On discrete-time state-space energy based system representation

M. Stork, J. Hrusak, D. Mayer

Abstract—This paper deals with stability and related structural properties of a relatively broad class of finite dimensional strictly causal continuous and discrete systems, which can be described in the state-space representation form. Dissipativity, instability, asymptotic stability as well as stability in the sense of Lyapunov is analyzed by a new approach based on an abstract state energy concept. The resulting energy metric function is induced by the output signal power and determines both, the structure of a digital system representation as well as the corresponding system state space topology. A special form of physically correct internal structure of an equivalent state space representation has been derived for discrete-time signals as a natural consequence of strict causality, signal energy conservation, dissipativity and state minimality requirements. New discretization approach of discrete systems based on energy preservation is presented. Results of simulation examples are shown for illustration of fundamental ideas and basic attributes of the proposed method.

Keywords—Continuous, discretization, dissipativity, energy, state-space.

I. INTRODUCTION

In this paper, a simple linear and nonlinear continuous and discrete systems are investigated from the energy point of view. As an alternative to the method of Liapunov functions a conceptually different approach can be based on the idea that, in fact abstract state space energy can be measured as distance from the system equilibrium to the actual state \( x(t) \), what is needed also for stability analysis. Thus, instead of the physical energy a metric \( \rho[ x(t), \bar{x} ] \) will be defined in a proper way, and for an abstract energy \( E(x) \) we then put formally (for continuous systems) \[1, 2, 3, 4\]. The energy is given by (1).

\[
E(x) = \frac{1}{2} \rho^2 \left[ x(t), \bar{x} \right] = \frac{1}{2} \sum_{i=1}^{n} \chi_i^2(t)
\]

where \( \bar{x} \) is an equilibrium state and \( \rho \) is scale. Within the state space paradigm the concept of an abstract energy seems to be one of the most natural means describing the internal system topology. A measure of distance of actual state from an equilibrium point or, more generally from an invariant set can be thought as a measure of energy accumulated in the state space of the given system. To avoid confusion an abstract system energy concept and the concept of signal power for both the continuous- and discrete-time system representations will be defined first.

We start with a natural assumption that every real signal must be generated by a realizable system. Let such a continuous system, be given in the form \[5, 6, 7\]:

\[
\mathcal{R}[\Sigma]: \quad \dot{x}(t) = A \cdot x(t) + B \cdot u(t), \quad x(t_0) = x^0, \quad y(t) = C \cdot x(t),
\]

In discrete-time case we proceed conceptually by exactly the same way as before. The signal generating system (SGS) is now represented by:

\[
\mathcal{R} \{\Sigma\}: \quad x(k+1) = A \cdot x(k) + B \cdot u(k), \quad x(k_0) = x^0, \quad y(k) = C \cdot x(k),
\]

and following definitions:

1. The immediate value of the state space energy

\[
E(x(k)) = \frac{1}{2} \|x(k)\|^2
\]

2. The immediate value of the output dissipative power

\[
P_o(k) = \gamma^2(k)
\]

3. The corresponding value of the system energy, accumulated in the state, be defined by

\[
\Delta E(x(k)) = P_i(k) - P_o(k)
\]

where for "zero input"

\[
\forall k \in \{0, 1, 2, \ldots\} : P_i = 0
\]

and the immediate change of state space energy is defined

\[
\Delta E(x(k)) = E(x(k+1)) - E(x(k))
\]

Putting \( u(k) = 0 \) \( \forall k \geq 0 \) and computing the difference of the energy function \( E(k) \) along any trajectory we get the signal power balance relation:
\[ \Delta E[x(k)]_{R(c)} = \frac{1}{2} \left[ x^T(k+1)x(k+1) - x^T(k)x(k) \right] \quad (9) \]

\[ = \frac{1}{2} x^T(k) [A'Ax(k) - x^T(k)] = \frac{1}{2} x^T(k) [A' - I] x(k) \]

From the energy conservation principle it follows that another special form of structurally dissipative state equivalent representation called discrete-time dissipation normal form must exist and can be specified by triplex matrix as follows

\[
A = \begin{bmatrix}
\Delta_1 & \delta_1 \Delta_2 & \delta_1 \delta_2 \Delta_3 & \ldots & \delta_1 \ldots \delta_{n-1} \Delta_n \\
-\Delta_2 & -\delta_2 \Delta_2 & -\delta_2 \delta_3 \Delta_3 & \ldots & -\delta_2 \ldots \delta_{n-1} \Delta_n \\
0 & \delta_2 & -\Delta_3 & -\delta_3 \Delta_3 & \ldots & -\delta_3 \ldots \delta_{n-1} \Delta_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -\Delta_{n-1} \Delta_n
\end{bmatrix}
\]

\[ B = [\beta_1, \beta_2, \ldots, \beta_n]' \quad (10) \]

\[ C = [\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \delta_n] \quad (11) \]

It is important to notice, that matrix \( A \) is Hessenberg type. A Hessenberg matrix is zero below the first subdiagonal. If the matrix is symmetric or Hermitian, the form is tridiagonal. This matrix has the same eigenvalues as the original, but less computation is needed to reveal them.

It is easy to show that the set of real basic (direct) design parameters \( \delta_i \) and the set of real complementary (feed-back) parameters \( \Delta_i \) must satisfy the following consistency conditions:

\[ 0 < \delta_i \leq 1, \quad \delta_i^2 + \Delta_i^2 = 1, \quad i \in \{1, 2, \ldots, n\}, \quad \delta_n = \gamma \quad (13) \]

having two important consequences:

1. Structural asymptotic stability

\[ \forall i, \quad i \in \{1, 2, \ldots, n\}: \quad |\Delta_i| < 1 \quad (14) \]

2. Structural minimality

\[ \forall i: 0 < \delta_i \leq 1, \quad \gamma \neq 0, \quad \beta_n \neq 0 \quad (15) \]

The derived structure of the discrete-time system representation in dissipation normal form corresponding to the Eqns. (10, 11, 12) is shown at the Fig. 1.

II. DISCRETIZATION METHODS

The integral approximation method for discretization of continuous-time linear system (2) is based on the assumption that the system input is constant during the given sampling period. The matrix of discrete system is computed by usually used principle

\[ A_D = e^{A_T} = L^{-1} \left\{ (sI - A_C)^{-1} \right\}_{s=T} \quad (16) \]

where \( T_s \) is sampling period and \( B \) and \( C \) are given by

\[ B_D = \left( \int_{\tau=0}^{T_s} e^{A_T} d\tau \right) B_C, \quad C_D = C_C \quad (17) \]

On the end of discretization, the matrix \( A_D \) eq. (16) must be transformed to \( A \) eq. (10) - which is Hessenberg type. The linear matrix transformation for this type of matrix is based on unitary matrix \( P \)

\[ A = P^{-1} A_D P = P^T A_D P \quad (18) \]

and

\[ B = P^T B_D, \quad C = C_D P \quad (19) \]

Unitary matrix \( P \) has orthonormal columns \( p_i = p_i^T \) and \( det(P) = 1 \).
III. SIMULATION RESULTS – LINEAR EXAMPLE

Let us consider a simple 4-th order linear continuous system in special form of a structurally dissipative state equivalent system representation called dissipation normal form, given by the following state-space equations

\[
\begin{align*}
\dot{x}_1(t) &= -\alpha_1 x_1(t) + \alpha_2 x_2(t) + u \\
\dot{x}_2(t) &= -\alpha_2 x_1(t) + \alpha_3 x_3(t) \\
\dot{x}_3(t) &= -\alpha_3 x_2(t) + \alpha_4 x_4(t) \\
\dot{x}_4(t) &= -\alpha_4 x_3(t) \\
y(t) &= x_4(t)
\end{align*}
\]

or in matrix form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
-0.3 & 0.4 & 0 & 0 \\
-0.4 & 0 & 0.2 & 0 \\
0 & -0.2 & 0 & 0.3 \\
0 & 0 & -0.3 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[C_x = [1 \ 0 \ 0 \ 0], \quad x(0) = 1\] (21)

The zero-pole map is shown in Fig. 2 and step response in Fig. 3.

![Pole-Zero Map](image)

**Fig. 2. Zero-pole map of continuous system (21).**

![Impulse Response](image)

**Fig. 3. Impulse response of continuous system (21).**

The state space energy of continuous system (21) is given in (22) and time evolution of energy is displayed in Fig. 4.

\[
E(x) = \frac{1}{2} \sum_{i=1}^{4} x_i^2(t)
\]

\[\text{Fig. 4. State space energy of continuous dissipative 4-th order system.}\]

The continuous system was discretized according (16) and (17) to obtain discrete version. After discretization (zero-order-hold type and sampling period \(T_s = 0.5\)) the discrete system is

\[
x(k + 1) = A_D x(k) + B_D u(k) \\
y(k) = C_D x(k)
\]

where rounded numerical values are

\[
A_D = \begin{bmatrix}
0.84 & 0.18 & 9e-3 & 5e-4 \\
-0.18 & 0.976 & 0.099 & 7e-3 \\
9e-3 & -0.099 & 0.983 & 0.149 \\
-5e-4 & 7e-3 & -0.149 & 0.988
\end{bmatrix} \\
B_D = [0.46 \ -0.047 \ 1.6e-3 \ -6e-5]^T
\]

\[C_D = [1 \ 0 \ 0 \ 0]\] (24)

and \(A_D\) (24) is transformed to Hessenberg form (25).

\[
A = \begin{bmatrix}
\Delta_1 & \delta_1 & \delta_2 & \delta_3 & \delta_4 \\
\delta_1 & -\Delta_1 & \delta_2 & \delta_3 & \delta_4 \\
0 & \delta_1 & -\Delta_1 & \delta_3 & \delta_4 \\
0 & 0 & \delta_1 & -\Delta_1 & \delta_4 \\
0 & 0 & 0 & \delta_1 & -\Delta_1
\end{bmatrix}
\]

Rounded numerical values after transformation of matrix \(A_D\) to \(A\) are

\[
A = \begin{bmatrix}
0.84 & -0.16 & -0.0157 & 2.4e-3 \\
0.18 & 0.976 & 0.097 & -0.0147 \\
0 & -0.1 & -0.984 & -0.149 \\
0 & 0 & 0.149 & 0.989
\end{bmatrix}
\]

From (26) and (26) is possible evaluate parameters \(\Delta_1 \div \Delta_4\).
and $\delta_i + \delta_e$. The simulation results of the discrete system (according eq. (3)) with initial conditions $x(0) = [1, 0, 0, 0]$ are displayed in following figures. The zoom of pole-zero map is shown in Fig. 5, impulse response in Fig. 6 and time evolution of state-space energy in Fig. 7, where energy is given by (4).

### IV. SIMULATION RESULTS – NONLINEAR EXAMPLE

The presented approach is possible used also in nonlinear systems. In next example, a simple nonlinear electrical circuit consisting of two coupled subsystems as depicted in the Fig.8 is investigated. The first subsystem is nonlinear antidissipative LC oscillator and the other is linear and dissipative. The negative impedance converter was used as a negative resistance element, thus giving rise to undamped oscillations in the first oscillator. Both the LC circuits are coupled by a nonlinear element $N$. The dead zone nonlinearity is used in this example (continuous version of this example is described in [8]). Dead zone nonlinear function is defined by eq. (27), where $p$ is dead zone parameter and $N$ is shown in Fig. 9.

$$N(x_1, x_2, p) = k_N \left[ (x_1 - x_2 - p) + \text{abs}(x_1 - x_2 - p) \right]$$
$$+ (x_1 - x_2 + p) - \text{abs}(x_1 - x_2 + p) \right] =$$
$$k_N \left[ 2 \times (x_1 - x_2) + \text{abs}(x_1 - x_2 - p)$$
$$- \text{abs}(x_1 - x_2 + p) \right] \quad (27)$$

The discrete system state equations are given in the form:

\[
x_i(k+1) = \Delta_i x_i(k) + \delta_i \Delta_i x_i(k) - N(x_1, x_2, p)
\]

\[
x_i(k+1) = \delta_i x_i(k) - \Delta_i \Delta_i x_i(k)
\]

\[
x_i(k+1) = \Delta_i x_i(k) + \delta_i \Delta_i x_i(k) - N(x_1, x_2, p)
\]

\[
x_i(k+1) = \delta_i x_i(k) - \Delta_i \Delta_i x_i(k)
\]

where $\Delta_i = -1.05 + x_i^2(k)/3.10^4$

![Fig. 5. Zero-pole map of discrete system-according (3).](image)

![Fig. 6. Impulse response of discrete system -according (3).](image)

![Fig. 7. State space energy of discrete dissipative 4-th order system-according (3).](image)

![Fig. 8. The circuit diagram of nonlinearly coupled oscillators.](image)

![Fig. 9. The dead zone nonlinear function.](image)

![Fig. 10. The state trajectory of $x_1$ and $x_2$.](image)
The state trajectories of \( x_1, x_2 \) and \( x_1, x_3 \) in the 2-D projection are shown in Fig. 10 and 11.

The time evolution of state variables of \( x_1 \) and \( x_3 \) are shown in Fig. 12. The energy of antidissipative oscillator, dissipative oscillator and total energy of both oscillators with nonlinear coupling is displayed in Fig. 13. Energy of first oscillator is given by:

\[
E(x_1, x_3) = 0.5(x_1^2 + x_3^2)
\]  
(29)

and energy of second oscillator:

\[
E(x_3, x_4) = 0.5(x_3^2 + x_4^2)
\]  
(30)

V. CONCLUSIONS

In the contribution a new unifying and constructive approach to linear and nonlinear discrete systems based on a metric - energy concept of the system state space, has been presented. Dissipativity, instability, asymptotic stability as well as stability in the sense of Lyapunov is analyzed by a new approach based on an abstract state energy concept. The resulting energy metric function is induced by the output signal power and determines both, the structure of a digital system representation as well as the corresponding system state space topology. A special form of physically correct internal structure of an equivalent state space representation has been derived for discrete-time signals as a natural consequence of strict causality, signal energy conservation, dissipativity and state minimality requirements. Results of simulation examples were shown for illustration of fundamental ideas and basic attributes of the proposed method.

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REFERENCES


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