A Computer Optimization of a Set of EWMA Quality Control Charts

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Abstract: - Two main approaches are possible when several correlated variables must be monitored: one multivariate quality control chart or a set of univariate charts. This paper deals with the optimal design of a set of EWMA control charts to monitor the mean of several quality variables simultaneously. A specific Markov’s chain model has been developed to compute the ARL of a set of $p$ EWMA charts. An optimization is carried out using Genetic Algorithms in order to find the optimal parameters of the EWMA charts, implemented in friendly software. The result of the optimization are the values of the parameters of the EWMA charts that minimize the out-of-control ARL for a specified shift, while respecting the constraint of a specified in-control ARL.

Key-words: - Quality; control; SPC; joint EWMA; Optimization

1 Introduction

The simultaneous statistical control of several correlated quality variables has been widely studied [1]. Essentially, two main approaches can be employed: 1) To control each variable with one control chart. For example, a set of $X$ control charts, CUSUM or EWMA charts. 2) To employ a single multivariate control chart.

Multivariate control charts have the advantage of allowing setting easily the false-alarm rate. On the other hand, some disadvantages are found. The main disadvantage is that the use of multivariate charts is more complicated in comparison with univariate control charts. Another issue is the interpretation of the out-of-control signal. The multivariate chart signals, but there is no indication of the variable(s) that have changed. However, some research has been done to overcome this problem [2, 3, 4, 5, 6].

When comparing the performance of both schemes (multivariate vs. multiple) no clear winner is found. One scheme will be better than the other in some circumstances. For example, for a given shift in the process to be monitored a multivariate chart may be the best option. However, for a different shift a well designed set of univariate charts may outperform the equivalent multivariate chart. Therefore, there is interest in a good design of a set of univariate charts, because for some industries/processes this will be the best option.

A most usual measure of performance of a control charting procedure is the average run length (ARL). It is defined as the expected number of samples (points on the chart) until an out-of-control signal. The out-of-control ARL (ARL1) is the average number of samples until a true alarm; the in-control ARL (ARL0) is the average number of samples until a false alarm. A good design of a process control scheme is one that keeps the in-control ARL at an acceptable level and achieves small out-of-control ARLs for shifts that are considered relevant and should be quickly detected.

This paper deals with the design of a set of EWMA control charts for the statistical control of the mean vector of a set of correlated variables. The first work has been developing a Markov’s chain model to obtain the ARL of a set of EWMA control charts. When the model was finished, the optimization of the parameters
of two EWMA charts is posed. Genetic Algorithms has been employed to solve this difficult optimization problem. The objective is to find the values for the charts’ parameters that result at the same time in a specified ARL0 and in the minimum ARL1 for a specified shift in the mean vector. The algorithm has been implemented in friendly software available from the authors. In addition, a comparison of performance is carried out against the MEWMA control chart [7].

2 Problem Context and Hypotheses

In this study it is assumed that the p variables to be monitored simultaneously by p joint EWMA charts, X1, X2, ..., Xp, follow a p-variate Normal distribution. Let \( \mu_0 = (\mu_1, \mu_2, ..., \mu_p)^T \) be the in-control mean of the process, \( \mu_1, \mu_2, ..., \mu_p \) the in-control standard deviations of the variables, and \( p_{ij} \) the correlations between any two variables \( X_i \) and \( X_j \). The objectives are to compute the ARL of the set of EWMA charts that minimize the ARL1 for a specified shift in the vector of means and at the same time achieve a specified value for the ARL0. Only sustained shifts in the mean vector are considered, with no change in the variance-covariance matrix. Such a shift may be represented by the vector \( \delta = (\delta_1, \delta_2, ..., \delta_p)^T \), \( \delta \neq 0 \), with at least one of the \( \delta_i \)’s different from zero.

3 Multiple EWMA Charts

The univariate EWMA (Exponentially Weighted Moving Average) control chart was originally introduced by Roberts (1959) with the aim of detecting small shifts in the process mean faster than the traditional \( \bar{X} \) control chart. Lucas and Saccucci (1990) have analyzed the performance of the EWMA chart, showing its advantage over Shewhart’s charts in the detection of small to moderate changes in the process mean.

Let \( \bar{X}_t \), \( t = 1, 2, ..., \) be the averages, in successive samples of size \( n \), taken periodically from a process, of a quality variable \( X \) with in-control mean \( \mu_0 \) and constant standard deviation \( \sigma \). The particular case of \( n=1 \) (individual observations) is quite common with EWMA charts. The EWMA statistic is calculated as follows:

\[
Y_t = \lambda \bar{X}_t + (1-\lambda)Y_{t-1}, \quad \text{where} \quad Y_0 = \mu_0 \quad \text{and} \quad 0 < \lambda \leq 1
\]

To detect shifts in the mean of \( X \), the upper control limit (UCL) and lower control limit (LCL) of the EWMA chart are calculated as follows:

\[
UCL = \mu_0 + L\sigma \sqrt{\frac{\lambda}{n(\lambda-1)}}
\]

\[
LCL = \mu_0 - L\sigma \sqrt{\frac{\lambda}{n(\lambda-1)}}
\]

where \( L \) is a design parameter and \( \sqrt{\frac{\lambda}{1-\lambda}}(\sigma/\sqrt{n}) \) is the asymptotic standard deviation of the EWMA statistic \( Z_t \). Alternatively, control limits based on the exact standard deviation of \( Z_t \) may be used. This standard deviation is \( \sqrt{\frac{\lambda(2-\lambda)}{2-\lambda}(\sigma/\sqrt{n})} \), so the resulting limits are narrower for the first few samples. When \( Z_t > UCL \) or \( Z_t < LCL \), the process is declared out-of-control. Aparisi and García-Díaz (2004) provide a software to obtain the best couple \((\lambda, L)\) given the desired ARL0 and the shift for which the out-of-control ARL should be minimized.

It is possible to design a set of \( p \) univariate EWMA charts with a particular sample size for each variable for the simultaneous monitoring of \( p \) cross-correlated variables. In the mathematical development, we consider this most general case (which, of course, encompasses as particular instances the case of equal sample sizes and even the case of unit sample sizes) that is, the case of EWMA charts on individual observations. Let therefore \( n_i \) represent the sample size of variable \( X_i \) \((i=1, 2, ..., p)\) and \( \bar{X}_{i(t)} \) represent the average this variable in the \( t \)-th sample. Furthermore, for simplicity, we will consider that the joint EWMA charts are applied to the standardized variables in the form

\[
Z_i(t) = \frac{\bar{X}_{i(t)} - \mu_{0i}}{\sigma_i/\sqrt{n_i}}
\]

rather than to the \( \bar{X}_{i(t)} \) values directly. Since standardization is just a matter of scale, the results will be the same that in the case of non-standardized variables.

The \( i \)-th EWMA chart monitors the statistic:

\[
Y_i(t) = \lambda Z_i(t) + (1-\lambda)Y_i(t-1)
\]

where \( Z_i(t) \) is the value of \( Z_i \) calculated with the values from the \( t \)-th sample. The initial value \( Y_i(0) \) is set equal to zero, since this is the mean of \( Z_i \) when the process is in control.

Since \( Z_i \sim N(0, 1) \), the central lines of the EWMA charts for \( Y_i, i=1, 2, ..., p \), are all set equal to zero, and the control limits of the \( i \)-th chart are given by

\[
LCL_i = -L\sqrt{\frac{\lambda}{2-\lambda}}
\]

\[
UCL_i = +U\sqrt{\frac{\lambda}{2-\lambda}}
\]
Note that these limits are not necessarily symmetrical. Symmetric limits would be the particular case for the i-th chart when \( L_i = U_i \).

4 Optimization of the EWMA Control Charts Parameters.

The problem of finding the best scheme, i.e., the best parameters of the multiple EWMA control charts, can be posed as the following optimization problem:

Given:
- Number of variables to monitor (number of charts): \( p \)
- Sample size to be employed: \( n \)
- Desired in-control ARL: \( \text{ARL}_0 \)
- Shift that it is important to detect as soon as possible (for which the out-of-control ARL should be minimized): \( \delta^* \)

Find: Control Limits coefficients of the charts: \( U_1, L_1, ..., U_p, L_p \) (see equations (2) and (3))
- Smoothing constants of EWMA charts: \( \lambda_1, \lambda_2, ..., \lambda_p \)

Such that:
- Minimizes ARL(\( \delta = \delta^* \))

Subject to:
- ARL(\( \delta = 0 \)) = ARL0
- And possible control limits restrictions (described below).

Genetic Algorithms could theoretically solve this optimization problem for whatever number of variables, \( p \). However, the ARL calculations using the Markov’s chain model presented in this paper are prohibitively time-consuming. The optimization for \( p = 2 \) in a very fast computer needs about 30 hours. Therefore, given the current speed of computers we do not solve the optimization for larger values of \( p \), and the results shown in this paper are for the case \( p = 2 \). However, the model is ready and waiting for faster computers to solve the optimization for larger values of \( p \).

As mentioned before, in Section 2, three different forms have been considered of the optimization problem, in what concerns the parameters of the charts. They result in three different types of symmetry of the ARL surface. Specifically:

**Problem Form A:** The optimization is carried out forcing the control limits of each chart to be symmetric, and to be equal for all charts, i.e., the coefficients of equations (2) and (3) have all the same value \( \lambda_1 = \lambda_2 = \lambda_p \), resulting in \( L_i = U_1 = L_2 = U_2 \). The smoothing constants are forced to be equal as well, i.e., \( \lambda_1 = \lambda_2 \). This leads to a large ARL symmetry in the plane of possible shifts. Namely, the ARL surface is symmetric with respect to the origin and also with respect to the main axes of the correlation structure. An important implication in practical terms is that the ARL for a shift of absolute size of \( \delta \) standard deviations in the mean of one of the variables alone does not depend on the sense (positive or negative) of the shift or on the variable that shifted either. So, Form A is the most indicated when shifts can be expected and in any one of the variables alone and in any sense (positive or negative) it is important to detect them quickly.

**Problem Form B:** The optimization is carried out forcing the control limits of each chart to be symmetric, and not necessarily equal between the charts, i.e., \( L_i = U_1 \) and \( L_2 = U_2 \) in expressions (2) and (3) respectively, which results in \( LCL_1 = -UCL_i \) and \( LCL_2 = -UCL_2 \). The smoothing constants are not necessarily equal either. This results in some ARL symmetry in the plane of possible shifts. Namely, the ARL surface is symmetric with respect to the origin (0, 0) but is no longer symmetric with respect to axes of the correlation structure. The ARL for a generic shift \( (\delta_1, \delta_2, ..., \delta_p) \) is the same as for the symmetric shift \( (-\delta_1, -\delta_2, ..., -\delta_p) \), but the ARL for a shift of size of \( \delta \) standard deviations in the mean of one of the variables alone, while remaining independent of the sense (positive or negative) of the shift, is no longer the same for different variables.

**Problem Form C:** In this form of the problem, all the control limits and smoothing constants can be different; no restriction is applied to their possible values. This will yield a very asymmetric ARL performance in the plane of possible shifts. We can imagine, as later results will show, that Form C is the one that will achieve, (with the same ARL0), the lowest out-of-control ARL for specified shifts. On the other hand, for shifts located in other areas of the plane of possible shifts it will give larger out-of-control ARLs in comparison with the ones given by Forms A and B. When the direction (and sense) of the typical shifts is known and unique, and/or the specifications are one-sided, however, this Form may be preferable. This issue will be discussed later.

This optimization problem is not an easy one. The out-of-control ARL has to be minimized, knowing that the ARL has not a closed form, as its values are computed employing a Markov chain approach.
5 Performance Comparison

We now compare the performance of the optimized couple of EWMA charts to the performance of the MEWMA control chart, for one direction of shift, three values of the Mahalanobis’ distance and $\rho=0.7$. Table 1 shows the results of this comparison.

The sizes of shift selected for the comparison are $d = 0.25$, 0.5 and 1 Mahalanobis’ units. However, there are infinite points in the plane of shifts that correspond to a given Mahalanobis’ distance. As it is well known (see, for instance, Aparisi et al., 2009), selecting a shift where there is only a change in one of the variables will favor the set of individual EWMA charts in the comparison, and choosing a shift against the axis of correlation of the variables (large Mahalanobis’ distance with small Euclidean distance) will favor the MEWMA chart. For these reasons, we have selected a shift for the comparison that has coordinates $(3x, x)$. That means that the shift in variable one is exactly three times the shift in variable two, and following the positive correlation between the variables. This is a shift that is not a priori favoring any scheme. The in-control ARL value ($\text{ARL}_0$ or still $\text{ARL}(d = 0)$) has been specified as 200. The software by Aparisi and García-Díaz [8] has been employed for optimizing the parameters of the MEWMA chart.

The conclusions are very similar for cases $d = 0.25$, 0.5 and 1. If the case $d = 0.5$ is analyzed, the MEWMA control chart has an out-of-control ARL of 26.66, — a value very similar to the ARL of the set of EWMA charts optimized in Form B (control limits different for each chart, although symmetric), which equals 26.12. Very close values are obtained between the ARLs of the set of EWMA in Form B and the MEWMA for the other Mahalanobis distances considered. The difference is only around 2% for the three cases.

Form C designs (in which all the control limits can be different) show always the lowest out-of-control ARL for the set of EWMA control charts. Form C improves the ARL of MEWMA chart in about 35% for $d = 0.25$, in about 17% for $d = 0.5$ and in about 8% for $d = 1$. As commented before, allowing all the control limits to be different yields this large improvement in the ARL. However, as it will be shown later, the performance for other areas of the plane of possible shifts is worsened.

The strict constraints of Problem Form A (symmetric control limits and same control limits for shifts of different magnitude) yield a worse performance of the multiple charts in comparison with the MEWMA control chart, for the shift for which the charts are optimized. The difference, however, is not large. The ARL values are between 12.5% and 14% higher than the MEWMA ARL values.

As a summary, optimization in the Form B for a given shift seems to produce designs with an ARL that shift very similar to the one of the MEWMA control chart. Optimization in Form C produce charts that clearly outperform the MEWMA chart for the given shift, and Form A charts (due to the high number of restrictions) have poorer performance than the MEWMA chart, although the loss in performance seems to be quite constant and only around 14%.
Fig 1. Software for optimizing the parameters of the EWMA charts.

Table 1. Results of the performance comparison.
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References: