Dynamics Analysis in an Economic Growth Model with Logistic Population and Delay

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Abstract: In this paper, we will analyze the mathematical model associated to an economic growth process with logistic population growth and delay. Mathematical modeling of this economical growth process leads to an optimal control problem with delay. We study the Hopf bifurcation of this growth model in which production occurs with delay while new capital is installed (time-to-build). The time-to-build technology is shown to yield a system of differential functional equations with a steady state. This steady state exhibits the Hopf bifurcation and we determine the direction and stability of the bifurcating periodic solutions by applying the normal form theory and the center manifold theorem.

Key–Words: mathematical models applied in economies, endogenous growth, logistic population, optimization problems.

1 Introduction

The purpose of this paper is to study the Hopf bifurcation of an economical growth model with logistic population growth and with a delay between investment and production (time-to-build).

In [3] we studied the Hopf bifurcation for an economical growth model in which production occurs with delay while new capital is installed and the growth rate of population is constant.

In [4] we studied the Hopf bifurcation for an economical growth model in which production and tax evasion occur with delay while new capital is installed.

In this paper, we consider an economical growth model with logistic population growth as in [10] in which production occurs with delay while new capital is installed. The optimality conditions, due to the introduction of the time delay, lead to a system of functional differential equations.

In [10], the authors determine the steady state of this system and they investigate the local stability of the steady state by analyzing the corresponding transcendental characteristic equation of its linearized system. By choosing the delay as a bifurcation parameter, in [10] the authors show that this model with a delay exhibits a Hopf bifurcation. Therefore, the dynamics are oscillatory and this is entirely due to time-to-build production. In this paper, we discuss the direction and stability of the bifurcating periodic solutions by applying the normal form theory and the center manifold theorem [11].

The outline of this paper is as follows. In Section 2, we formulate the economical growth model with logistic population growth and delay between investment and production (time-to-build). In Section 3, choosing the delay as a bifurcation parameter, sufficient conditions for the existence of the Hopf bifurcation are derived. In Section 4, the direction of the Hopf bifurcation is analyzed by the normal form theory and the center manifold theorem introduced by Hassard and some criteria for the stability of the bifurcating periodic solutions are obtained. In Section 5, the conclusions are presented.
2 Setup of the model

Consider an economy that is inhabited by infinitely-lived households that, for simplicity, is normalized to one. Each household has access to a technology that transforms labor $L$ and capital $K$ into output $Y$ by a neoclassical production function $F : \mathbb{R}_+ \rightarrow \mathbb{R}$.

We assume that at time $t$ the household uses capital goods produced at time $t - \tau$, therefore the production at time $t$ is given by

$$Y(t) = F(K(t - \tau), L(t)).$$

Denoting the capital per unit of labor by $k = \frac{K}{L}$ for any $L \neq 0$, we define the production function in intensive form as $f(k)$. Therefore, $f$ is a $C^2$ class function, strictly increasing, strictly concave, linearly homogeneous, satisfying $f(0) = 0$, and the Inada conditions

$$\lim_{k \to 0} f'(k) = \infty, \quad \lim_{k \to \infty} f'(k) = 0.$$

The representative household’s preferences are represented by a continuous, strictly increasing and concave instantaneous utility function $U$ and subjective to a discount rate $\rho$.

Considering the aggregate consumption $C(t)$, the capital accumulation equation is:

$$\dot{K}(t) = F(K(t - \tau), L(t)) - \delta K(t - \tau) - C(t) $$

where $\delta \in [0, 1]$ is the rate at which capital depreciates.

Following Brida and Accinelli [3], function $L$ is assumed to evolve according to the logistic law

$$\dot{L}(t) = aL(t) - bL^2(t) $$

with $a > b > 0$. For simplicity, the initial population has been normalized to one, $L_0 = 1$.

Using the properties of production function, we can rewritten the capital accumulation equation in intensive form, thus

$$\dot{k}(t) = f(k(t - \tau)) - (a - bL(t) + \delta)k(t - \tau) - c(t).$$

(4)

In this economy the representative household chooses at each moment in time the level of consumption $c(t)$ so that to maximize the global utility

$$\int_0^\infty U(c(t)) e^{-\rho t} dt$$

subject to the constraint (3) and the following constrains

$$\begin{align*}
\dot{k}(t) &= f(k(t - \tau)) - (a - bL(t) + \delta)k(t - \tau) - c(t) \\
\dot{L}(t) &= aL(t) - bL^2(t), \quad L(0) = L_0, \\
k(t) &= \varphi(t); t \in [-\tau, 0].
\end{align*}$$

where $0 < c(t) \leq f(k(t - \tau))$, $k(t - \tau)$ is the productive capital at time $t$, and $\varphi : (-\infty, 0] \rightarrow \mathbb{R}_+$ is the initial capital function; it needs to be specified in order to identify the relevant history of the state variable.

That economical problem leads us to the following mathematical optimization problem denoted by $(P)$.

**Problem P.** Determine $(c^*, k^*, L^*)$ which maximizes the following functional

$$\int_0^\infty U(c(t)) e^{-\rho t} dt$$

and which verifies

$$\begin{align*}
\dot{k}(t) &= f(k(t - \tau)) - (a - bL(t) + \delta)k(t - \tau) - c(t) \\
\dot{L}(t) &= aL(t) - bL^2(t), \quad L(0) = L_0, \\
k(t) &= \varphi(t); t \in [-\tau, 0].
\end{align*}$$

To solve this optimization problem, apply the generalized Maximum Principle for time lagged optimal control problems (see Pontryagin et al 1962, [15]). As in [1], the first order conditions are obtained.

Using the first order conditions for the optimization problem $(P)$ we obtain:

**Remark 1.** The optimal solution of the problem $(P)$ is a solution of the following system of differential equations

$$\begin{align*}
\dot{c}(t) &= \frac{U'(c(t))}{U''(c(t))} [\delta + \rho + a - bL(t) - f'(k(t - \tau))] \\
\dot{k}(t) &= f(k(t - \tau)) - (a - bL(t) + \delta)k(t - \tau) - c(t) \\
\dot{L}(t) &= aL(t) - bL^2(t).
\end{align*}$$

3 Local stability analysis and Hopf bifurcation

In [10], the authors determine the steady state of the system (6) and they investigate the local stability of this steady state by analyzing the corresponding transcendental characteristic equation of its linearized system.

They set the following results:
Proposition 1 The system of functional differential equations (6) has a unique steady state \((c^*, k^*, L^*)\) which is determined by the following equations:

\[
f'(k^*) = \delta + \rho, \quad c^* = f(k^*) - \delta k^*, \quad L^* = \frac{a}{b}.
\]

The associated characteristic equation of the linearized system of system (6) is given by

\[
\lambda^3 - c_{001}\lambda^2 - (b_{010}\lambda^2 + (b_{100}a_{010} - c_{001}b_{010})\lambda - b_{100}a_{010}c_{001})e^{-\tau \lambda} = 0 \quad (7)
\]

where

\[
g(c^*) = \frac{U'(c^*)}{U''(c^*)},
\]

\[
a_{010} = -g(c^*)f''(k^*), \quad a_{001} = -bg(c^*),
\]

\[
a_{110} = -g'(c^*)f''(k^*), \quad a_{011} = -g'(c^*)f'''(k^*),
\]

\[
a_{101} = -bg'(c^*)f''(k^*),
\]

\[
a_{012} = -g''(c^*)f''(k^*),
\]

\[
a_{120} = -g''(c^*)f'''(k^*),
\]

\[
a_{020} = f''(k^*) - \delta, \quad a_{002} = bk^*,
\]

\[
c_{001} = -a, \quad c_{002} = -2b.
\]

Proposition 2 If \(\tau = 0\), then the characteristic equation (7) is given by

\[-\lambda^3 + (\rho - a)\lambda^2 + (a\rho - a_{010})\lambda - a_{010}a_{001} = 0. \quad (8)
\]

The equation (8) has one positive eigenvalue and two eigenvalues with negative real part.

Also, in [10] the existence of Hopf bifurcation for system (6) is studied by choosing the delay \(\tau\) as the bifurcation parameter. The following result is obtained

Proposition 3 Let \(\lambda = \lambda(\tau)\) be a solution of (7). If \(\tau_c, \omega_c\) are given by

\[
\omega_c = \frac{1}{\sqrt{2}} \sqrt{b_{010}^2 + \sqrt{b_{010}^4 + 4a_{010}^2}}
\]

\[
\tau_c = \frac{1}{\omega} \arctan \frac{\rho\omega}{f''(k^*)g(c^*)}
\]

and

\[
\text{Re} \left( \frac{d\lambda}{d\tau} \right)_{\lambda = \omega_c, \tau = \tau_c} \neq 0
\]

then a Hopf bifurcation occurs at the steady state given by \((c^*, k^*, L^*)\) when \(\tau\) passes through \(\tau_c\).

4 Direction and local stability of the Hopf bifurcation

In this section, we study the direction, the stability and the period of bifurcating periodic solutions. The method that we used is based on the normal form theory and the center manifold theorem introduced by [9]. From the previous section, we know that if \(\tau = \tau_c\), then all the roots of (7), other than \(\pm i\omega_c\), have negative real parts and any root of the form \(\lambda(\tau) = \alpha(\tau) + i\omega(\tau)\) satisfies \(\alpha(\tau_c) = 0, \omega(\tau_c) = \omega_c\) and

\[
\frac{d\alpha(\tau_c)}{d\tau} \neq 0.
\]

For notational convenience, let \(\tau = \tau_c + \mu, \mu \in \mathbb{R}\). Then \(\mu = 0\) is a Hopf bifurcation value of system (6). In the study of the Hopf bifurcation problem, we transform system (6) into an operator equation of the form

\[
x_t = A(\mu) x_t + \mathcal{R}(\mu) x_t \quad (9)
\]

where \(x = (x_1, x_2, x_3)^T, \quad x_t = x(t + \theta), \quad \theta \in [-\tau, 0] \).

The operators \(A\) and \(\mathcal{R}\) are defined as

\[
A(\mu) \phi(\theta) = \begin{cases} 
\frac{d\phi(\theta)}{d\theta}, & \theta \in [-\tau, 0) \\
A\phi(0) + B\phi(-\tau), & \theta = 0
\end{cases}
\]

where \(\phi \in C^1([-\tau, 0], \mathbb{C}^2)\) and \(A, B\) are given by

\[
A = \begin{bmatrix}
0 & 0 & a_{001} \\
b_{100} & 0 & b_{001} \\
0 & 0 & c_{001}
\end{bmatrix},
\quad B = \begin{bmatrix}
0 & a_{010} & 0 \\
b_{010} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and

\[
\mathcal{R}(\mu) \phi(\theta) = \begin{cases} 
(0, 0)^T, & \theta \in [-\tau, 0) \\
(F_1(\mu, \phi), F_2(\mu, \phi), F_3(\mu, \phi))^T, & \theta = 0
\end{cases}
\]

where

\[
F_1(\mu, \phi) = \frac{1}{2!}[a_{020}\phi_2^2(\tau) + 2a_{110}\phi_1(0)\phi_2(\tau) + 2a_{100}\phi_1(0)\phi_1(\tau) + 4a_{010}\phi_1(0)\phi_1(\tau) + 4a_{011}\phi_1(0)\phi_1(\tau) + 4a_{002}\phi_3^2(\tau)]
\]

\[
F_2(\mu, \phi) = \frac{1}{2!}[b_{020}\phi_2^2(\tau) + 2b_{011}\phi_2(\tau)\phi_3(\tau) + 2b_{010}\phi_2^2(\tau) + 2b_{001}\phi_1^2(\tau) \phi_2(\tau) + 2b_{002}\phi_3^2(\tau)]
\]

\[
F_3(\mu, \phi) = \frac{1}{2!}c_{002}\phi_3^2(0).
\]
For $\psi \in C^1([0, \tau], \mathbb{C}^2)$, the adjoint operator $A^*$ of $A$ is defined as

$$A^* (\mu) \psi (s) = \begin{cases} - \frac{d\psi (s)}{ds}, & s \in [0, \tau) \\
 A^T \psi (0) + B^T \psi (\tau), & s = \tau \end{cases}$$

(12)

For $\phi \in C^1([-\tau, 0], \mathbb{C}^2)$ and $\psi \in C^1([0, \tau], \mathbb{C}^2)$, we define the bilinear form

$$\langle \psi, \phi \rangle = \int_{-\tau}^{0} \left( \int_{0}^{\theta} \overline{\psi}^T (\xi - \theta) B \phi (\xi) d\xi \right) d\theta.$$ 

(13)

To determine the Poincare normal form of the operator $A (\mu)$, we need to calculate the eigenvector $\phi$ of $A$ associated with eigenvalue $\lambda_1$ and the eigenvector $\phi^*$ of $A^*$ associated with eigenvalue $\lambda_2 = \overline{\lambda_1}$.

**Proposition 4** (i) The eigenvector $\phi$ of $A(0)$ associated to the eigenvalue $\lambda_1 = i \omega$ is given by $\phi (\theta) = \omega e^{\lambda_1 \theta}$, $\theta \in [-\tau, 0]$ where $w = (w_1, w_2, v_0)^T$ and $v_1 = a_{010}^2 e^{\lambda_2 \tau c}, \lambda_2 = \lambda_1, v_2 = \lambda_1, v_3 = 0$.

(ii) The eigenvector $\phi^*$ of $A^*(0)$ associated to the eigenvalue $\lambda_2 = \lambda_1$ is given by $\phi^* (s) = w e^{\lambda_2 s}, s \in [0, \tau], w = (w_1, w_2, v_3)^T$, and $w_1 = \eta, w_2 = \lambda_1 \eta$.

$$\eta = \frac{\lambda_2}{2} \overline{\lambda_1} (\bar{v}_1 + \lambda_1 \bar{v}_2) + a_1$$

where

$$a_1 = \bar{v}_2 (a_{010} + \lambda_1 b_{010}) (1 - \tau_c e^{\lambda_1 \tau c} - e^{\lambda_2 \tau c})$$

(iii) With respect to (13) we have

$$\langle \phi^*, \phi \rangle = 1, \langle \phi^*, \overline{\phi} \rangle = \langle \overline{\phi}, \phi^* \rangle = 0, \langle \overline{\phi^*}, \overline{\phi} \rangle = 1.$$ 

Next, we determine the coordinates of the center of the manifold $\Omega_0$ at $\mu = 0$ as in [8], [11].

We consider

$$z (t) = \langle \phi^*, x_1 \rangle, \quad w (t, \theta) = x_t - 2 \text{Re} \{z (t) \phi (\theta)\}.$$ 

On the center manifold $\Omega_0$,

$$w (t, \theta) = w (z (t), \bar{z} (t), \theta)$$

where

$$w (z, \bar{z}, \theta) = w_20 (\theta) \frac{z^2}{2} + w_{11} (\theta) \bar{z} \bar{z} + w_{02} (\theta) \frac{\bar{z}^2}{2} + ...$$

(14)

and $z, \bar{z}$ are the local coordinates of the center manifold $\Omega_0$ in the direction of $\phi$ and $\phi^*$, respectively.

For the solution $x_t \in \Omega_0$ of (9) notice that for $\mu = 0$, we have

$$\dot{z} (t) = \lambda_1 z (t) + \langle \phi^*, R (w (t, \theta) + 2 \text{Re} \{z (t) \phi (\theta)\}) \rangle.$$ 

(15)

We rewrite (15) as $\dot{z} (t) = \lambda_1 z (t) + g (z, \bar{z})$ with

$$g (z, \bar{z}) = \langle \phi^*, (0) \rangle R (w (z, \bar{z}, \theta) + 2 \text{Re} \{z \phi (\theta)\})$$

(16)

We expand the function $g (z, \bar{z})$ on the center manifold $\Omega_0$ in powers of $z$ and $\bar{z}$

$$g (z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{12} \frac{z^2 \bar{z}}{2} + \ldots$$

(17)

**Proposition 5** For the system (6) we have

$$g_{20} = \bar{w}_1 F_{120} + \bar{w}_2 F_{220}, g_{11} = \bar{w}_1 F_{111} + \bar{w}_2 F_{211},$$

$$g_{02} = \bar{w}_1 F_{102} + \bar{w}_2 F_{202}, g_{21} = \bar{w}_1 F_{121} + \bar{w}_2 F_{221}$$

where

$$F_{120} = \frac{a_{010} \nu_2 e^{\lambda_1 \tau c} + 2 a_{110} \nu_1 \nu_2 e^{\lambda_2 \tau c}},$$

$$F_{202} = \bar{w}_1 F_{102} + \bar{w}_2 F_{202},$$

$$F_{111} = \frac{a_{110} (\nu_1 \nu_2 e^{\lambda_1 \tau c} + \nu_1 \nu_2 e^{\lambda_2 \tau c}) + a_{020} \nu_2 \nu_2}{2},$$

$$F_{121} = \frac{a_{110} (\nu_1 \nu_2 e^{\lambda_1 \tau c} + \nu_1 \nu_2 e^{\lambda_2 \tau c}) + a_{010} \nu_1 \nu_2}{2} + a_{110} (\nu_1 \nu_2 e^{\lambda_1 \tau c} + \nu_1 \nu_2 e^{\lambda_2 \tau c}) + a_{010} \nu_1 \nu_2 + a_{010} \nu_1 \nu_2 e^{\lambda_2 \tau c},$$

and

$$w_{20} (\theta) = (w_{20} (\theta), w_{20} (\theta))^T, \quad w_{11} (\theta) = (w_{11} (\theta), w_{11} (\theta))^T$$

are given by

$$w_{20} (\theta) = \frac{g_{20}}{\lambda_1} e^{\lambda_1 \theta} - \frac{g_{20}}{3 \lambda_1} \bar{v}_1 e^{\lambda_2 \theta} + E_1 e^{2 \lambda_1 \theta},$$

$$w_{11} (\theta) = \frac{g_{11}}{\lambda_1} e^{\lambda_1 \theta} - \frac{g_{11}}{\lambda_1} \bar{v}_1 e^{\lambda_2 \theta} + E_2, \quad \theta \in [-\tau, 0]$$

where

$$F_{12} = (E_{11}, F_{12})^T, \quad E_2 = (E_{21}, E_{22})^T,$$

$$E_{11} = F_{220} + \left( b_{010} e^{\lambda_1 \tau c} - 2 \lambda_1 \right) E_{12},$$

$$E_{12} = \frac{F_{220} - 2 \lambda_1 F_{220}}{2 \lambda_{010} - a_{010}} e^{\lambda_1 \tau c} - 4 \lambda_1^2$$

$$E_{21} = b_{010} E_{22} + F_{211}, \quad E_{22} = \frac{F_{211}}{a_{010}}$$
Therefore, we can compute the following parameters

\[
C(0) = \frac{i}{2\omega} \left( g_{20} g_{11} - 2 |g_{11}|^2 - \frac{1}{3} |g_{02}|^2 \right) + \frac{g_{21}}{2}
\]

\[
\mu_0 = -\frac{\text{Re} (C(0))}{M_1}, \quad \beta_0 = 2 \text{Re} (C(0)) (21)
\]

\[
T_0 = -\frac{\text{Im} (C(0)) + \mu_0 N_1}{\omega}
\]

where \( N_1 = \text{Im} \left( \frac{d\lambda}{d\tau} \right) \bigg|_{\lambda = i\omega, \tau = \tau_c} \).

**Proposition 6** In formulas (21), \( \mu_0 \) determines the direction of the Hopf bifurcation: if \( \mu_0 > 0 \) (< 0) the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exist for \( \tau > \tau_c \) (< \( \tau_c \)); \( \beta_0 \) determines the stability of the bifurcation periodic solutions: the solutions are orbitally stable (unstable) if \( \beta_0 < 0 \) (> 0). \( T_0 \) determines the period of the bifurcating periodic solutions: the period increases (decreases) if \( T_0 > 0 \) (< 0).

## 5 Conclusions

In this paper, we formulate a growth model with delay for capital and logistic population growth. Using the delay as a bifurcation parameter we have shown that a Hopf bifurcation occurs when this parameter passes through a critical value. The direction of the Hopf bifurcation, the stability and the period of the bifurcating periodic solutions are also discussed and characterized.

The parameters of the real models are often subject to perturbations that can be considered as stochastic or uncertain. Starting with these considerations, as in [13, 14], the associated stochastic and fuzzy models can be taken into consideration.

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**References**


