Controlled Stochastic Jump Processes

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Abstract—Such processes are widely used in Engineering Sciences, Biology, Insurance, Inventory Theory and so on. We present a series of meaningful examples and describe rigorous mathematical models. One of the most powerful methods of obtaining an optimal control strategy is Dynamic Programming. Another modern method of attack is so called Convex Analytic Approach. In this report, the both methods will be briefly discussed. The second half of the article will be devoted to more specific models. First, we concentrate on the processes with local transitions, like birth-and-death processes. They are known to be successfully approximated by deterministic differential equations under the so called ‘fluid scaling’. In this connection, a new look at the well known ‘μC-rule’ in the Queuing Theory will be discussed: we shall compare the stochastic and deterministic versions. Finally, we shall stop on applications of the theory to the optimal buffer sizing for Internet routers.

Keywords—Communication networks, Constrained optimization, Continuous-time Markov decision process, Optimal control, Queuing theory.

I. INTRODUCTION

The trajectories of a purely jump process are piecewise constant functions of time, $\xi_t \in S$, $t \in \mathbb{R}_+$, where $S$ is the state space. If $\xi_t = x$ and action $a \in A$ is applied then $q(i|x, a)$ is the jump intensity (transition rate) to a new state $y \in \Gamma \subseteq S$, meaning that the sojourn time is exponential with parameter $q(S \setminus \{x\}|x, a)$, and the new state $y$ has probability distribution $q(dy|x, a)/q_x(a)$, where $q_x(a) \equiv q(S \setminus \{x\}|x, a)$.

Very often it is sufficient to consider “discrete case”, when the space $S$ is no more than countable. Then the jump (transition) intensity is given by function $q(j|i, a)$, where $i, j \in S$ and $\sum_{j\in S} q(j|i, a) = 0$. The sojourn time is $\exp(q_x(a))$ with $q_x(a) = \sum_{j \neq i} q(j|i, a) = -q(i|i, a)$, and the new state $j$ is realized with probability $q(j|i, a)/q_x(a)$.

If $\pi$ is a control strategy, for example in the form $a_t = \varphi(\xi_t)$, and $c_0(x, a)$ is the loss rate, then the total discounted loss equals

$$V_0(\pi) = E^\pi \left[ \int_0^\infty e^{-\alpha t} c_0(\xi_t, a_t) dt \right],$$

and one is looking for a strategy solving problem $V_0(\pi) \to \min_{\pi}$. Here $\alpha > 0$ is the discount factor.

There exist many other versions:

- multiple-objective constrained setting, when there are several different loss rates $c_0, c_1, \ldots, c_N$, and the problem looks like

$$V_0(\pi) \to \min_{\pi} \quad V_n(\pi) \leq d_n, \quad n = 1, 2, \ldots, N;$$

- long-run average loss, when

$$V_0(\pi) = \lim_{T\to\infty} \frac{1}{T} E^\pi \left[ \int_0^T c_0(\xi_t, a_t) dt \right];$$

- total (undiscounted) loss, e.g. up to absorption:

$$V_0(\pi) = E^\pi \left[ \int_0^\infty c_0(\xi_t, a_t) dt \right];$$

- optimal stopping when the process under control can be stopped at a (random) time moment $\tau$:

$$V_0(\pi, \tau) = E^\pi \left[ \int_0^{\tau} c_0(\xi_t, a_t) dt + \hat{c}(\xi_\tau) \right] \to \min_{\pi, \tau}.$$

Of course, one can investigate the constrained versions of the latter models.

The first publications devoted to continuous-time jump models appeared in the 1960-ies. (See e.g. [1, 2].) It seems, A. Yushkevich was first to consider past-dependent strategies in the 1970-ies [3], and M. Kitaev generalized the theory in the 1980-ies to randomized past-dependent strategies [4]. We mention only the most influencing papers; the first rigorous monograph on the Dynamic Programming Approach to jump models appeared in 1995 [5]. All these early publications, except for the pioneer article [25], considered the totally bounded transition rate $q_{i,j}(a) \leq K < \infty$ and a single objective functional $V_0$ to be minimized; the loss rate $c_0$ was usually bounded, as well.

To the best of our knowledge, the constrained modification was first studied in [7, 8] using the convex analysis. At the beginning of 2000-ies, many articles dealing with the unbounded transition rate started to appear (e.g. [9]). Recent book [10] is devoted to such models with different performance criteria, including the constrained modifications. But the authors considered only countable state space and restricted themselves to Markov strategies. The more general constrained discounted models, where randomized past-dependent strategies are allowed, were developed in [11, 12].

In the next sections, we give an overview of the current state-of-the-art: after several real-life examples, we briefly present the Dynamic Programming and the Convex Analytic Approach to discounted models, describe a very effective method for the investigation of the processes with local transitions (‘fluid approximation’) and demonstrate applications of the theory to Information Transmission in Internet.

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II. EXAMPLES

Of course, in each area mentioned below, there exist plenty of different particular models. We present only typical examples for illustration purposes.

A. Queues

Suppose there are \( m > 1 \) types of jobs to be served by a single server. Arrival and service rates for type \( j \) equal \( \Lambda_j \) and \( M_j \). Assume, there is infinite space for waiting. The holding cost of one type \( j \) job equals \( C_j \) per time unit. At any moment, the server should choose a job for service from the queue: that is, the action \( a = j \in \{1, 2, \ldots, m\} \) means that a type \( j \) job is under service. Note that we allow to switch the server to another job even before the service is completed. The goal is to minimize the total discounted holding cost over the infinite horizon. It is natural to assume that the system is stable: \( \sum_{j=1}^{m} \frac{\Lambda_j}{M_j} < 1 \). As is known [13], the optimal strategy, so called \( \mu C \)-rule, looks like follows. One should order all the job types in such a way that \( M_1C_1 \geq M_2C_2 \geq \ldots \geq M_mC_m \); at any time moment, the server must serve the queue with the highest \( M_jC_j \), if it is non-empty, of course.

B. Epidemics

Consider a closed population of size \( n \), where each individual can be either susceptible, or infective, or ‘removed’. The latter means either death, or recovery with immunization, so that the removed individuals cannot become infective. Let \( s(t) \) and \( i(t) \) denote the number of susceptibles and infectives, so that the state \( (s(t), i(t)) \) is two-dimensional. Now, if the current state is \( (s, i) \), then only the following two transitions can occur in a time interval \((t, t + \delta t)\):

\[
egin{align*}
(s, i) \rightarrow (s-1, i+1) & \quad \text{with probability } (k_1/n)s \delta t + o(\delta t), \\
(s, i) \rightarrow (s, i-1) & \quad \text{with probability } k_2i \delta t + o(\delta t),
\end{align*}
\]

where \( k_1 \) and \( k_2 \) are some positive coefficients. Any one susceptible meets \( k_1 \) individuals per time unit, and the chance to catch the disease equals \( s/n \). We suppose that the cost of an individual being infected is fixed. Without interventions, the process can terminate at the states of the form \((s, 0)\), but at any moment \( \tau \) we can stop the process by immunizing all the susceptibles at cost \( B_1 + s(\tau)B_2 \), where \( B_1 \) and \( B_2 \) are known constants. Clearly, this model is an example of the optimal stopping problem; it was investigated in [14].

C. Inventory

Usually, Inventory Theory deals with deterministic models [15]. If the inventory level is \( y \geq 0 \) then the demand rate is \( \mu(y) > 0 \), so that

\[
\frac{dy}{d\tau} = -\mu(y).
\]

At the moment \( \tau^* \) when \( y(\tau^*) \) reaches zero, the cycle is over and \( y(\tau^* + 0) = b > 0 \), i.e. the replenishment is instantaneous, the set-up cost being \( K \). Holding \( y \) units results in the cost \( yg(y) \) per time unit, and the order size \( b > 0 \) is under control. One is interested in the minimal cost per time unit.

Below, we formulate the corresponding (more accurate) stochastic version of the problem (see [16]). The \( \xi_t \) process describes the stock level and takes values in the state space \{0, 1, 2, \ldots\}. We assume that only the following transitions can occur from the current state \( \xi_t = i \) in a time interval \((t, t + \delta t)\):

\[
\begin{align*}
i \to i-1 & \quad \text{with probability } M_i i \delta t + o(\delta t) \quad \text{if } i > 0, \\
0 \to B & \quad \text{with probability } \lambda i \delta t + o(\delta t) \quad \text{if } i = 0.
\end{align*}
\]

Here, only the value of \( B \in \{1, 2, \ldots\} \), order size, is under control.

Remark 1 The lead time is exponential. If the replenishment is instantaneous (\( \lambda = 0 \)) then one should put the transition rate \( 1 \to B \) equal \( M_1 \).

If, like previously, \( K \) is the set-up cost, and \( G(i) \) is the holding cost per time unit, then we deal with the model with the long-run average loss, \( c_0(i, a) = G(i) + K M_1 I(i = 1) \). Here and below, \( I \) is the indicator function.

D. Reliability

Suppose a device can be in one of the states \{0, 1, \ldots, L\}, where 0 means ‘new’ and \( L \) means ‘broken’. One can consider the state \( i \) as the age of the device. The broken device is immediately replaced by a new one, but the decision maker can at any moment go for the preventive maintenance which takes exponentially distributed time. Suppose the sojourn time in state \( i \) is also exponential with parameter \( \lambda(i) \). Then transition rates from state \( i \) equal

\[
\mu \text{ for transition } i \to 0, \text{ if } i < L \text{ and the preventive maintenance is undertaken;}
\]

\[
\lambda(i) \text{ for transition } i \to i+1 (i \to 0) \text{ if } i < L - 1 (i = L - 1) \text{ and no preventive maintenance.}
\]

The case of instantaneous preventive maintenance (\( \mu = 0 \)) can be described similarly to Remark 1. Clearly, the state \( L \) is never observed: the process spends zero time in that state.

The reward rate in state \( i \) equals \( r(i) \), the cost of preventive maintenance in state \( i \) equals \( m(i) \), and the replacement of the broken device leads to the loss \( R \). Now the loss rate equals

\[
c_0(i, a) = I\{a = 1\} \mu m(i) - I\{a = 0\}(r(i) + \lambda(i)R),
\]

and one can investigate the model with the total discounted or long-run average loss.

Remark 2 Suppose \( \lambda(i) \equiv \lambda \) and the age of the device is unobservable, so that one can only see the failures and instantaneous replacements. The lifetime, being the sum of exponential random variables, has gamma distribution with density \( f(t) = \frac{\lambda^L}{\Gamma(L)} t^{L-1}e^{-\lambda t} \). Now, one can apply the theory of controlled jump processes with partial information. In a nutshell, one should make decisions about
III. DESCRIPTION OF THE CONTROLLED PROCESS

In this section, we shall present Kitaev’s construction of controlled jump processes, firstly introduced in [4] (see also [5, Chap.4] and [8]).

The following denotations are frequently used throughout this work: \( I \) stands for the indicator function; \( \delta_\cdot(x) \) is the Dirac measure concentrated at \( x \); \( B(X) \) is the Borel \( \sigma \)-algebra of the set \( X \); \( F_1 \cup F_2 \) is the smallest \( \sigma \)-algebra containing the two \( \sigma \)-algebras \( F_1 \) and \( F_2 \); \( X^c \) is the complement of the set \( X \); \( \mathbb{R}_+^\infty \triangleq (0, \infty) \), \( \mathbb{R}_0^\infty \triangleq [0, \infty) \), and for any signed measure \( \mu \), \( |\mu| \) denotes its total variation. The abbreviation \( s.t. \) stands for “such that” or “subject to”.

Let \( (S, \mathcal{B}(S)) \) be a Borel space, with \( S \) the state space of the process under control, and \( \mathcal{B}(S) \) its Borel \( \sigma \)-algebra. Having joined the isolated point \( x_{\infty} \) to \( S \), let us set \( S_{\infty} \triangleq S \cup \{x_{\infty}\} \). Let us denote \( (\Omega^N, \mathcal{F}^N) \triangleq \left( (S \times \mathbb{R}_+^\infty)^\infty, \sigma( (S \times \mathbb{R}_+^\infty)^\infty) \right) \), and join all the sequences of the form \( (x_0, \ldots, x_{m-1}, x_{m-1}, \infty, x_{\infty}, \ldots, x_{\infty} \ldots) \) to \( \Omega^N \), where \( m \geq 1 \) is some integer (we omit \( \theta_m \) in case \( m = 1 \), \( \theta_1 \in \mathbb{R} \) and \( x_1 \neq x_{\infty} \) for all non-negative integers \( l \leq m - 1 \). After the corresponding modification of the \( \sigma \)-algebra \( \mathcal{F}^N \), we obtain the basic measurable space \( (\Omega, \mathcal{F}) \). Put \( T_0 \triangleq 0 \), \( T_m \triangleq \theta_1 + \theta_2 + \ldots + \theta_m \), \( T_{\infty} \triangleq \lim_{m \to \infty} T_m \), and

\[
\xi_t(\omega) \triangleq \sum_{m \geq 0} \left[ I\{T_m \leq t < T_{m+1}\} x_m + I\{T_{\infty} \leq t\} x_{\infty}\right].
\]

In what follows, \( \omega = \{x_0, \theta_1, x_1, \ldots\} \) is often omitted, and \( h_m(\omega) = (x_0, \theta_1, \ldots, \theta_m, x_m) \) is referred to as an in-component history. Note, \( \{\xi_t, t \geq 0\} \) is the jump process of interest. Here \( \theta_m \) (resp. \( T_m, x_m \)) can be understood as the inter-jump intervals or sojourn times (resp. the jump moments, the state of the process on the interval \( [T_m, T_{m+1}) \)). We do not intend to consider the process after \( T_{\infty} : \) the isolated point \( x_{\infty} \) will be regarded as absorbing. Note, \( \{T_m, x_m\}, m \geq 0 \) is called marked point process, where \( \{x_m, m \geq 0\} \) marks the point process \( \{T_m, m \geq 0\} \) [5, Chap.4, Sec.4.3]. Clearly, \( \Omega \) is the canonical space of the jump process.

Associated with the marked point process \( \{T_m, x_m\}, m \geq 0 \) is the random measure

\[
\mu(\omega, dt, dy) \triangleq \sum_{m \geq 1} I\{T_m < \infty\} I\{x_m \in dy\} I\{T_m \in dt\},
\]

where \( \omega \in \Omega \). Denote by \( F_t = \sigma(\mu([0, s] \times \Gamma_s) : \Gamma_s \in \mathcal{B}(S), s \leq t) \) the natural filtration.

So far, we have described \( \{\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}\} \). We need now construct a probability measure on this triplet to build a stochastic basis. To do so, let us firstly define strategies (policies), on which the probability measure will be dependent. Let \( A \) be the action space, and assume \( (A, \mathcal{B}(A)) \) to be Borel. We also join the isolated point \( a_{\infty} \) to \( A \) and thus define \( A_\infty = A \cup \{a_{\infty}\} \). Let \( A(x) \), a set-valued function, be the admissible action space, when the current state of the jump process \( \{\xi_t, t \geq 0\} \) is \( x \). We shall assume that for any \( x \in S \), \( A(x) \in \mathcal{B}(A) \). Here we put \( A(x_{\infty}) \triangleq \{a_{\infty}\} \).

Definition 1 Let \( \mathcal{P} \) be the predictable (with respect to \( \{F_t\}_{t \geq 0} \)) \( \sigma \)-algebra on \( \Omega \times \mathbb{R}_+^\infty \). A \( \mathcal{P} \)-measurable transition probability function \( \pi(\cdot, \omega, t) \) on \( (A_\infty, \mathcal{B}(A_\infty)) \), concentrated on \( A(\xi_{-}) \), is called a strategy (policy). A strategy is called non-randomized and denoted as \( \phi \) whenever there exists a predictable \( A_\infty \)-valued process \( \phi(\omega, t, g, \cdot) \), \( t \geq 0 \) such that \( \pi(\Gamma^A(\omega, t), t) = I\{\Gamma^A \supseteq \phi(\omega, t)\} \) for all \( \Gamma^A \in \mathcal{B}(A_\infty) \). A strategy is called stationary if it has the form \( \pi(\cdot, \xi_{-}, \omega) \), where \( \pi \) is a stochastic kernel. A non-randomized strategy with \( \phi \) taking the form of \( \phi(\xi_{-}, \omega) \), where \( \phi \) is measurable, is called deterministic stationary, and we often simply signify such strategies with \( \phi \). A strategy is called Markov if it has the form \( \pi(\xi_{-}, \cdot, \omega) \).

Having defined \( K \triangleq \{(x, a) : x \in S, a \in A(x)\} \), let us assume that the set \( K \) is Borel-measurable and admits a graph of a measurable mapping \( \psi : S \to A \). Time-homogeneous controlled transition rate \( q(dy|x,a) \) is a finite measurable signed kernel on \( B(S) \) given \( (x, a) \in K \), so that \( q(\Gamma|S, x, a) \in [0, \infty) \) for all \( \Gamma \in \mathcal{B}(S) \), subject to \( \Gamma \cap \{x\} = \emptyset \) and \( q(S, x, a) = 0 \). The latter condition says that \( q(dy|x, a) \) is conservative, which implies that \( q(\{x\}, x, a) = -q((S \setminus \{x\}) \setminus S, x, a) \) and \( q_{\infty}(x, a) \triangleq \sup_{x \in A(x)} q_{\infty}(x, a) < \infty \). Additionally, \( q_{\infty} = sup_{x \in A(x)} q_{\infty}(x, a) < \infty \), meaning that \( q(dy|x, a) \) is stable. We put \( q(dy|x, a, \infty) \equiv 0 \). Finally, let us denote the cost rates, assumed to be measurable in \( (x, a) \in K \), by \( c_n(x, a), n = 0, 1, \ldots, N \), and the fixed constraints on the corresponding performance functionals by \( d_n, n = 1, 2, \ldots, N \).

Under any fixed strategy \( \pi \), the following predictable random measure reflects the desired infinitesimal behaviour of the above introduced process \( \xi_t \):

\[
\nu^\pi(\omega, dt, \Gamma_S) \triangleq \int_A \pi(da|\omega, t)q(\Gamma_S \setminus \{\xi_{-}(\omega), a\}) dt = \left\{ I\{t = 0\} \Lambda(\Gamma_S|x_0) + \sum_{m \geq 0} I\{T_m < t \leq T_{m+1}\} \right. \times \left. \Lambda^m(\Gamma_S|x_0, \ldots, x_m, x_{m}, t - T_m) \right\} dt,
\]

where \( \Gamma_S \in \mathcal{B}(S) \) and

\[
\Lambda^m(dy|x_0, \theta_1, x_1, \ldots, \theta_m, x_m, u) = \int_A \pi(da|x_0, \theta_1, x_1, \ldots, \theta_m, x_m, u + T_m) \times q(dy \setminus \{x_m\}|x_m, u). \]

Clearly, \( \Lambda^m \) also depends on \( \pi \). We write \( \pi(dy|x_0, \theta_1, x_1, \ldots, \theta_m, x_m, t) \) instead of \( \pi(da|\omega, t) \) because any \( \mathcal{P} \)-measurable function has that form on the interval \( T_m < t \leq T_{m+1} \).

Now instead, measure \( \nu^\pi \) is absolutely continuous with respect to the Lebesgue measure, under any strategy \( \pi \).
Moreover, it is known that, for a given initial distribution \( \gamma \) on \( S \), there exists a unique probability measure \( P_\infty^\gamma \) on \( (\Omega, \mathcal{F}) \) such that, restricted on \( \mathcal{F}_0 \), \( P_0^\gamma = \gamma \), and \( \nu^\gamma \) defined in (3) is the dual predictable projection of measure \( \mu \) defined in (2) [5, Chap.4.5]. In what follows, \( \mathbb{E}^\gamma \) is the corresponding expectation operator. If \( \gamma(\cdot) = \delta_x(\cdot) \) is concentrated at point \( x \in S \), we use denotations \( P_x^\gamma \) and \( \mathbb{E}_x^\gamma \).

Now the performance functionals briefly described in the Introduction, can be rigorously defined, and we shall concentrate on the discounted model with

\[
V_n(x, \pi) \overset{=}\equiv \mathbb{E}_x^\gamma \left[ \int_0^\infty e^{-\alpha t} \int_A c_n(\xi_{t-}, a) \pi(da|\omega, t) dt \right]
\]

and

\[
V_n(\pi) \overset{=}\equiv \mathbb{E}_x^\gamma \left[ \int_0^\infty e^{-\alpha t} \int_A c_n(\xi_{t-}, a) \pi(da|\omega, t) \right] = \int_S V_n(x, \pi) \gamma(dx).
\]

We shall shortly introduce conditions which guarantee that all the integrals are well defined. Thus, given \( \{S, A, \{A(x)\} x \in S\}, \{q_i, c_i\}, i = 1, \ldots, N \), \( \gamma, \alpha \), the constrained optimization problem looks like

\[
V_0(\pi) \to \min \pi \quad \text{s.t.} \quad V_n(\pi) \leq d_n, \quad n = 1, 2, \ldots, N.
\]

Let us denote \( \Delta \overset{=}\equiv \{\pi | V_n(\pi) \leq d_n, n = 1, \ldots, N\} \) the set of feasible strategies, and \( V_0^* \overset{=}\equiv \inf_{\pi \in \Delta} V_0(\pi) \) the constrained-optimal value. In what follows, we always assume \( \Delta \neq \emptyset \) : the concerned problem is consistent. If \( N = 0 \) the problem is unconstrained.

Definition 2 A strategy \( \pi^* \in \Delta \) is called (constrained)-optimal if the infimum in (7) is achieved at: \( V_0(\pi^*) = V_0^* \).

IV. MAIN CONDITIONS AND AUXILIARY RESULTS

Condition 1 There exist a measurable (weight) function \( w(x) \geq 1 \) on \( S \) and constants \( \rho, b \geq 0, b \geq 0 \) such that

(a) \( \bigcup_{i=0}^\infty S_i = S \), \( \sup_{x \in S_i} w(x) < \infty \) as well as \( \lim_{I \to \infty} \inf_{x \in A \cap S_i} w(x) = \infty \) for an increasing system of measurable subsets \( S_i \subseteq S \).

(b) \( \int_{S_i} q(d\gamma(x), a) w(y) \leq \rho w(x) + b, \forall x \in S, a \in A(x) \).

(c) For any \( I \), \( \sup_{x \in S_i} q_{x_i} < \infty \), where \( S_i \) has been defined in part (a), and \( q_{x_i} \overset{=}\equiv \sup_{a \in A(x)} q_{x_i}(a) \).

Condition 1 is of a Lyapunov type. Theorem 1 shows that it guarantees the \( \xi_t \) process to be non-explosive.

Condition 2 (a) \( \int_{S} \gamma(dy) w(y) < \infty \), where \( \gamma \) is the given initial distribution.

(b) \( \alpha > \rho \), where \( \alpha \) is the discount factor, and \( \rho \) is as in Condition 1.

(c) There exist constants \( M \geq 0 \) and \( c \geq 0 \) such that \( \inf_{a \in A(x)} c_n(x, a) \leq M w(x) + c, \forall x \in S, n = 1, 2, \ldots, N \).

This condition guarantees that the performance functionals (6) are well defined. Condition 2(c) is a version of the one imposed in [17].

Theorem 1 Suppose Condition 1 is satisfied. Then under any strategy \( \pi \), the following assertions hold:

(a) For any initial distribution \( \gamma \), \( P_x^\gamma(\mathcal{T}_\infty = \infty) = 1 \), and hence \( \forall t \geq 0, P_x^\gamma(\xi_t \in S) = 1 \). So explosion does not occur. Moreover, for all \( x \in S, t \geq 0 \),

\[
E_x^\gamma[w(\xi_t)] \leq e^{\rho t} w(x) + \frac{b}{\rho}(e^{\rho t} - 1).
\]

(b) If additionally Condition 2 is satisfied, then for any \( \gamma \), inequality

\[
V_0(\pi) \geq - \frac{M(\alpha \int_S \gamma(dy) w(y) + b)}{\alpha(\alpha - \rho)} - \frac{c}{\alpha} > -\infty
\]

holds.

Generally speaking, \( q_x \) may be not measurable. However, according to [18, D.5 Prop.], under Condition 3 below, \( q_x \) is measurable on \( S \).

Condition 3 (a) \( A(x) \) is compact, \( \forall x \in S \).

(b) \( q_x(a) \) is upper semicontinuous on \( A(x) \), \( \forall x \in S \).

Condition 4 There exists a constant \( L > 0 \) such that \( 0 < q_x < L w(x) \), \( \forall x \in S \).

Theorem 2 (a) Suppose Condition 1 and Condition 3 are satisfied. Then under any fixed strategy \( \pi \), \( \forall x \in S, t \geq 0 \), \( \forall i \in \mathcal{B}(S) \) such that \( \exists l : \Gamma \subseteq S_i \), with \( S_i \) being defined in Condition 1, Kolmogorov's forward equation (in the integral form) holds:

\[
P^\pi_x(\xi_t \in \Gamma) = \mathbb{I}(x \in \Gamma)
\]

\[
+E_x^\gamma \left[ \int_0^t \int_A \pi(da|\omega, u) q(\Gamma \setminus \{\xi_u\}|\xi_u, a) du \right]
\]

\[
-E_x^\gamma \left[ \int_0^t \int_A \pi(da|\omega, u) q_{\xi_u}(a) I(\{\xi_u \in \Gamma\}) du \right].
\]

(b) In part (a), if we replace Condition 1(c) by Condition 4, whereas all the other conditions are still satisfied, then we have the following stronger statement: \( \forall \Gamma \in \mathcal{B}(S) \),

\[
P^\pi_x(\xi_t \in \Gamma) = \mathbb{I}(x \in \Gamma)
\]

\[
+E_x^\gamma \left[ \int_0^t \int_A \pi(da|\omega, u) q(\Gamma \setminus \{\xi_u\}|\xi_u, a) du \right]
\]

\[
-E_x^\gamma \left[ \int_0^t \int_A \pi(da|\omega, u) q_{\xi_u}(a) I(\{\xi_u \in \Gamma\}) du \right].
\]

The expectations that appear in the above formulae are finite.

Condition 5 There exist a measurable function \( w'(x) \geq 1 \) on \( S \) and nonnegative constants \( L', \rho' \) and \( b' \) such that the following assertions hold:

(a) \( q_x + 1)w'(x) \leq L'w(x) \), where \( w \) comes from Condition 1.

(b) \( \int_{S} q(dy|x, a) w'(y) \leq \rho' w'(x) + b', \forall x \in S, a \in A(x) \).

(c) \( \alpha > \rho' \).

(d) There exist constants \( M' \geq 0 \) and \( c' \geq 0 \) satisfying \( \inf_{a \in A(x)} c_n(x, a) \leq M' w'(x) + c', \forall x \in S, n = 1, 2, \ldots, N \).
Definition 3 A measurable function \( u \) on \( S \) satisfying 
\[
\sup_{x \in S} \frac{|u(x)|}{w(x)} < \infty \quad (\text{resp. } \sup_{x \in S} \frac{|u(x)|}{w'(x)} < \infty)
\]
is said to have a bounded \( w \)-\,(resp. \( w' \))-weighted norm, with the norm 
\[
\|u\|_w = \sup_{x \in S} \frac{|u(x)|}{w(x)} \quad (\text{resp. } \|u\|_{w'} = \sup_{x \in S} \frac{|u(x)|}{w'(x)}).
\]
The collection of all functions \( u \) on \( S \) with a bounded \( w \)-\,(resp. \( w' \))-weighted norm is denoted by \( B_w(S) \) (resp. \( B_{w'}(S) \)).

Theorem 3 Suppose Condition 1, Condition 3 and Condition 5(a,b) are satisfied. Then \( \forall u \in B_w(S) \), the following two versions of Dynkin's formula hold:
\[
E^x [u(\xi_t)] - u(x) = \int_0^t \int_A \pi(da|\omega, \nu)q(dy|\xi_s, a)u(y)dv,
\]
\[
E^x [u(\xi_t)]e^{-\alpha t} - u(x) = \int_0^t e^{-\alpha t} \left\{ \left\{-\alpha u(\xi_v)\right\} \right. \right. + \left. \left. \int_A \pi(da|\omega, \nu)q(dy|\xi_v, a)u(y) \right\} dv. \tag{9}
\]

V. Dynamic Programming Approach

In this section, we investigate the unconstrained model when \( N = 0 \). The proofs of the presented statements can be found in [12], they generalize the results given in [10, 11].

Condition 6 (a) For any bounded nonnegative measurable function \( u(y) \) on \( S \) and fixed \( x \in S \), \( u'(x, a) \equiv \int_S u(y)q(dy|x, a) \) is lower semicontinuous in \( a \in A(x) \).
(b) \( \int_S w(y)q(dy|x, a) \) is continuous in \( a \in A(x) \), \( \forall x \in S \), where \( w \) comes from Condition 1.
(c) \( c_0(x, a) \) is lower semicontinuous in \( a \in A(x) \), \( \forall x \in S \).
(d) \( A(x) \) is compact, \( \forall x \in S \).

Theorem 4 Suppose Condition 1(b), Condition 2(b,c) and Condition 6 are satisfied. Then the Bellman equation
\[
\alpha U(x) = \inf_{a \in A(x)} \left\{ c_0(x, a) + \int_S q(dy|x, a)U(y) \right\}. \tag{10}
\]

admits a solution \( U^* \in B_w(S) \), which is given by the point-wise limit of the following non-increasing sequence of measurable functions \( \{U^{(n)}\}, n = 0, 1, \ldots \):
\[
U^{(0)}(x) \equiv \frac{M(\alpha w(x) + b)}{\alpha (\alpha - \rho)} + \frac{c}{\alpha},
\]
\[
U^{(n+1)}(x) \equiv \inf_{a \in A(x)} \left\{ \frac{c_0(x, a)}{\alpha} + \frac{1 + \bar{q}_x}{\alpha + 1 + \bar{q}_x} \times \int_S U^{(n)}(y) \left( \frac{q(dy|x, a)}{1 + \bar{q}_x} + I\{x \in dy\} \right) \right\}. \tag{11}
\]

For each \( n = 0, 1, 2, \ldots \)
\[
|U^{(n)}(x)| \leq \frac{M(\alpha w(x) + b)}{\alpha (\alpha - \rho)} + \frac{c}{\alpha}. \]

Theorem 5 Suppose Condition 1, Condition 2(a,b), Condition 5 and Condition 6 are satisfied. Then the following assertions hold:
(a) Suppose function \( U^* \in B_{w'}(S) \) solves the Bellman equation (12), then, for some deterministic stationary strategy \( \phi^* \)
\[
\int_S \gamma(dy)U^*(y) = \inf_{\pi} V_0(\pi) = V_0(\phi^*).
\]

If a measurable map \( \phi^* : x \to \phi^*(x) \in A(x) \) provides the infimum in (12) then strategy \( \phi^* \) is optimal.
(b) The Bellman equation (12) has a unique solution \( U^* \) in the class \( B_{w'}(S) \) which can be constructed using iterations (13), where \( w, M, c, \rho \) and \( b \) should be replaced with \( w', M', c', \rho' \) and \( b' \).
(c) The Bellman equation \( U^* \) solves the following dual linear program (DLP) in the space of measurable functions on \( S \):
\[
\int_S \gamma(dy)v(y) \to \max_{v} \tag{13}
\]
\[
s.t. \quad \forall (x, a) \in K,
\]
\[
\frac{1}{\alpha} c_0(x, a) - v(x) + \frac{1}{\alpha} \int_S v(y)q(dy|x, a) \geq 0; \quad v \in B_{w'}(S). \]

(d) Suppose \( v \) is feasible for DLP (14). Then it solves the DLP if and only if \( v(x) = U^*(x) \) a.s. (with respect to \( \gamma \)).

VI. Convex Analytic Approach

This method of attack proved to be effective for constrained problems when \( N > 0 \). In this section, we present briefly the results announced in [20]. Condition 2(c) must be modified in the following way:
\[
|c_n(x, a)| \leq Mw(x) + c \quad \text{for all } (x, a) \in K, \quad n = 0, 1, \ldots, N.
\]
The same concerns Condition 5(d). Condition 6 must be satisfied for all \( n = 0, 1, \ldots, N \). The new versions will be referred to as conditions 2', 5' and 6'.

Definition 4 The occupation measure of a strategy \( \pi \) is a probability measure on \( \mathcal{B}(S \times A) \) defined by
\[
\eta^\pi(G_S \times \Gamma_A) \equiv \alpha \int_0^\infty e^{-\alpha t} E^\pi \left[ I\{\xi_t \in G_S\}\pi(\Gamma_A|\omega, t) \right] dt,
\]
\[
\Gamma_S \in \mathcal{B}(S), \Gamma_A \in \mathcal{B}(A). \tag{14}
\]

Below, \( D \) denotes the set of all occupation measures.

Theorem 6 Suppose Condition 1(a,b), Condition 2(a,b), Condition 3 and Condition 4 are satisfied. Then the following assertions hold:
(a) For any fixed $\pi$, $\eta^\pi$ satisfies the following two relations:
\[ \eta(S \times A) = \gamma(S) + \frac{1}{\alpha} \int_{S \times A} q(s A) \eta(dy \times da) \] (15)
\[ \eta(S \in B(S)) , \]
and
\[ \int_M w(x)\eta(dx \times A) \leq \alpha \int_M w(x)\gamma(dx) + b \]
\[ \alpha - \rho < \infty , \] (16)

where $w$ comes from Condition 1.
(b) Every occupation measure $\eta^\pi(dx \times da)$ is concentrated on $K$, i.e., $\eta^\pi(K) = 1$.
(c) If a probability measure on $S \times A$ concentrated on K, namely, $\eta$, satisfies the two relations in part (a), then there exists a stationary strategy $\pi$ such that $\eta = \eta^\pi$. One can take $\pi$ as in the following formula, whose validity is guaranteed by [18, D.8 Prop.].
\[ \eta(S \times A) = \int_{S \times A} \pi(S \times A) \eta(dy \times A) . \] (17)

(d) Consider the probability measure $\eta$ as in part (c). If a stationary strategy $\tilde{\pi}$ is such that $\tilde{\eta}^\pi = \eta$, then $\tilde{\pi}$ is a version of the $\pi$ in (17), and in fact,
\[ \eta(S \times A) = \int_{S \times A} \tilde{\pi}(S \times A) \eta^\pi(dy \times A) . \]

(e) Suppose a stationary strategy $\pi$ is fixed. Then equation
\[ \tilde{\eta}(S) = \gamma(S) + \frac{1}{\alpha} \int_S \int_A q(S A) \pi(da A) \eta(dy) \]
has a unique solution in the class of probability measures on $S$ s.t. $\int_S w(y)\eta(dx) < \infty$. Moreover, the unique solution is provided by $\tilde{\eta}(dx) = \eta^\pi(dx \times A)$.

Clearly, under the conditions in Theorem 6, $D$ is convex.

Suppose $f(x) \geq 1$ is some given measurable function on $S$. Denote by $M^*_f(K)$ the (linear) space of finite signed measures $M$ on $K$ with a finite $f$-norm, i.e.,
\[ \int_K f(x) M(dx \times da) < \infty , \]
where $|M|$ denotes the total variation of $M$. For such $f$ and $M^*_f(K)$, we can define the concept of $f$-weak convergence.

Definition 5 For all $M, M_n \in M^*_f(K)$, $n = 1,\ldots$, we call $M_n$ to be $f$-weakly convergent to $M$, if for every continuous function $u(x, a)$ on $K$ with a finite $f$-norm, i.e.,
\[ \sup_{x, a} \left( \sup_{x, a} |u(x, a)| \right) \frac{f(x)}{f(x)} < \infty , \]
the following holds:
\[ \lim_{n \to \infty} \int_K u(x, a) M_n(dx \times da) = \int_K u(x, a) M(dx \times da) . \]
The $f$-weak convergence of $f$-bounded signed measures on Borel spaces other than $K$ can be defined similarly.

Condition 7 (a) For any bounded continuous function $u$ on $S$, function $\int_S w(y) q(dy|x, a)$ is continuous on $K$.
(b) $w$ coming from Condition 1 is continuous on $S$. The Convex Analytic Approach to the optimization problem (7) is to formulate it as a (primal) linear program, PLP:
\[ \frac{1}{\alpha} \int_K c_0(x, a) \eta(dx \times da) \to \min_{\eta, \tilde{\eta}_1, \ldots, \tilde{\eta}_N} \] (18)
\[ \text{s.t.} \]
\[ \frac{1}{\alpha} \int_K c_0(x, a) \eta(dx \times da) = \gamma(S) \forall \Gamma \in B(S) ; \]
\[ \int_K c_n(x, a) \eta(dx \times da) + \beta_n = \alpha d_n ; \]
\[ \eta(dx \times A) \tilde{w}(x) < \infty ; \]
\[ \gamma \text{ is a probability measure on } K , \]
\[ \beta_n \geq 0 , n = 1, \ldots, N , \]
\[ \tilde{w} \in B_{\tilde{w}}(S) , \]
which obviously coincides with (14) if $N = 0$. In what follows, the optimal value of a PLP (resp. DLP) is denoted as $\inf(PLP)$ (resp. $\sup(DLP)$).

Condition 8 $c_n(x, a)$ is continuous on $K$, $\forall n = 0, 1, \ldots, N$.

Theorem 8 Suppose Condition 1(a,b), Condition 2, Condition 4, Condition 5, Condition 6, Condition 7 and Condition 8 are satisfied. If additionally, the inequalities in (7) are all strict for some $\pi$ (the so called Slater’s condition), then there is no duality gap, i.e., $-\infty < \sup(DLP) (19) = \inf(PLP) (18) < \infty$.

Condition 9 (a) For any bounded continuous function $u$ on $S$, function $\int_S u(y) q(dy|x, a)$ is continuous on $K$, and $\sup_{x, a} \frac{u(x)}{w(x)} < \infty$.
(b) (i) The multifunction $x \to A(x)$ is compact-valued and upper semicontinuous (see [18, Appendix D]).
(ii) $w^\prime$ is continuous, and for each $r \geq 0$, a compact set $S_r \subseteq S$ exists such that $\frac{w(x)}{w^\prime(x)} \geq r \forall x \notin S_r$.
(iii) $S$ and $A$ are $\sigma$-compact.
(c) $c_n(x, a)$ is upper semicontinuous on $K$, $\forall n = 0, 1, \ldots, N$.

Theorem 9 Suppose Condition 1(a,b), Condition 2’, Condition 3, Condition 4, Condition 5’ and Condition 9 are satisfied. Then there is a randomized stationary optimal strategy solving problem (7)
We emphasize that in contrast with the case of unconstrained problems, it is well known that the class of deterministic stationary strategies are not sufficient for solving constrained optimization problems, see [8].

VII. FLUID APPROXIMATION TO THE PROCESSES WITH LOCAL TRANSITIONS

The simplest example of a jump process with local transitions is the pure birth process (Poisson process). Suppose it is not controlled and $\Lambda$ is its rate. Then, according to the law of large numbers, if $\Lambda = n\lambda$, then, for any finite $T > 0$ sup$_{0 \leq t \leq T} \left| \frac{1}{n} \xi_t - \lambda t \right|$ converges in probability to zero. Here $\xi_t$ is the Poisson ($n\lambda$) process under consideration. Moreover,

$$E \left[ \sup_{0 \leq t \leq T} \left| \frac{1}{n} \xi_t - \lambda t \right| \right] \leq \frac{\text{const}}{\sqrt{n}}.$$  

This idea is used for complicated queuing systems. (See e.g. [21].) In principle, for good enough functions $g(x)$, one can deduce the convergence

$$E \left[ \int_0^T g \left( \frac{\xi_t}{n} \right) dt \right] \rightarrow \int_0^T g(y_t) dt \text{ as } n \rightarrow \infty,$$  

(20)

where

$$y_0 = 0, \quad \frac{dy}{dt} = \lambda,$$  

(21)

and the rate is $\frac{1}{n^2}$. But it turns out that, due to integrating wrt time, the rate of convergence in (20) is $\frac{1}{n}$. Rigorous reasoning for specific cases can be found in [22]; stronger results for controlled models will appear in [23].

It should be emphasized that ‘fluid models’ like (21) are widely used for the study of complicated stochastic models, often without justification. For example, the epidemic model from part B, Section II, can be described similarly to (21):

$$\begin{align*}
\frac{dy^1}{dt} &= -k_1y^1y^1; \\
\frac{dy^2}{dt} &= k_1y^1y^2 - k_2y^2.
\end{align*}$$

Note, $y^*$ and $y^t$ correspond to the proportions $s/n$ and $i/n$ of susceptibles and infectives in the population of size $n$. Successful solution of an optimal control problem for deterministic (‘fluid’) epidemic model can be found in [24].

The deterministic inventory model (1) from Section II is another fluid limit of a stochastic model: one should put $M_i \cong n\nu (i/n), B = nb$ and $G(i) = g(i/n)$, where $n$ (‘fluid scaling parameter’) corresponds to the units in which the inventory level is measured.

Now, let us look at the queuing system described in part A, Section II. If the jobs are small and the arrival and service rates are big, we can apply the fluid scaling and put

$$A_j = n\lambda_j, \quad M_j = n\mu_j, \quad C_j = c_j/n.$$

The action space $A = \{1, 2, \ldots, m\}$ is as before. Suppose the initial state

$$Y = \{Y_1, Y_2, \ldots, Y_m\} = \{ny_1, ny_2, \ldots, ny_m\} = ny,$$

the numbers of jobs of different types, is given, and consider the optimal control problem without discounting up to the first moment when the system is empty:

$$nV(Y, \pi) = E^Y_0 \left[ \int_0^\infty \sum_{a \in A} G(\xi_t, a) \pi(a, t) dt \right] \rightarrow \min, \pi$$

(22)

where $\tau = \min \{t : \xi_t = 0\}$. Here $\xi_t$ is vector, describing the numbers of jobs of different types:

$$G(Y, a) = \sum_{j=1}^m C_j Y_j = \sum_{j=1}^m c_j y_j.$$  

It is known that the $\mu C$-rule is optimal for problem (22). It is also optimal for the fluid model which has the form:

$$v(y, \varphi) = \int_0^\infty \sum_{j=1}^m c_j y_j(t) dt \rightarrow \min \varphi,$$

subject to

$$\frac{dy(t)}{dt} = \begin{cases} \lambda_j, & \text{if } j \neq a(t) \text{ or if } y_j = 0; \\
\lambda_j - \mu_j, & \text{if } j = a(t) \text{ and } y_j > 0.
\end{cases}$$

(23)

Namely, the feedback control strategy

$$a^\ast(t) = \varphi^\ast(y(t)) = \left( k + 1 \right) \cdot I \{y_1(t) = 0, y_2(t) = 0, \ldots, y_m(t) = 0, y_{k+1}(t) > 0 \}$$  

solves problem (23). (We assume that $\mu_1 c_1 \geq \mu_2 c_2 \geq \ldots \mu_m c_m$.) The minimal value (Bellman function)

$$u^\ast (Y) = \min_{\pi} nV(Y, \pi)$$

is unknown, but the fluid model provides its approximation.

Theorem 10 (a) The Bellman function $u^\ast (y) \triangleq \min_{\varphi} v(y, \varphi)$ for problem (23) is defined by the following formulae

$$u^\ast (y_1, y_2, \ldots, y_m) = \sum_{k=1}^m c_k \left[ y_k T_k + \frac{\lambda_k T_k^2}{2} \right. + \frac{(y_k + \lambda_k T_k - 1)^2}{2(\mu_k (1 - \sum_{j=1}^k \frac{\lambda_j}{\mu_j})} \right],$$

where

$$T_k \triangleq \sum_{i=1}^k \frac{y_i}{\mu_i} \left( 1 - \sum_{j=1}^k \frac{\lambda_j}{\mu_j} \right).$$

(b) For any vector $\hat{y} \in \mathbb{R}^m$, the following inequality holds:

$$\sup_{0 \leq (Y_1, Y_2, \ldots, Y_m) \leq \hat{y}} | nU^\ast (Y) - u^\ast (Y/n) | \leq \frac{mD}{n} \left( \text{max}_{1 \leq j \leq m} \max \left\{ \lambda_j, \mu_j - \lambda_j \right\} \right. \times \sum_{k=1}^m \frac{y_k}{\eta_k} \left( \frac{1}{\eta_k} - \frac{1}{\lambda_k} \right),$$

where $D = \max_{1 \leq j \leq m} \left| \frac{\partial u^\ast}{\partial y_j} \right|$ is a constant since function $u^\ast$ is quadratic; vector inequalities are component-wise.
One can show that
\[ D \leq \frac{1}{\mu \delta} + \frac{(\delta + 1) \sum_{k=1}^{m} \lambda_k c_k}{\mu^2 \delta^3}, \]
where \( \delta \triangleq 1 - \sum_{j=1}^{m} \frac{\lambda_j}{\mu_j} \) and \( \mu \triangleq \min_{1 \leq j \leq m} \mu_j \).

More details and the proofs can be found in [16].

VIII. Optimal buffer sizing for congestion control in Internet

Controlled queueing systems are widely used when studying information transmission. For example, an Internet router can be considered as a server which, after fast consideration, sends packets of information further. One of the main parameters that can be tuned is the buffer size. If it is small then many packets will get lost (and resent), but if a packet is accepted then it goes through the router very quickly. On the other hand, if the buffer size is big then the throughput is high, but, since the average buffer occupancy is big, there is a big delay in the information transmission. Below, we present the Pareto set for this bi-objective optimization problem for a rather standard model of a bottleneck router under Additive Increase Multiplicative Decrease (AIMD) congestion control. Of course, a stochastic jump model would have described the situation more accurately, but it will be too complicated for the analysis. Therefore, we study the fluid model which is usually very accurate if the packets are small and the arrival and service rates are high, see Section VII. Below, we briefly present some results published in [25].

The main dynamic variable is the so called ‘total congestion window’ \( w(t) \) which is connected to the total sending rate:
\[ \lambda(t) = \frac{w(t)}{T + x(t)/\mu}. \]
Here \( T \) is the propagation delay, \( \mu \) is the capacity of the router (service rate), and \( x(t) \) is the current amount of data in the bottleneck queue. Note, we consider a single information flow which can be a combination of many long-lived AIMD TCP connections. (TCP stands for Transmission Control Protocol.)

Suppose \( B \) is the size of the Drop Tail buffer. If \( x(t) < B \) then the congestion window increases linearly (‘Additive Increase’):
\[ \frac{dw}{dt} = \frac{m}{T + x(t)/\mu}, \]
where \( m \) is a known constant. The dynamics of the queue length \( x \) is given by
\[ \frac{dx(t)}{dt} = \begin{cases} \lambda(t) - \mu, & \text{if } 0 < x(t) < B, \\ \lambda(t) - \mu, & \text{if } x(t) = 0 \text{ and } \lambda(t) \geq \mu, \\ 0, & \text{otherwise.} \end{cases} \]
If \( x(t^*) = B \) (the buffer starts to overflow) then, after the time delay \( \delta = T + B/\mu \), the sender will notice that, so that upon the reception of the congestion signal at time \( t^* + \delta \), the congestion window is reduced according to equation (‘Multiplicative Decrease’):
\[ w(t^* + \delta + 0) = \beta^k w(t^* + \delta - 0). \]

Usually, \( k = 1 \), but sometimes it is necessary to send several congestion signals in order to reduce the sending rate below the transmission capacity of the bottleneck router. In TCP New Reno version, \( \beta = 1/2 \).

If all the parameters \( T, \mu, m, B \) are fixed, the system exhibits a cyclic behavior: after the time moment \( t^* + \delta + 0 \) the sending rate \( \lambda \) becomes below \( \mu \), and the queue length decreases. But, because of the additive increase of the congestion window \( w \), sooner or later the sending rate becomes bigger than \( \mu \), the variable \( x(t) \) starts to increase, reaches the value of \( B \), and, after delay \( \delta \), the new cycle begins. The average buffer occupancy equals
\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t x(s) ds, \]
and the average goodput equals
\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t g(s) ds, \]
where \( g(t) = \begin{cases} \lambda(t), & \text{if } x(t) < B, \\ \mu, & \text{if } x(t) = B. \end{cases} \)

When the buffer size \( B \) increases, the both performance criteria increase. The Pareto set is presented on Fig.1. On the same picture, the simulation results with the help of NS-2 are given. One can see that the theoretical curve is a good approximation to the real trade-off ‘buffer occupancy against goodput’.

![Figure 1: Pareto set: numeric calculations and NS-2 simulations.](image)

More details can be found in [25].

IX. Conclusion

In the current report, some modern trends in the optimal control of jump stochastic processes are presented. Nowadays, it is not a problem to study processes with unbounded transition rate. Perhaps the next big step will be the theory of stochastically discontinuous processes, when, at some time moments jumps can occur with positive probability. A very interesting and rich branch is ‘piece-wise deterministic processes’, but it is beyond the
scope of the current work. As for approximations, one can develop fluid models and estimate their accuracy for the discounted case, for long-run average loss, for optimal stopping etc. The next order approximation (‘diffusion approximation’) is also rapidly developing, but again we were not able to mention it here. Finally, a very important area is numerical methods, especially for solving multiple-objective (e.g. constrained) problems.

References