A method based on the Monte Carlo optimization schemes for the control of nonlinear polynomial systems

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Abstract: The present work proposes a new method to the problem of implementation of Input-State feedback linearization. Such method is based on the search of a diffeomorphism which transforms the original nonlinear system into a linear one in the controllable canonical form with an external reference input, and the subsequent using of linear pole-placement techniques. The problem is solved without need of the differential geometric complexity of the feedback linearization technique. It is based however, on an analytical method using the development in to generalized Taylor series of vectorial nonlinear functions and Kronecker product tools. The algebraic developments, we present in this paper, use the Monte Carlo optimization method to choose the best parameters of the polynomial feedback control in order to ensure the effectiveness of the diffeomorphism. A simulation study is synthesized to show the effectiveness of the proposed approach.

Key-words: Nonlinear system control, Input-State feedback linearization, Analytic representations, Kronecker product, Monte Carlo optimization.

1 Introduction

The problem of Input-State feedback linearization method amounts to finding a nonlinear change of coordinates along with a nonlinear feedback, like in the new coordinates, the system in question is a stable linear one [1–3]. The exact state feedback linearization problem was pioneered and elegantly solved in [4–9], and for single input systems necessary and sufficient conditions for exact feedback linearization were established and documented in texts [4, 5]. However, these conditions, mainly based on the differential geometric techniques, return the practical exploitation and implementation of the synthesized method which appears as complicated and non sure task [10–12]. To solve this problem, a new approach has been developed, in this work, by using approximate polynomial developments of the functions characterizing the nonlinear model. These analytical representations consist in determining a polynomial control law via the synthesis of a polynomial transformation leading to a stable linear model. Certain parameters of the polynomial control are selected arbitrary, which makes it possible to have several solutions. In order to obtain the optimal one, we use the Monte Carlo method optimization. The paper is organized as follows. In section II, the practical implementation of exact Input-State feedback linearization problem is formulated and solved. Section III presents the optimization problem in the choice of the control polynomial parameters. The results of a numerical simulation study are presented in section IV. Section V concludes this paper.

2 Problem formulation

We consider single-input nonlinear systems with the following state-space representation:

\[
\dot{x} = \sum_{i \geq 1} F_i x^{[i]} + g_0 u \\
= \sum_{i \geq 1} f_i x^{[i]} + g_0 u
\]

(1)

where \( X \in \mathbb{R}^n \) is the vector of state variables and \( u \in \mathbb{R} \) the input variable. It is assumed that \( F(X) \) is analytic vector fields on \( \mathbb{R}^n \) and can be developed as generalized Taylor series using the Kronecker product and power of vectors. \( X^{[i]} \) is the \( i\)-th redundant Kronecker power of a \( n \)-dimensional vector \( x \) [13], and \( x^{[i]} \in \mathbb{R}^n \) is the \( i\)-th non-redundant Kronecker power of vector \( X \) [14, 15]. The relation between the nonredundant Kronecker power \( x^{[i]} \) and the redundant Kronecker power \( X^{[i]} \) of \( n \)-dimensional vector \( X \) can be written as [14]:

\[
X^{[i]} = R_{n,i} x^{[i]} \quad \text{or} \quad x^{[i]} = (R_{n,i}^\dagger X^{[i]})
\]

where \( (R_{n,i}^\dagger) \) is the Moore-Penrose pseudo-inverse of
The proposed approach, to solve this problem, needs the state feedback as:

\[ u = \alpha (x) + \beta (x) v \]  

with \( \alpha (x) \) and \( \beta (x) \) defined on \( U \) and \( v \) being an external reference input like the original system (1) can be transformed to a linear one:

\[ \dot{z} = Az + bv \]  

where \((A; b)\) is a controllable pair of constant matrices of appropriate dimensions.

For conditions under which the nonlinear system (1) is feedback linearizable can be found in texts such as [4, 5].

Note that, in literature [4, 16, 17], the new linear system have a Brunowsky canonical form. In the present work, the original system (1) is transformed to a linear one provided that it be stable.

Since the transformed system (4) is linear, the next step is to employ linear pole-placement techniques in order to arbitrarily assign the poles of the closed-loop system [18]. In the particular case, one can calculate a constant gain vector \( \mathbf{v} \), like the state static feedback law:

\[ v = -kz \]  

induces the closed-loop dynamics:

\[ \dot{z} = (A - bk) z \]  

when applied to the linear system (4).

Using the generalized Taylor series development in the static state feedback and in the diffeomorphism, one obtains:

\[
\begin{align*}
\mathbf{u} &= \sum_{i \geq 1} \alpha_i \mathbf{x}^{[i]} + \sum_{j \geq 0} \beta_j \mathbf{x}^{[j]} v \\
\mathbf{z} &= \phi (\mathbf{x}) = \sum_{k \geq 1} \phi_k \mathbf{x}^{[k]}
\end{align*}
\]

Then, the goal will consist in the determination of the nonlinear transformation \( S(\mathbf{z}) \) defined as:

\[ x = \phi^{-1} (\mathbf{z}) = S(\mathbf{z}) \]  

Then, the static state feedback which is characterized by the coefficients \( \alpha_i \) and \( \beta_j \), will be determined. The canonical form of (8) can be written as:

\[ x = \sum_{i \geq 1} S_1^i \mathbf{z}^{[i]} \]  

The Kronecker k-power of vector \( \mathbf{x} \) can be expressed by:

\[ \mathbf{x}^{[k]} = \sum_{i \geq k} S_i^k \mathbf{z}^{[i]} \]  

where:

\[ S_p^n = \sum_{j=1}^{p-n+1} (S_{p-j}^{n-1} \otimes S_j^1) \]  

Then, in the one hand, the expression (1) will be written as:

\[ \dot{x} = \sum_{i \geq 1} f_1 S_1^i \mathbf{z}^{[i]} + \sum_{i \geq 2} f_2 S_2^i \mathbf{z}^{[i]} + \ldots \]

\[ + \sum_{i \geq p} f_p S_p^i \mathbf{z}^{[i]} + g_0 u \]

On the other hand, if we use the following derivative of \( i \)-th Kronecker’s power of vector \( x \) [13]:

\[ \left( \mathbf{x}^{[i]} \right)_x = \frac{dx^{[i]}_x}{dt} = v^{[i]} \left( I_n \otimes \mathbf{x}^{[i-1]} \right) \]

where \( v^{[i]} = \sum_{j=0}^{i-1} (U_{n \times n} \otimes I_{n(i-j-1)}) \) and \( U_{p \times q} \) designates the Kronecker permutation matrix [13], then one obtains another expression of the derivative given by:

\[ \dot{x} = \sum_{i \geq 1} S_1^i v^{[i]} \left( I_n \otimes \mathbf{z}^{[i-1]} \right) \mathbf{z} \]  

Replacing \( \dot{z} \) by its expression (6), one has:

\[ \dot{x} = \sum_{i \geq 1} S_1^i v^{[i]} \left( (A - bk) \otimes I_{n(i-1)} \right) \mathbf{z}^{[i]} \]  

When expressing the input control \( u \) in terms of \( z \); we obtain the new following expression:

\[ u = \sum_{i \geq 1} \alpha_1 S_1^i \mathbf{z}^{[i]} + \ldots + \sum_{i \geq p} \alpha_p S_p^i \mathbf{z}^{[i]} - \left( \beta_0 k z + \ldots + \sum_{i \geq p} (\beta_p S_p^i \otimes k) \mathbf{z}^{[i+1]} \right) \]

Replacing \( u \) by its new expression (16) in (12), yields the expression of the derivative \( \dot{x} \) in terms of \( z \). The identification of elements of this equation with those of equation (15) leads to the following recurrent algorithm which exposes the solution of the studied problem:
1. For \( p = 1 \)
\[
\begin{align*}
S_1^1 &= I_n \\
\bar{f}_1 + g_0 \alpha_1 &= A - bk \\
A - bk &\text{ is stable}
\end{align*}
\] (17)

\( \beta \) arbitrary

2. For \( p \geq 2 \)
\[
\begin{align*}
\text{vec}(S_p^1) &= -\text{pinv} \begin{pmatrix} A_p \end{pmatrix} \text{vec} \left( \sum_{i=2}^{p} f_i S_p^1 R_n^p \right) \\
\alpha_p &= \text{pinv} \begin{pmatrix} g_0 \end{pmatrix} g_0 \left[ - \sum_{i=2}^{p-1} \alpha_i S_p^1 R_n^p \right] \\
&\quad + \left( \sum_{i=1}^{p} \left( \beta_i S_{p-1}^i \otimes k \right) R_n^p \right) \text{pinv} \begin{pmatrix} S_p^2 R_n^p \end{pmatrix} \\
A_p &= \begin{pmatrix} (R_n^p)^T \otimes (f_1 + g_0 \alpha_1) \\
&\quad - \left[ v(\beta) \right] (A - bk) \otimes (I_{n-p+1}) \right] R_n^p)^T \otimes I_n \\
\beta_{p-1} &\text{ arbitrary}
\end{align*}
\] (18)

where \( \text{vec}(\cdot) \) designates the vectorization operator and \( \text{pinv}(\cdot) \) designates the Moore-Penrose inverse.

Using this algorithm, we can deduce the components of the nonlinear transformation defined by (7) as:
\[
\begin{align*}
\phi_1^1 &= (S_1^1)^{-1} \\
\phi_k^1 &= - (S_1^1)^{-1} \left( \sum_{i=2}^{k} S_1^1 \phi_k^i \right) \\
\phi_k^i &= \sum_{j=1}^{k-i+1} (\phi_{k-j}^{i-1} \otimes S_1^j)
\end{align*}
\] (19)

A problem which can be studied, now, is the influence of parameters \( \beta \) on the effectiveness of the solution. Since the choice of parameters \( \beta_p \) is arbitrary, we resort to an optimization method to determine the optimal values of these parameters.

3. Optimization of parameters \( \beta_p \)

In this section, we will study the problem of the choice of parameters \( \beta_p \) using the Monte Carlo method. The Monte Carlo (MC) optimization technique as defined by Conley for mathematical programming problems consists of generating a random sample of many feasible solutions and selecting the best one. In our problem, we generate randomly and independently 50 vectors of \( \beta_p \). In each simulation, we calculate the Normalized Square Error (NMSE) defined as:
\[
\text{NMSE}_z = \frac{\| z - \hat{z} \|_2^2}{\| z \|_2^2}
\] (20)

where \( z \) is the exact solution given by:
\[
\begin{align*}
\dot{z} &= Az + bv \\
z(0) &= \phi(x(0)) \\
v &= -kz
\end{align*}
\] (21)

and \( \hat{z} \) is the approximate solution given by (7) and (19):
\[
\hat{z} = \sum_{k=1}^{\infty} \phi_k x[k]
\] (22)

\( \| \cdot \|_2 \) denotes the norm 2.

4. Illustrative example and simulation results

Consider the following single-input nonlinear dynamic system [19]:
\[
\begin{align*}
\dot{x}_1 &= 3 \sin x_2 \\
\dot{x}_2 &= -x_1 x_2 + u
\end{align*}
\] (23)

The study is considered in the neighborhood of the equilibrium point \( x_0 = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \). The system (23) developed as generalized Taylor series, truncated to the third order, yields the following polynomial system:
\[
\begin{align*}
\dot{x}_1 &= 3 x_2 - 0.5 x_2^3 \\
\dot{x}_2 &= -x_1 x_2 + u
\end{align*}
\] (24)

The analytical form of the system (24) is given by:
\[
\dot{x} = f_1 x + f_2 x^2 + f_3 x^3 + g_0 u
\] (25)

with
\[
\begin{align*}
f_1 &= \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}; f_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}; \\
f_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and}
\end{align*}
\]

\[
g_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

This nonlinear system will be changed to a linear one given by (4) with: \( A = \begin{pmatrix} 0 & 3 \\ 1 & -0.5 \end{pmatrix} \); \( b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). and the new control law \( v = -k \hat{z} \) is chosen like the linear system is defined by poles -1.5 and -2 which correspond to a gain matrix \( k = \begin{pmatrix} 2 & 3 \end{pmatrix} \).

The polynomial feedback control and the state transformation are truncated to the order 3. Then, according to equation (9) the analytical state expression transformation is given by:
\[
x = S(z) = S_1^1 z + S_2^1 z^2 + S_3^1 z^3
\] (26)
and the polynomial feedback control law is written as:
\[ u = \alpha(x) + \beta(x)v \]
\[ = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \left( \beta_0 + \beta_1 x + \beta_2 x^2 \right)v \]

(27)

The initial conditions have been chosen as:
\[ x(0) = [0.5 - 0.5]^T \]

For arbitrary choices of \( \beta_p \) such as:
\[ \beta_0 = 0.0879; \quad \beta_1 = [9.8323 \ 0.1868] \quad \text{and} \quad \beta_2 = [2.8142 \ 1.9009 \ 6.0243 \ 5.3630]. \]

We simulate the variations of the exact and approximate states \((z_1; \hat{z}_1)\) and \((z_2; \hat{z}_2)\), represented, respectively, in figure 1 and figure 2. It is obvious, in these figures, that the exact variables of system (21) and the approached one of system (22) have not the same dynamical behavior. However, one can note that the variables of the two systems are asymptotically stable although an arbitrary choice of parameters \( \beta_1 \) and \( \beta_2 \). The important difference between \((z_1; \hat{z}_1)\) and \((z_2; \hat{z}_2)\) will cause the change of the operating conditions for the state vector and then the whole local performances are not achieved.

In figure 3, we represent the variation of the variables \((z_1; \hat{z}_1)\) and in figure 4, we represent the variables \((z_2; \hat{z}_2)\). These figures show the good concordance between the state variables given by (21) and those defined by (22). This confirms the validity of the approximate transformation.

In table 1, we compare the NSE, defined by (20) for the parameters \( \beta_p \) \((p = 0, 1, 2)\) determined arbitrarily in a first study, and by the means of MC in a second one. It appears that the NSE is very reduced.
Table 1: Comparison of NSE

<table>
<thead>
<tr>
<th></th>
<th>Arbitrary choices</th>
<th>MC choices</th>
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</thead>
<tbody>
<tr>
<td>$N_{SE_z}$</td>
<td>5.1503</td>
<td>0.0033</td>
</tr>
</tbody>
</table>

in the case of MC optimization which explain the accurate concordance of variables of systems (21) and (22). This second comparison study will conclude the big advantage of implementing MC optimization routine when determining the $\beta_p$ parameters.

Figure 5 represents the dynamical evolution of the control inputs respectively obtained with variable of transformation (21) and (22). An important overshoot is observed in the case of arbitrary choice of $\beta_p$ parameters. This will induce saturation behavior of control variable in the case of physical processes. However, a satisfactory behavior is obtained with optimized $\beta_p$ parameters.

5 Conclusion

In this paper, we propose a new technique to solve practical implementation difficulties of Input-State feedback linearization method. We have used analytical representations and the Kronecker product tools. Firstly, we determine a nonlinear change of coordinates to a linear stable system. Secondly, we derive the polynomial control law via this transformation. In order to ensure the effectiveness of such transformation and to obtain a satisfactory dynamical behavior of the control signal, we exploit the Monte Carlo optimization method to determine the optimal parameters which achieve this goal. Our future work will focus on the design of a stability analysis by the search of a Lyapunov function. This will ensure the maximization of asymptotic stability regions around the considered operating points.

References:


