A novel fuzzy measure and its extensional signed fuzzy measure

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Abstract: In this paper, a novel fuzzy measure, rho-measure, was proposed, and its extensional signed fuzzy measure satisfying Liu’s revised monotonicity, called signed rho-measure, was also proposed, some relative properties were discussed.

Key-Words: - Fuzzy density function, Fuzzy measure, Signed fuzzy measure, revised monotone, Rho-measure

1 Introduction

When there are interactions among the independent variables, the performances of the most often used multiple regression models and ridge regression model are not well enough. In contrast, the Choquet integral takes into account the interactions among independent variables, the discrete Choquet integral regression models based on some non-negative valued fuzzy measures or monotonic measure can be considered for using to overcome the undesirable phenomenon of dependence[1-2,6-9]. Therefore, the most important key issue in the application of Choquet integral is how to decide an adequate fuzzy measure with the determination of fuzzy density function

If the interactions among the independent variables are not homogeneous, the correlation values of some independent variables are negative, in other words, the fuzzy density function is no more a non-negative valued function, and Choquet integral regression models based on some signed fuzzy measures are needed.

In this study, a novel fuzzy measure, ρ-measure, is proposed, and its extensional signed fuzzy measure satisfying Liu’s revised monotonicity [4-5], called signed ρ-measure, is also proposed, some relative properties were discussed.

This paper is outlined as follows: Section 2 reviews fuzzy measures and normalized fuzzy measures. Signed fuzzy measures and revised monotonicity are introduced in section 3. The new fuzzy measures, ρ-measure, signed ρ-measure, and some theorems of them are provided in Section 4. A artificial examples of ρ-measure and signed ρ-measure are provided in Section 5. Finally, conclusions are summarized in Section 6.

2 Fuzzy Measures

The fuzzy measures, and normalized fuzzy measures are concise introduced as follows.

2.1 Fuzzy Measures [1-2,6-9]

Definition 1: fuzzy measure, monotone measure
A fuzzy measure or monotone measure, $\mu$-measure, on a finite set $X$, if its measure function $g_\mu : 2^X \to [0, \infty)$ satisfying the following axioms:

1) $g_\mu(\emptyset) = 0$ (vanishing at the empty set)  \hspace{0.5cm} (1)
2) $A \subseteq B \Rightarrow g_\mu(A) \leq g_\mu(B)$ (monotonicity)  \hspace{0.5cm} (2)

2.2 Normalized Fuzzy Measures [1, 7-9]

**Definition 2:** Normalized fuzzy measure, Normalized monotone measure

A fuzzy measure or monotone measure, $\mu$-measure, on a finite set $X$ is called a normalized fuzzy measure or normalized monotone measure, if its measure function $g_\mu : 2^X \to [0, 1]$ satisfies the normalized condition:

$$g_\mu(X) = 1$$  \hspace{0.5cm} (3)

2.3 Fuzzy density function [7-9]

**Definition 3:** Fuzzy density function

A fuzzy density function, or singleton measure, of a fuzzy measure $\mu$ on a finite set $X$ is a function $d(X) \to [0, 1]$ satisfying:

$$d(x) = g_\mu(\{x\}), x \in X$$

$d(x)$ is called the fuzzy density of singleton $x$.

If $\sum_{x \in X} d(x) = 1$, then the fuzzy density function is called normal density function.

2.4 Classifications of the fuzzy measures

**Definition 4:**

Let $\mu$ be a fuzzy measure on a finite set $X$, $\forall A, B \in X, A \cap B = \emptyset$
1) If $g_\mu(A \cup B) \leq g_\mu(A) + g_\mu(B)$, then $\mu$ is called a sub-additive fuzzy measure
2) If $g_\mu(A \cup B) = g_\mu(A) + g_\mu(B)$, then $\mu$ is called an additive fuzzy measure
3) If $g_\mu(A \cup B) \geq g_\mu(A) + g_\mu(B)$, then $\mu$ is called a super-additive fuzzy measure

Note that; For any given fuzzy density function $d(x)$ on a finite set $X = \{x_1, x_2, \ldots, x_n\}$, the additive measure, $\alpha$-measure, can easily be obtained by following measure function $g_\alpha : 2^X \to [0, 1]$ satisfying

$$g_\alpha(A) = \sum_{x \in A} d(x), \forall A \subseteq X$$  \hspace{0.5cm} (5)

2.5 $\lambda$-measure [2]

For any given fuzzy density function on a finite set $X = \{x_1, x_2, \ldots, x_n\}$, to construct a non-additive fuzzy measure on $X$, we need consider to specify $2^n - n - 1$ coefficients satisfying $n^{2^n - 1}$ monotone conditions, it is not an easy thing, $\lambda$-measure is the well-known one, it was proposed by Sugeno in 1974 [2], its formal definition is list as follows;

**Definition 5:** $\lambda$-measure [2]

For a given fuzzy density function $d(x)$ on a finite set $X$, a $\lambda$-measure, $g_\lambda$, is a fuzzy measure on $X$, its measure function satisfying:

1) $g_\lambda(\emptyset) = 0$ (vanishing at the empty set)
2) $A, B \in 2^X, A \cap B = \emptyset, A \cup B \neq X$
$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A)g_\lambda(B)$$  \hspace{0.5cm} (6)
3) $\prod_{i=1}^{n} [1 + \lambda d(x_i)] = \lambda + 1 > 0, d(x_i) = g_\lambda(\{x_i\})$  \hspace{0.5cm} (7)

Note that $\lambda$-measure has a unique solution without closed form.

Sugeno (1974) proposed an important theorem as follows;

**Property 1**

Let $d(x)$ be the fuzzy density function $s(x)$ of $\lambda$-measure on a given finite set $X$
1) If $\sum_{i \in X} d(x) > 1$, then $\lambda$-measure is a sub-additive fuzzy measure
2) If $\sum_{i \in X} d(x) = 1$, then $\lambda$-measure is just an additive fuzzy measure
3) If $\sum_{i \in X} d(x) < 1$, then $\lambda$-measure is a supper-additive fuzzy measure

3 Signed Fuzzy Measures

A generalization of fuzzy measures by allowing that the measure function can take negative values,
leads to the notion of signed measures for using to non-homogeneous interactions among independent variables.

The signed fuzzy measure was introduced by B. Jiao [3], investigated and modified by X. Liu [4], E. Pap emphasized the revised monotonicity of Signed fuzzy measure [5].

3.1 Signed Fuzzy Measures

Definition 5: Signed fuzzy measure [5-6]

A signed fuzzy measure, \( \mu \)-measure, on a finite set \( X \), if its measure function \( g_\mu : 2^X \to (-\infty, \infty) \) satisfying the following axioms:

1) \( g_\mu (\emptyset) = 0 \) (vanishing at the empty set)
2) \( \forall A, B \subseteq X, A \cap B = \emptyset \) (revised monotonicity)
   i) \( g_\mu (A) \cdot g_\mu (B) \geq 0, g_\mu (A) \lor g_\mu (B) > 0 \)
   \[\Rightarrow g_\mu (A \cup B) \geq g_\mu (A) \lor g_\mu (B)\] (8)
   ii) \( g_\mu (A), g_\mu (B) \leq 0, g_\mu (A) \land g_\mu (B) < 0 \)
   \[\Rightarrow g_\mu (A \cup B) \leq g_\mu (A) \land g_\mu (B)\] (9)
   iii) \( g_\mu (A) < 0 < g_\mu (B) \Rightarrow g_\mu (A) \leq g_\mu (A \cup B) \leq g_\mu (B)\) (10)

4 Signed \( \rho \)-fuzzy measure

Before to construct any signed fuzzy measure, we need consider the signed fuzzy density function first, the signed fuzzy density function and normalized signed fuzzy density function are introduced as bellows;

4.1 Signed Fuzzy Density Function

Definition 6: Signed fuzzy density function

A signed fuzzy density function of a signed fuzzy measure \( \mu \) on a finite set \( X \) is a function \( d : X \to (-\infty, \infty) \) satisfying:

1) \( d(x) = g_\mu \left( \{x\} \right), x \in X \) (11)
2) \( |X| = n = n_+ + n_- \)
   \[\text{Where } n, n_+ \in N, n_- \in N \setminus \{0\}\] (12)
3) \( d(x) = \begin{cases} d(x_i^+) > 0, x = x_i^+, i = 1, 2, \ldots, n_+ \\ d(x_j^-) < 0, x = x_j^-, j = 1, 2, \ldots, n_- \end{cases}\) (13)

\( d(x) \) is called the signed fuzzy density of singleton \( x \).

Definition 7: Signed normalized fuzzy density function

A signed normalized fuzzy density function of a signed fuzzy measure \( \mu \) on a finite set \( X \) is a function \( d : X \to [-1,1] \) satisfying:

\[d(x) = g_\mu \left( \{x\} \right), x \in X\] (14)

For any given signed fuzzy density function \( d : X \to [-1,1] \) on a finite set \( X = \{x_1, x_2, \ldots, x_n\} \), To construct a signed fuzzy measure on \( X \), we need consider to specify coefficients satisfying \( n2^{n-1} + m2^{n-1} + mn2^{n+m-2} \) revised monotone conditions, it is more difficulty than the case of fuzzy measure. A novel fuzzy measure, called \( \rho \)-measure and its signed fuzzy measure, called signed \( \rho \)-measure, are proposed as follows;

4.2 \( \rho \)-measure

Definition 8: \( \rho \)-measure

A set function \( g_\rho : 2^X \to [0, \infty) \) is called \( \rho \)-measure, if it satisfies:

1) \( g_\rho (\emptyset) = 0 \) (vanishing at the empty set)
2) \( \forall A, B \in 2^X, A \cap B = \emptyset, A \cup B \neq X \)
   \[\Rightarrow g_\rho (A \cup B) = g_\rho (A) + g_\rho (B) - g_\rho (A)g_\rho (B)\] (15)

Theorem 1 \( \rho \)-measure is a fuzzy measure

Proof:

Support \( \forall A, B \in 2^X, A \subseteq B \) to prove \( g_\rho (A) \leq g_\rho (B) \) (16)

Let \( B = A \cup (B - A), A \cap (B - A) = \emptyset \) (17)

Then

\[g_\rho (B) = g_\rho (A \cup (B - A)) = g_\rho (A) + g_\rho (B - A) - g_\rho (A)g_\rho (B - A)\]
\[= g_\rho (A) + g_\rho (B - A) \geq g_\rho (A)\geq g_\rho (A)\]
\[\Rightarrow g_\rho (A) \leq g_\rho (B)\]

Hence \( \rho \)-measure is a fuzzy measure.

4.3 Signed \( \rho \)-measure

\( \rho \)-measure can easily be extended to its signed \( \rho \)-measure as follows;
Definition 9: Signed $\rho$-measure

For a given signed fuzzy density function $d : X \rightarrow [-1,1]$ on a finite set $X$, the set function $g_\rho : 2^X \rightarrow [-1,1]$ is called a signed $\rho$-measure, if it satisfies

1) $g_\rho(\emptyset) = 0$ (vanishing at the empty set)

2) $\forall A, B \in 2^X, A \cap B = \emptyset, A \cup B \neq X$

\[ g_\rho(A \cup B) = g_\rho(A) + g_\rho(B) - g_\rho(A)g_\rho(B) \quad (19) \]

3) $\forall A \in 2^X, g_\rho(A) = 1 - \prod_{x \in A} (1 - d(x)) \quad (20)$

Theorem 2 signed $\rho$-measure is well-defined

Proof:

Here we need only to prove that formula (19) is the necessary and sufficient conditions of formula (20) I) first to prove the necessary condition by using the mathematical induction, that is, let

$A = \{x_1, x_2, ..., x_n\} \in 2^X$, using the condition (19), to prove that $\forall A \in 2^X, g_\rho(A) = 1 - \prod_{x \in A} (1 - d(x))$

i) Let $m=2$ then $A = \{x_1, x_2\}$, we can obtain

\[ g_\rho(A) = g_\rho(\{x_1\}) + g_\rho(\{x_2\}) - g_\rho(\{x_1\})g_\rho(\{x_2\}) = 1 - \prod_{x \in A} (1 - d(x)) \]

\[ g_\rho(A) = 1 - \prod_{x \in A} (1 - d(x)) \quad (21) \]

ii) if $m=k$, $A = \{x_1, x_2, ..., x_k\} \in 2^X$ satisfying

\[ g_\rho(A) = 1 - \prod_{x \in A} (1 - d(x)) \]

that $A \cup \{x_{k+1}\} = A \cup \{x_{k+1}\} \in 2^X$ satisfying

\[ g_\rho(A) = 1 - \prod_{x \in A} (1 - d(x)) \]

Then

\[ g_\rho(A) = g_\rho(\{x_{k+1}\}) - g_\rho(A\cup\{x_{k+1}\}) + g_\rho(A\cup\{x_{k+1}\})g_\rho(\{x_{k+1}\}) \]

\[ = 1 - \prod_{x \in A} (1 - d(x)) \]

\[ = 1 - \prod_{x \in A} (1 - d(x_{k+1})) \]

\[ = 1 - \prod_{x \in A} (1 - d(x_{k+1})) \]

i) from i) and ii), it completed the proof

II) Second to prove to prove the sufficient condition, Since $g_\rho(A) = 1 - \prod_{x \in A} (1 - d(x)), \forall A \subset X$, then

\[ g_\rho(A) + g_\rho(B) - g_\rho(A)g_\rho(B) = \left[\prod_{x \in A} (1 - d(x))\right] + \left[\prod_{x \in B} (1 - d(x))\right] - \left[\prod_{x \in A} (1 - d(x))\right] \left[\prod_{x \in B} (1 - d(x))\right] \]

\[ = \prod_{x \in A} (1 - d(x)) \prod_{x \in B} (1 - d(x)) \quad (24) \]

\[ g_\rho(A) + g_\rho(B) - g_\rho(A)g_\rho(B) = g_\rho(A \cup B) \]

Note that the formula (19) provides a closed form representation of signed $\rho$-measure, it is more convenient for computation than formula (18)

Theorem 3 Signed $\rho$-measure is a signed fuzzy measure with closed form

Proof:

Here we need only to prove the revised monotonicity as follows;

Let $\forall A, B \subseteq X, A \cap B = \emptyset$

i) If $g_\rho(A), g_\rho(B) \geq 0, g_\rho(A) \vee g_\rho(B) > 0$

Since

\[ g_\rho(A \cup B) = g_\rho(A) + g_\rho(B) - g_\rho(A)g_\rho(B) \]

\[ = g_\rho(A) + g_\rho(B) \left[1 - g_\rho(A)\right] \geq g_\rho(A) \quad (25) \]

And

\[ g_\rho(A \cup B) = g_\rho(A) + g_\rho(B) - g_\rho(A)g_\rho(B) \]

\[ = g_\rho(A) \left[1 - g_\rho(B)\right] + g_\rho(B) \geq g_\rho(B) \quad (26) \]

Hence $g_\rho(A \cup B) \geq g_\rho(A) \vee g_\rho(B)$

ii) If $g_\rho(A), g_\rho(B) \leq 0, g_\rho(A) \wedge g_\rho(B) < 0$

Since

\[ g_\rho(A \cup B) = g_\rho(A) + g_\rho(B) - g_\rho(A)g_\rho(B) \]

\[ = g_\rho(A) + g_\rho(B) \left[1 - g_\rho(A)\right] \leq g_\rho(A) \quad (27) \]

And

\[ g_\rho(A \cup B) = g_\rho(A) + g_\rho(B) - g_\rho(A)g_\rho(B) \]

\[ = g_\rho(A) \left[1 - g_\rho(B)\right] + g_\rho(B) \leq g_\rho(B) \quad (28) \]

Hence $g_\rho(A \cup B) \leq g_\rho(A) \wedge g_\rho(B)$

iii) If $g_\rho(A) < 0 < g_\rho(B)$

Since
\[ g_\rho(A \cup B) = g_\rho(A) + g_\rho(B) - g_\rho(A)g_\rho(B) \]
\[ = g_\rho(A) + g_\rho(B)[1 - g_\rho(A)] \geq g_\rho(A) \]

And

\[ g_\rho(A \cup B) = g_\rho(A) + g_\rho(B) - g_\rho(A)g_\rho(B) \]
\[ = g_\rho(A)[1 - g_\rho(B)] + g_\rho(B) \leq g_\rho(B) \]

Hence \[ g_\rho(A) \leq g_\rho(A \cup B) \leq g_\rho(B) \]

5 Artificial examples

Example 1

\[ X = \{x_1, x_2, x_3, x_4\} \]
\[ d : X \rightarrow [0, 1], \]
\[ d(x_1) = 0.1, d(x_2) = 0.2, d(x_3) = 0.3, d(x_4) = 0.4, \]
\[ g_\rho : 2^X \rightarrow [0, 1], g_\rho(A) = 1 - \prod_{i \in A}^{\cap} d(x_i) \]
\[ g_\rho(\emptyset) = 0 \]
\[ g_\rho(\{x_1\}) = 0.1, g_\rho(\{x_2\}) = 0.2, g_\rho(\{x_3\}) = 0.3, g_\rho(\{x_4\}) = 0.4, \]
\[ g_\rho(\{x_1, x_2\}) = 0.28, g_\rho(\{x_1, x_3\}) = 0.37, g_\rho(\{x_1, x_4\}) = 0.46, \]
\[ g_\rho(\{x_2, x_3\}) = 0.44, g_\rho(\{x_2, x_4\}) = 0.52, g_\rho(\{x_3, x_4\}) = 0.58, \]
\[ g_\rho(\{x_1, x_2, x_3\}) = 0.496, g_\rho(\{x_1, x_2, x_4\}) = 0.558, \]
\[ g_\rho(\{x_1, x_3, x_4\}) = 0.622, g_\rho(\{x_2, x_3, x_4\}) = 0.664 \]
\[ g_\rho(\{x_1, x_2, x_3, x_4\}) = 1 - 0.35 + 0.05 - 0.024 = 0.696 \]

Example 2

\[ X = \{x_1, x_2, x_3\} \]
\[ d : X \rightarrow [0, 1], \]
\[ d(x_1) = 0.5, d(x_2) = 0.4, d(x_3) = 0.3, \]
\[ g_\rho : 2^X \rightarrow [0, 1], g_\rho(A) = 1 - \prod_{i \in A}^{\cap} d(x_i) \]
\[ g_\rho(\emptyset) = 0 \]
\[ g_\rho(\{x_1\}) = 0.5, g_\rho(\{x_2\}) = 0.4, g_\rho(\{x_3\}) = 0.3, \]
\[ g_\rho(\{x_1, x_2\}) = g_\rho(\{x_1\}) + g_\rho(\{x_2\}) - g_\rho(\{x_1\})g_\rho(\{x_2\}) = 0.7, \]
\[ g_\rho(\{x_1, x_3\}) = g_\rho(\{x_1\}) + g_\rho(\{x_3\}) - g_\rho(\{x_1\})g_\rho(\{x_3\}) = 0.35, \]
\[ g_\rho(\{x_2, x_3\}) = g_\rho(\{x_2\}) + g_\rho(\{x_3\}) - g_\rho(\{x_2\})g_\rho(\{x_3\}) = 0.22 \]
\[ g_\rho(\{x_1, x_2, x_3\}) = 0.61 \]

6 Conclusion

In this paper, a novel fuzzy measure, \( \rho \)-measure, was proposed, and its extensional signed fuzzy measure satisfying Liu’s revised monotonicity, signed \( \rho \)-measure, was also proposed, some relative properties were discussed, two artificial example of these two new measure are given, respectively.

In the future, we consider to use Choquet integral with respect to the proposed signed \( \rho \)-measure for multi-decision making problems.

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