On Lowen's fuzzy compact spaces

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Abstract- In this paper, we obtain an axiomatic characterization of Lowen's fuzzy compactness.

Keywords- Mathematics, fuzzy sets, Topology, Lowen's compactness, operators.

I. INTRODUCTION

There exist various notions on compactness in fuzzy topological spaces (see the books by Liu and Luo [1] or Palaniappan [5]).

This fact is due to problems as the conservation of compactness fuzzy notions by products, or the other problem on some kinds of fuzzy compactness which are good extensions of compactness and other are not. The concept of fuzzy compactness defined by R. Lowen is a good extension of compactness and this notions is productive.

On the other hand there exists an interesting characterization of compact Hausdorff topological spaces (due to De Groot) using five axioms.

In this paper, we obtain an axiomatic characterization of Lowen's fuzzy compactness.

II. BASIC DEFINITIONS

We use fuzzy topological spaces in Lowen's sense [3].

First, we present some basic definitions:

Definition 1. I denotes the unit interval, and I_r is I equipped with the topology $T_r = \{]\alpha, 1] \mid \alpha \in I \} \cup \{ I \}$.

Definition 2. [2] If δ is a fuzzy topology on X then $\iota(\delta)$ is the initial topology on X for the family of functions δ and the topological space I_r .

Definition 3. [2] Let **Top** denote the category of topological spaces and **Fuz** the category of fuzzy topological spaces. We define the functor $\omega: \mathbf{Top} \rightarrow \mathbf{Fuz}$ which to each object (X, T) associates $(X, \omega(T))$ where $\omega(T)$ is the fuzzy topology consisting of all continuous functions from (X, T) to I_r . A fuzzy topological space $(X, \omega(T))$ is called topologically generated.

Definition 4. When introducing a generalization, in **Fuz**, of a notion in **Top** it is natural to request of the generalization that when restricted to **Top** it should give the original notion. Such an extension is called a *good extension* ([4]).

Definition 5. [3] A fuzzy set ν is fuzzy compact if for all family β of open fuzzy sets such that $\sup_{\mu \in \beta} \mu \geq \nu$ and for all $\varepsilon > 0$, there exists a finite subfamily $\beta_0 \subset \beta$ such that $\sup_{\mu \in \beta_0} \mu \geq \nu - \varepsilon$.

Definition 6. [3] The fuzzy topological space (X, δ) is fuzzy compact if each constant fuzzy set in (X, δ) is fuzzy compact.

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III. RESULTS

Theorem. Let Γ be a class of fuzzy topological spaces which satisfies the following conditions:

- (a) The topological product of any family of members of Γ is a member of Γ .
- (b) Every closed fuzzy subspace of a member of Γ is a member of Γ .
- (c) If $X \in \Gamma$ and $Y \in \Gamma$ and if X is a fuzzy subspace of Y , then X is a closed fuzzy subspace of Y .
- (d) Every closed fuzzy continuous image of a member of Γ is a member of Γ .
- (e) The class Γ contains a fuzzy topological space consisting of more than one point.

Then, Γ is precisely the class of all fuzzy compact T_2 spaces.

Proof. Let

$\Gamma^* = \{ (X, T) \text{ fuzzy topological space} / (X, \omega(T)) \in \Gamma \}$, i.e.

$\Gamma^* = \{ (X, T) \mid T = \iota(\tau), (X, \tau) \in \Gamma \}$.

i) Let $\{ (X_j, T_j) \}_{j \in J} \subset \Gamma^*$ then

$T_j = \iota(\tau_j)$ for every $j \in J$ such that

$(X_j, \tau_j) \in \Gamma$. Then,

by (a), $(\prod X_j, \prod \tau_j) \in \Gamma$ and

$(\prod X_j, \iota(\prod \tau_j)) \in \Gamma^*$. Thus

$(\prod X_j, \prod \iota(\tau_j)) = (\prod X_j, \prod T_j) \in \Gamma^*$

ii) Let $Y \subset X$, such that Y is closed in (X, T) . Then Y is closed fuzzy in

$(X, \tau) \in \Gamma$. From (b), the fuzzy subspace

$(Y, \tau|_Y) \in \Gamma$, and $(Y, \iota(\tau|_Y)) \in \Gamma^*$, i.e.

$(Y, \iota(\tau)|_Y) = (Y, T|_Y) \in \Gamma^*$.

iii) Let $(X, T), (Y, S) \in \Gamma^*$ such that X is a subspace of Y . Then, X is a fuzzy subspace of $(Y, \omega(S)) \in \Gamma$, and, by (c), X is a closed fuzzy of $(Y, \omega(S))$. Thus X is closed in (Y, S) .

iv) Let an onto closed continuous map

$f : (X, T) \rightarrow (Y, S)$, such that

$(X, T) \in \Gamma^*$. Then, if $\tau = \omega(T)$, and

$\varsigma = \omega(S)$, we have an onto closed continuous

fuzzy map $f : (X, \tau) \rightarrow (Y, \varsigma)$, with

$(X, \tau) \in \Gamma$. By (d), $(Y, \varsigma) \in \Gamma$. Then

$(Y, S) \in \Gamma^*$.

v) From (e) there exists a space $(X, \tau) \in \Gamma$ such that X has of more than one point. Then $(X, \iota(\tau)) \in \Gamma^*$ and X has more than one point.

From a Theorem of de Groot (see [6]) it follows that Γ^* is precisely the class of all compact Hausdorff topological spaces. Thus Γ is the class of all compact Hausdorff fuzzy topological spaces.

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