sequences are not mentioned even if they could contribute to better student motivation. However, a good way exists how to present more applications and more recent results that involve recurrence relations. With the help of IT and especially with the help of CAS we can present such substantial related concepts as stationary point, cobweb plot, stability, parameter sensitivity, bifurcation or chaos for instance. These concepts could be presented graphically via animation and we would like to emphasize that one of the main advantage of the graphical approach is to see how the behaviour of the problem solution is closely dependent on parameters. The aim of this paper is to provide ideas how to enrich student knowledge of recursively defined sequences.

Key-Words: visualization, motivation, CAS, Maple, **s**equence, recursively defined sequence, linear recursive equation, logistic equation, cobweb plot, bifurcation diagram

1 Introduction

It is well known that *sequences* are defined either by explicit *formulas* for the *n*-th terms or *recursively* which means that the previous values of terms of the sequence are used to generate the value of the next term of the sequence. In other words one needs to know the first couple of values of the given sequence and the recurrence construction to be able to find subsequent terms of the sequence. Recursively defined sequences have a lot of interesting applications in mathematics, particularly in numerical mathematics e.g. different methods of iterations for solving equations, but also in economy, biology and social sciences. Unfortunately in many classes of secondary schools in the Czech Republic and also in basic mathematics courses at universities there is not enough space to show these interesting applications. However, there exists a quick way how to present more applications and recent results that involve recurrence relations, cf. [3]. With the help of IT and especially with the help of CAS we can present such concepts as stationary point, cobweb plot, stability, parameter sensitivity, bifurcation or chaos for instance, for more details see [4]. These concepts could be presented graphically and we would like to emphasize that one of the main advantages of the graphic approach is to see parameter dependence behaviour of problems. The aim of this paper is to describe a kind of animation that we prepared for students to enrich their knowledge about recursively defined sequences and, in this way, to motivate them for deeper interest in mathematics problems.

First we briefly remind a basic mathematical formulation of recursively defined sequences. Given a real function f: $N_0 \times R \to R$ and an *initial value* $x_0 \in R$, consider the sequence of *iterates* of x_0 under the given function f:

$$x_0,$$
$$x_1 = f(0, x_0),$$
$$x_2 = f(1, f(0, x_0)),$$
$$x_3 = f(2, f(1, f(0, x_0))),$$
$$\ldots$$

Such a sequence is called *recursively defined sequence* and it is evident that its terms can be written in a more concise form as

$$x_0,$$
$$x_1 = f(0, x_0),$$
$$x_2 = f(1, x_1),$$
$$x_3 = f(2, x_2),$$
$$\ldots$$

It means that for given x_0, we can compute an arbitrary term of sequence x_t, where $t = 0, 1, 2, \ldots$, according to the relation

$$x_{t+1} = f(t, x_t).$$

This relation, together with an initial value x_0 is called a *first order recurrence equation.* The *solution* to this equation is any sequence x_t generated by this equation. It is worthwhile to notice that the given formulation is not the most general one but for the purposes of our consideration it is fully sufficient.

2 Solving Recurrence Equation with CAS

Sometimes it is possible to find a simple explicit formula for solution x_t to a given recurrence equation, but more often it is not possible. Students of basic courses of mathematics are usually able to find these formulas only for arithmetic and geometric sequences. If we want to show them more, we can use CAS at the beginning. Particularly in the Maple environment it is possible to use the command called `rsolve`.

For instance we can consider the following recurrence equation

```
> eq1:=x(t+1)=a*x(t)+b,
```

where *a* and *b* are real constants. This equation is known as *linear recurrence equation* of the first order and the function *f*: $y = ax+b$ as *linear map*. If $a = 1$ we obtain an arithmetic sequence. If $b = 0$ and $a \notin \{0,1\}$ we obtain a geometric sequence. The given equation has the constant solution

$$p = \frac{b}{1-a}$$

that is called an *equilibrium state* or a *stationary point*. To find a general formula for solution to the linear recurrence equation we can simply use

```
> rsolve(eq1, x).
```

Particularly for $a \neq 1$ we get

$$x_t = a^t\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}.$$

Once we have a formula we can start to study how the behaviour of the solution changes when values of parameters are changed. The most straightforward graph is to plot x_t as a function of positive integer *t* – such graph is called a *time graph*. We prepared an animation of time graphs that demonstrates changes in the behaviour of the solution. One of these parameter dependence observations is shown in Fig.1. Please note that this figure is only one graph that is a part of the set of graphs generating animation.

With the prepared animation it can be easy to demonstrate that the sequence x_t that solves linear recurrence equation of the first order can either decrease monotonically and *converge to the equilibrium state p,* or it can exhibit *damped oscillations* around *p,* or it can *diverge* to $+\infty$, $-\infty$, or, finally, it can exhibit *explosive oscillations* around the stationary point *p*.

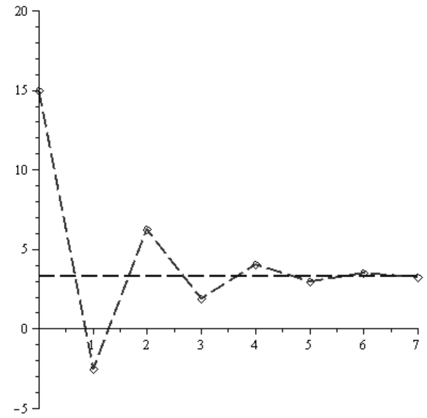

Fig.1: Solution to a linear recurrence equation. Sequence x_t exhibits damped oscillations around stationary point p, $x_0 > p = b/(1-a)$, $-1 < a < 0$, $b > 0$.

In this way it is possible to experiment with many recursively defined sequences and it is not difficult to find that recurrence equations exist for which it is not possible to find formulas for *n*-th term. One of them is the *logistic equation*; see Fig.2. We are going to deal with this equation later in this paper.

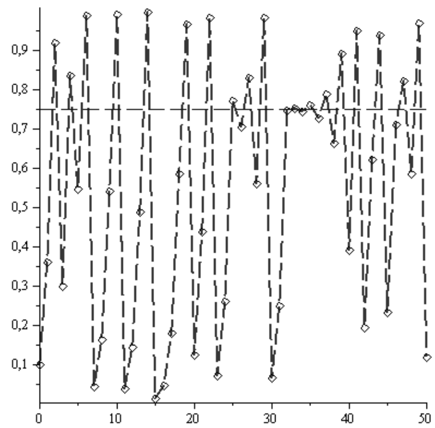

Fig.2: A turbulent development of solution x_t to the logistic equation $x_{t+1}=3.99027\, x_t(1-x_t)$, $x_0=0.1$.

3 Recurrence Equation and Cobweb Plot

A simple graphical technique called cobweb plot can help us to study the behaviour of solutions to recurrence equations. We prepared an animation that instructively demonstrates both the construction of the cobweb plot and the behaviour of the solution to a given recurrence equation. We considered a less general recurrence equation than in the previous part, particularly we consider that there is an initial value x_0 and a recurrence rule $x_{t+1} = f(x_t)$, where *f* is a map from [0, 1] on [0, 1], but not necessarily a linear one. Such a problem is called an *autonomous recurrence equation* and in this case it is possible to construct a plane *coordinate system* with the horizontal axis as x_t and the vertical axis as x_{t+1}. Now we can review the construction of a cobweb plot. First we plot the graph of the function f as well as the diagonal line $y = x$. Then we draw a vertical segment that is drawn from x_0 to the intersection with the graph of the function. This intersection has a corresponding ordinate $x_1 = f(x_0)$ it means we obtain a point (x_0, x_1). In the next step we draw a horizontal segment from this point to the diagonal. The point of intersection has the abscissa x_1, it means we obtain a point (x_1, x_1). Now we can repeat the same steps and we subsequently obtain terms of recurrently defined sequence x_2, x_3, x_4, ... A screen capture of the animation can be seen at Fig.3. More iteration for different map is given at Fig.4.

Notice that the intersection of the graph of the function f and the diagonal is a stationary point p that can be considered as a solution to the equation $x = f(x)$. If the cobweb spirals inward, then the stationary point is stable, and if it spirals outwards, then it is an unstable stationary point. If the stationary point of a given recurrence equation is its limit, the stationary point is called *asymptotically stable*.

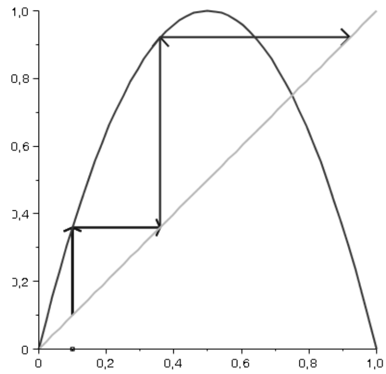

Fig.3: One screen of animation, that shows a construction of cobweb plot for the logistic equation $x_{t+1}=4x_t(1-x_t)$.

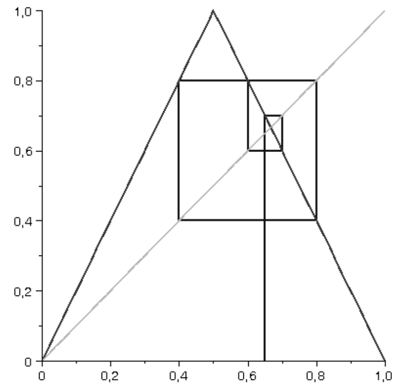

Fig.4: A more detailed cobweb plot for the recurrence equation $x_{t+1}=1-2|x_t-1/2|$, $x_0=0.64$, 2-cycle {0.8, 0.4}.

Moreover we can see that the iteration process is a way how to find a root of the equation $x = f(x)$. The approximation of the limit of the iteration process and therefore the root of the latter equation can be found with Maple. For evaluation we can use the first 100 iteration, then we simply use the command

```
> stationary_point:=(f@@100)(x0)
```

The sequence of iterates of x_0 under the function f: $x_0, x_1=f(x_0), x_2=f(x_1)=f^{(2)}(x_0), x_3=f(x_2)=f^{(3)}(x_0), ...$ can exhibit many different phenomena. To be able to study these phenomena of recurrence equations, we will need the following concept: a string of distinct points

$$p_1, p_2, ..., p_K,$$

where $K \in N$ and N is the set of all natural numbers, forms a *cycle of length K* or K-cycle if $f^{(K)}(p_i)=p_i$ for all $1 \le i \le K$, and points p_i are called *periodic points* of f with period K. It means that each periodic point with period K is a stationary point of the derived map $f^{(K)}$. This concept can be also visualized by cobweb plot, cf. Fig. 4, where we can observe 2-cycle. The derived map has very important consequences in behaviour patterns.

4 Logistic Equation

Let us consider the following recurrence equation

```
> eq2:=x(t+1)=a*x(t)*(1-x(t)),
```

which is known as *logistic equation* and the function f: $y = ax(1-x)$, $x \in [0,1]$ that generates its right hand side is known as *logistic map*. If we choose a = 2 then it is possible to find a formula for solution but for other values the Maple command `rsolve` does not find any formulas. To find a time plot of the solution to this equation it is necessary to use recurrence formula directly.

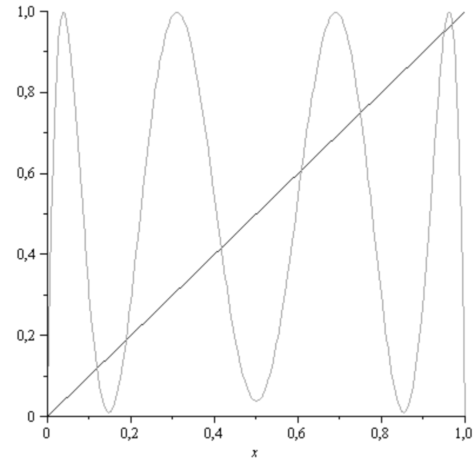

Fig.5: $f^{(3)}$, for f: y= 3.99027 x(1-x), except the stationary point we can see periodic points with period 3 that can be found on the line y=x.

We prepared an animation of several time plots which is suitable for studying of behaviour of solutions to logistic equation that depends on the value of the parameter *a*. The time plot for a=3.99027 can be find at Fig.2.

A logistic map is a simple quadratic function. This can be a reason that the logistic recurrence equation looks innocuous at the first glance. But in [2] it was noticed, that the sequence of iterates of logistic maps exhibit many unexpected phenomena. We concentrate here only to the phenomena that are concerned with the number of K-cycles depending up the values of parameter $a \in [0,4]$. For this reason, we prepared an animation of maps $f^{(k)}$, for different values of *k* and different values of *a*. Intersections graphs of $f^{(k)}$ and diagonal $y = x$ denote periodic points with period *k*, one of the screen can be seen at Fig.5.

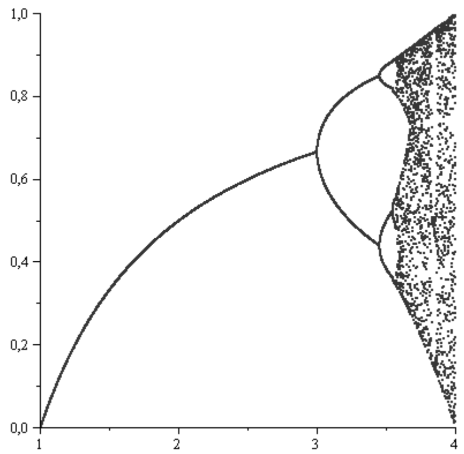

Fig.6: Bifurcation diagram for logistic map, it summarize number of either stable stationary points or k-periodic points for different values of parameter a.

The prepared animation can also help us to present a familiar statement that "period three implies chaos", cf. [1]. The number of *k* periodic points can be also depicted at bifurcation diagram. We prepared procedures that allow us to construct it, the result can be seen at Fig. 6. This animation allows us to study different maps with different domains, so we can freely see and study details of different bifurcation diagrams.

4 Conclusion

On one hand, computer experiments contribute crucially to new mathematical discoveries and development of the field called dynamical systems. On the other hand, computer animations can help us to introduce basic ideas of this field (see e.g. papers [5] - [6] of our colleague who has been interested in creation of multimedia applications for many years) . In the paper we described few animations that can approach ideas and concepts of discrete dynamical systems. The key mathematical concept that is connected with these ideas and that can be found in many basic courses of mathematics at universities or even at most secondary schools curricula is recursively given sequence. This concept was the initial point of our considerations and we therefore think that basic concepts of the theory of dynamical systems can be introduced to students very early in their mathematics education. R. May in [2] advises that people should be introduced to a simple nonlinear equation such as logistic equation early in their mathematical education, because "not only in research, but also in the everyday word of politics and economics, we would all be better off if more people realised that simple nonlinear systems do not necessary posses simple dynamical properties". The animations described in the paper can be considered as our attempt to contribute to fulfil of the advice and the wish of R. May. We hope that these animations could help teachers to transfer basic ideas of dynamical system from computer screens into visions of students.

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