Development of Wu-Ma Test of the Van Hiele Levels of Geometrical Thinking Based on Grey Relational Analysis

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Abstract: This study is going to use the Grey relational analysis (GRA) to analyze the Wu-Ma Test of the van Hiele levels of geometrical thought. There are five levels of the van Hiele’s geometric thought: “visual”, “descriptive”, “theoretical”, “formal logic”, and “the nature of logical laws.” This study is focus on the first three levels. The current results of this study is not only identify the easiest and the most difficult questions for students, but also displays that the proposed GRA provides a novel approach in handling education test. However, further studies may be conducted to explore new educational test models based on grey system theory.

Key-Words: - elementary school; geometric thought; Grey relational analysis (GRA); thinking level; van Hiele

1 Introduction

Geometry is one of the most important topics in mathematics [1-7]. Geometry curriculum is developed and designed according to the van Hiele model of geometric thought [1-3].

In 1957, the van Hiele model was developed by two Dutch mathematics educators, P. M. van Hiele, and his wife [8]. Several studies have been conducted to discover the implications of the theory for current K 12 geometry curricula, and to validate aspects of the van Hiele model [9-21]. Most of researchers focus on the geometry curricula of secondary school. To discover the implications of the van Hiele theory for elementary school students, however, it is also very important. The focus of this study is at the elementary level.

In the development of grey system theory, grey relational analysis (GRA) [21-26] has been one of the most effective analytical tools. In the recent years, many GRA models have been developed, those of which are well summarized in [24, 27]. The modeling based on GRA examines a rank of order of multiple objects with resemblance from an objective. GRA has been applied to a variety of applications, such as decision-making in economics, marketing research [27], medicine, computer science [22-27], system modeling, social science, geometry, chemistry, management, etc. [22-27]. Due to the effectiveness and robustness of GRA, this study proposes a new approach for analyzing the Wu-Ma Test of the first three van Hiele levels of geometrical thought based the modeling of GRA. The main objective of this study was as follows: The Grey relational analysis (GRA) is used to analyze the Wu-Ma Test of the first three van Hiele levels of geometrical thought.

2 Van Hiele Model

There are five levels of the van Hiele’s geometric thought: “visual”, “descriptive”, “theoretical”, “formal logic”, and “the nature of logical laws” [8]. These five levels have two different labels: Level 1 through Level 5 or Level 0 through Level 4. Researchers have not yet come to a conclusion of which one to use. In this study, these five levels
were called Level 1 through Level 5, and the focus of this study is on Level 1, 2, and 3. At the first level, students learned the geometry through visualization. “Figures are judged by their appearance. A child recognizes a rectangle by its form and a rectangle seems different to him than a square [8].” At this first level students identify and operate on shapes (e.g., squares, triangles, etc.) and other geometric parts (e.g., lines, angles, grids, etc.) according to their appearance [19-21]. At the second level, figures are bearers of their properties. That a figure is a rectangle means that it has four right angles, diagonals are equal, and opposite sides are equal. Figures are recognized by their properties. If one tells us that the figure drawn on a blackboard has four right angles, it is a rectangle even if the figure is drawn badly. But at this level properties are not yet ordered, so that a square is not necessarily identified as being a rectangle [8]. At the third level, Properties are ordered. They are deduced one from another: one property precedes or follows another property. At this level the intrinsic meaning of deduction is not understood by the students. The square is recognized as being a rectangle because at this level definitions of figure come into play [8].

2 Methods and Procedures

2.1 Participants
The participants were 2,848 elementary school students who were randomly selected from 23 elementary schools in 23 counties/cities in Taiwan. In this study, the 88 students of the 6th grades were selected as samples.

2.2 Instrument
The instrument used in this study, Wu-Ma’s Geometry Test (WMGT), was specifically designed for this project due to there were no suitable Chinese instruments available [19, 20, 21]. This instrument was designed base on van Hiele level descriptors and sample responses identified by Fuys, Geddes, and Tischler [12]. There are 25 multiple-choose questions of the first van Hiele level (Part 1); 20 multiple-choose questions of the second van Hiele level in the second (Part 2); and 25 multiple-choose questions of the third van Hiele level (Part 3). The test is focus on three basic geometric concepts: triangle, quadrilateral and circle.

2.3 Validity and Reliability
The attempt to validate the instrument (WGT) involved the critiques of a validating team. The members of this team included elementary school teachers, graduate students majored in mathematics education, and professors from Mathematics Education Departments at several pre-service teacher preparation institutes. The team members were given this instrument, and provide feedback regarding whether each test item was suitable or not. They also gave suggestions about how to make this test better.

In order to measure the reliability of the WGT, 289 elementary school students (Grades 1-6) were selected to take the WGT. These students were not participants in this study. The alpha reliability coefficient of the first part of WGT was .6754 (p < .001) using SPSS for Windows [19]-[21].

2.4 Procedure
The one-time WGT was given during April 2006. The class teachers of the participants administered the test in one mathematics class. The tests were graded by the project directors.

3 Grey Relational Analysis
The grey relational analysis is depicted in this section, starting with some essential definition. What follows are important theorems that involved in this model. The construction of GRA model is briefly described. More details are referred to the works of Yamaguchi et al. [32-33].

Definition 1. Let the set \( X \) be a vector space to apply grey relational analysis, and the vectors \( y, x \), are elements of \( X \). The inner product of \( x \) and \( y \) is defined as follows:
\[
\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \cos \theta
\]
Where \( n \in \mathbb{R}_+ \),
\[
x = (x_1, x_2, \ldots, x_n)^T
\]
\[
\|x\|_\xi = \sqrt{\sum_{i=1}^{n} x_i^\xi}
\]
The \( x \) is content with the vector space axiom. Eq. (1) is satisfied with the inner product axiom. Both axioms are in set theory [33].

Definition 2. The metric between two vectors \( x, y \) with the distinguish coefficient \( \xi \) is defined as follows:
\[
\|x - y\|_\xi = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^{\xi}}
\]
Where \( \xi \geq 1 \).

Eq. (4) is well known as Minkowski distance [34] or \( L_p \) norm [35]. The Euclidean distance is the special
Axiom 1. The $X$ is a norm space, as consisted with the following three properties.

1. $\|x\|_\xi \geq 0$
2. $\|a \cdot x\|_\xi = |a| \cdot \|x\|_\xi$, where $a \in R$
3. $\|x + y\|_\xi \leq \|x\|_\xi + \|y\|_\xi$

The third property in $L_p$ norm has been proved mathematically. Also, it has been proved that the norm space $X$ is Banach space in $PL$ norm.

There are two different approaches of metric calculations. One is the traditional GRA and the other is called “proposal GRA” in [33]. For the traditional GRA, metric is calculated at each attributes independent. On the other hand, proposal metric at two-dimensional space and $\xi = 2$, is the $\zeta$ root of a sum of difference at all attributes, and $\|x_1 - x_2\|_\xi = \|v_2 - x_1\|_\xi$. Traditional GRA’s metric is more suitable for time series data, while the proposal metric is capable to employ the data set without time series. In this paper, the proposal GRA is used.

Definition 3. The following two features that are able to extract from traditional GRA concept are describe as follows:

1. The metric between two sequences is calculated, and normalized the grey relational grade with distinguish coefficient.
2. Grey relational grade has the order relation of each sequence.

Definition 4. The $\Gamma$ is a grey relational space, such as $\Gamma \subset X \times X$. The current GRA is a process that transfers Banach space into the grey relational space, and is content with Def. 3. The former is described by

$$f : X \rightarrow \Gamma \quad \text{..................(5)}$$

Definition 5. The variables $x_0$ and $x_i$ are both $n$-dimensional vectors, such as $x_0, x_i \in X$, which is the replaced sequence in GRA. Note that $x_0$ is a reference vector, and $x_i$ is an inspected vector, where $i = 1,2,...,m$.

Definition 6. The grey relational grade $\gamma_{ij}$ is defined as a value obtained by grey relational analysis, which is given for the ordered pair $(X_i, X_j) \in \Gamma$.

Definition 7. The localized grey relational grade $\gamma_{0i}$ can be defined as follows:

$$\gamma_{0i} = \frac{\Delta_{\max} - \Delta_{0i}}{\Delta_{\max} - \Delta_{\min}} \quad \text{..................(6)}$$

where

$$\Delta_{0i} = \|x_0 - x_i\|_\xi$$
$$\Delta_{\max} = \max_{\forall i} \{\Delta_{0i}\}$$
$$\Delta_{\min} = \min_{\forall i} \{\Delta_{0i}\}$$

Theorem 1. The globalized grey relational grade $\gamma_{ij}$ can be represented as follows:

$$\gamma_{ij} = 1 - \frac{\Delta_{ij}}{\Delta_{\max}} \quad \text{..................(7)}$$

where $i, j = 1,2,...,m$.

$$\Delta_{ij} = \|x_i - x_j\|_\xi$$
$$\Delta_{\max} = \max_{\forall i} \max_{\forall j} \{\Delta_{ij}\}$$

Eq. (7) is equivalent Eq. (6), such as

$$\gamma_{ij} = \frac{\Delta_{\max} - \Delta_{ij}}{\Delta_{\max} - \Delta_{\min}} \quad \text{..................(8)}$$

In Eq. (8), $\Delta_{\min} = \Delta_{ii} = 0$ because $\Delta_{ii}$ becomes a oneself metric at $i = j$. Hence, Eq. (8) is described as follows:

$$\gamma_{ii} = \frac{\Delta_{\max} - \Delta_{ij}}{\Delta_{\max} - \Delta_{\min}} = 1 - \frac{\Delta_{ij}}{\Delta_{\max}} \quad \text{(Q.E.D.)}$$

Theorem 2. Localized grey relational grade has the following three properties:

1. Normality : $0 \leq \gamma_{0i} \leq 1(\gamma_{0i} \in [0,1])$
2. Isolation :
   $$\|x_0 - x_i\|_\xi = \Delta_{\max} \Leftrightarrow \gamma_{0i} = 0$$
3. Closeness :
   $$\|x_0 - x_i\|_\xi = \Delta_{\min} \Leftrightarrow \gamma_{0i} = 1$$

4 Data and Variables

This study focuses on the sample data of 88 students mentioned in the previous section. Based on the overview of prior literature [15-20], this study derives a broader coverage of significant variables for geometrical concepts, including WMGT questions as follows: Questions 1 to 25 (P1-P25) in Level 1, Questions 26 to 45 (P26-P45) in Level 2, and Questions 46 to 70 (P46-P70) in Level 3.
5 Results
The sample data of the above-mentioned 70 questions of WMGT are used to calculate the Minkowski distance of \( L_p \) norm. The grey relational grade (Gamma) values are computed using current GRA model in Table 1. The questions in this table are sorted by the average values of gamma.

From Table 1, based on the analysis of the GRA, Question 8 (identification on filled and hollow) is the easiest for students and Question 5 (convex and concave figures) is the most difficult one among these 25 questions of figures in the first van Hiele level. Next, in the second level, Question 30 is the easiest for students and Question 44 is the most difficult one among these 20 questions. Finally, Question 48 is the easiest for students and Question 50 is the most difficult one among these 25 questions of figures in the first van Hiele level.

Table 1 Gamma values for a part of the questions in the First Three van Hiele Levels, sorted by gamma

<table>
<thead>
<tr>
<th>Questions</th>
<th>Gamma</th>
<th>Questions</th>
<th>Gamma</th>
<th>Questions</th>
<th>Gamma</th>
</tr>
</thead>
<tbody>
<tr>
<td>P05</td>
<td>0</td>
<td>P44</td>
<td>0</td>
<td>P50</td>
<td>0</td>
</tr>
<tr>
<td>P17</td>
<td>0.159</td>
<td>P32</td>
<td>0.2698</td>
<td>P60</td>
<td>0.0127</td>
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<td>P35</td>
<td>0.3372</td>
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<td>0.3513</td>
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<td>0.0384</td>
</tr>
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<td>P42</td>
<td>0.3657</td>
<td>P57</td>
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<td>0.3804</td>
<td>P69</td>
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<td>P14</td>
<td>0.3884</td>
<td>P45</td>
<td>0.4262</td>
<td>P61</td>
<td>0.091</td>
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<tr>
<td>P18</td>
<td>0.4562</td>
<td>P38</td>
<td>0.4585</td>
<td>P68</td>
<td>0.1452</td>
</tr>
<tr>
<td>P02</td>
<td>0.5049</td>
<td>P39</td>
<td>0.5279</td>
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<td>0.159</td>
</tr>
<tr>
<td>P16</td>
<td>0.5305</td>
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<td>0.159</td>
</tr>
<tr>
<td>P03</td>
<td>0.5571</td>
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<td>0.5657</td>
<td>P67</td>
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</tr>
<tr>
<td>P21</td>
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<td>P33</td>
<td>0.7236</td>
<td>P49</td>
<td>0.1871</td>
</tr>
<tr>
<td>P25</td>
<td>0.6926</td>
<td>P37</td>
<td>0.8488</td>
<td>P64</td>
<td>0.1871</td>
</tr>
<tr>
<td>P15</td>
<td>0.7633</td>
<td>P43</td>
<td>0.8488</td>
<td>P66</td>
<td>0.1871</td>
</tr>
<tr>
<td>P22</td>
<td>0.7653</td>
<td>P27</td>
<td>0.8893</td>
<td>P70</td>
<td>0.2446</td>
</tr>
<tr>
<td>P23</td>
<td>0.7852</td>
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<td>0.8893</td>
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</tr>
<tr>
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<td>0.9374</td>
<td>P53</td>
<td>0.3353</td>
</tr>
<tr>
<td>P10</td>
<td>0.8762</td>
<td>P28</td>
<td>0.9374</td>
<td>P52</td>
<td>0.3827</td>
</tr>
<tr>
<td>P12</td>
<td>0.8762</td>
<td>P29</td>
<td>0.9374</td>
<td>P59</td>
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<tr>
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<td>1</td>
<td>P62</td>
<td>0.4828</td>
</tr>
<tr>
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<td>P36</td>
<td>0.5657</td>
<td>P65</td>
<td>0.5002</td>
</tr>
<tr>
<td>P24</td>
<td>0.9317</td>
<td>P46</td>
<td>0.5179</td>
<td>P49</td>
<td>0.1871</td>
</tr>
<tr>
<td>P06</td>
<td>0.9637</td>
<td>P51</td>
<td>0.6942</td>
<td>P56</td>
<td>0.159</td>
</tr>
<tr>
<td>P09</td>
<td>0.9637</td>
<td>P63</td>
<td>0.775</td>
<td></td>
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</tr>
<tr>
<td>P08</td>
<td>1</td>
<td>P48</td>
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<td></td>
</tr>
</tbody>
</table>

The graph of average values of gamma is shown in Fig. 1-3, in which each point represents a question studied. The diagram depicts a smooth distribution of gamma values from 0 to 1.

To compare the patterns of gamma values of different levels, values of gamma are plotted in Figs. 1-3. These curves are smoothly distributed and they have similar patterns. From the comparison of Figs. 1-3, it is obvious that the gamma values are satisfactory.

6 Conclusions
A new approach for the analysis of the WMGT of van Hiele levels based the modeling of Grey Relational Analysis (GRA) has been presented. Overall, the empirical results show that the WMGT is a good Test for measure the geometrical concepts as well as the proposed GRA model demonstrates satisfactory prediction accuracy. The current results show that the proposed GRA provides a novel approach in handling education test. However, further studies may be conducted to explore new educational test models based on grey system theory.

References:


