Adaptive synchronization of chaotic systems with time changing parameters

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Abstract—This paper presents chaos synchronization between different chaotic and hyper-chaotic systems with uncertain and time variable parameters. Chaos synchronization of them by using adaptive control is achieved. Based on the Lyapunov stability theory an adaptive control law is derived such that the two different systems with unknown parameters are to be synchronized. Moreover the adaptive laws of parameters are developed. Two different chaotic systems and a hyper-chaotic system are employed as the examples Simulation results are also given to demonstrate the effectiveness of the proposed principles.

Keywords—Chaotic, adaptive, synchronization, parameters.

I. INTRODUCTION

SINCE Pecora and Carroll [1] introduced a method to synchronize two identical chaotic systems with different initial conditions, chaos synchronization have attracted a great deal of attention from various fields and has been studied extensively by many researchers during the last decades. Many effective synchronization schemes have been proposed since the time. However, most of these methods mainly concerned the synchronization of two identical systems with certain or identical uncertain parameters. But, in real-life applications, it is hardly the case that the structure and the dimension of master and slave chaotic or hyper-chaotic systems can be assumed to be identical. Moreover, the parameters of chaotic systems are perturbed by external inartificial factors and can not be known exactly in priori. Therefore, synchronization of two different chaotic and hyper-chaotic systems with uncertain parameters is more essential and realistic in actual applications.

Chaos synchronization is one of the critical issues in nonlinear science, for its various potential applications in physics, secure communications, chemical reactors, control theory, biological networks, and artificial neural networks, etc. Particularly, in recent neuroscience research, synchronization plays a very important role in the analysis of migraine and in the application of epilepsy.

More and more applications of chaos synchronization are also used in secure communication make it much more important to synchronize two different chaotic systems. In this paper, for chaotic and hyper-chaotic systems a unified mathematical approach is used. This method enables to adaptive synchronization of two chaotic and hyperchaotic systems with different structure, different dimension and uncertain parameters.

The paper is organized as follow. On the beginning (section II) the short theory is described. In section III the uncoupled systems are shown, section IV presents the synchronization of 2 uncertain systems. Section V presents the result of synchronization of systems with fixed uncertain parameters. In section VI the systems with time changing parameters are simulated. Conclusions are finally drawn in Section VII.

II. SYNCHRONIZATION PRINCIPLES

Consider the following system described by

\[ \dot{x} = f(x) + F(x)P \] (1)

where \( x \in \mathbb{R}^m \) is the state vector of the system, \( f: \mathbb{R}^m \rightarrow \mathbb{R}^m \) is a continuous vector function including nonlinear terms, and \( F: \mathbb{R}^m \rightarrow \mathbb{R}^{m \times k} \), and \( P \in \mathbb{R}^{k \times k} \) is the vector of system parameters. Eq. (1) is considered as the master system. The many well-known chaotic (hyperchaotic) systems belong to Eq. (1). Similarly, the slave system is given by

\[ \dot{y} = g(y) + G(y)Q + U \] (2)

where \( y \in \mathbb{R}^n \) is the state vector of the system, \( g: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous vector function, \( G: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times l} \), and \( Q \in \mathbb{R}^{l \times l} \) is the vector of system parameters. The purpose of chaos synchronization is to design a controller \( U (U \in \mathbb{R}^l) \), which is able to synchronize the state of the master system and the slave system.

When order \( n = m, k = l \) and the functions \( f = g, F = G \), the slave system is identical to the master system, and the complete synchronization problem has been well studied.

When two systems satisfy the condition \( n < m \) (of course \( f \neq g, F \neq G \)), that is, the order of the slave oscillator is lower than that of the master system, the synchronization is only attained in reduced order. Actually, reduced-order synchronization is the problem of controlling a slave system to be the projection of the master system. With a suitable controller, the reduced-order synchronization between two different systems can be achieved, i.e.,

\[ \lim_{t \to \infty} \| y - x \| = 0 \] (3)

Defining the error vector \( e = y - x \) (\( e \in \mathbb{R}^n \)), we subtract (1) from (2) and get
\[ \dot{e} = g(y) + G(y)Q + U - f(x) - F(x)P \]
\[ = h(e, x) + G(e, x)\hat{Q} - F(x)\hat{P} + U \]  

(4)

where \( h : R^n \times R^m \to R^n \) is a continuous vector function. In practical situations, the parameters belonging to the master and the slave system are always unknown. Therefore, by using the adaptive control and the parameters identification techniques [2], the controller can be decided as

\[ U = H(e, x) - G(e, x)\hat{Q} + F(x)\hat{P} \]

(5)

where \( H : R^n \times R^m \to R^n \), \( \hat{Q} \) and \( \hat{P} \) are the estimated vector of unknown parameters, and the updating laws of the estimated parameters are given by

\[
\begin{align*}
\dot{\hat{Q}} &= G^T(y)e^T = G^T(e, x)e^T \\
\dot{\hat{P}} &= -F^T(x)e^T
\end{align*}
\]

(6)

Assume a positive Lyapunov function

\[ V = \frac{1}{2}(e^T e + \hat{Q}^T \hat{Q} + \hat{P}^T \hat{P}) \]

(7)

where

\[ \hat{P} = \hat{P} - \tilde{P} \quad \text{and} \quad \hat{Q} = \hat{Q} - \tilde{Q} \]

(8)

With the choice of the updating laws above and reasonable control function \( H(e, x) \), the time rate of change of \( V \) along the solution in Eq. (4) will be smaller than zero. In other words, the error vector will approach to zero as time goes on and the states of the slave system and projected master system are synchronized asymptotically.

### III. UNCOUPLED CHAOTIC SYSTEMS

In this section, the 2 chaotic systems are described. Booth systems are simple three-dimensional chaotic systems [2, 3].

The equations of master system \( M \) are

\[
\begin{align*}
\dot{x}_m &= b_m + 0.5x_m z_m - 0.5a_m y_m z_m \\
\dot{y}_m &= 0.5a_m x_m z_m - c_m y_m \\
\dot{z}_m &= 1 - x_m^2 - y_m^2
\end{align*}
\]

(9)

and equations of slave system \( S \)

\[
\begin{align*}
\dot{x}_s &= -a_s x_s + a_s y_s + u_1 \\
\dot{y}_s &= x_s z_s - y_s + u_2 \\
\dot{z}_s &= b_s - x_s y_s - c_s z_s + u_3
\end{align*}
\]

(10)

For uncoupled systems the \( u_1 = u_2 = u_3 = 0 \) (in equation 10). The phase portrait of master system for \( a_m = 16; b_m = 1; c_m = -1 \) and initial condition \( [0.1 \ 0.1 \ 0.1] \) is shown in Fig. 1 and slave system for \( a_s = 4; b_s = 16; c_s = 1 \) and initial condition \( [0.1 \ 0.1 \ 0.1] \) in Fig. 2.

### IV. SYNCHRONIZED CHAOTIC SYSTEMS

In this section, synchronization between two different chaotic systems is achieved. To realize synchronization of the two systems, an effective controller will be designed via active control technique [4, 5]. We assume that M system drives the S system Subtracting system (10) from the system (9), we obtain the error dynamical system between the drive system and the response is given by

\[
\begin{align*}
\dot{e}_x &= a_s(e_y - e_z + y_m - x_m) - b_m \\
&\quad - 0.5z_m(a_m x_m + c_m y_m) + u_1 \\
\dot{e}_y &= e_y(e_z + z_m) + x_m(e_z + z_m) - e_y \\
&\quad - y_m - 0.5z_m(a_m x_m + c_m y_m) + u_2 \\
\dot{e}_z &= b_s - e_z(e_y + y_m) - x_m(e_y + y_m) \\
&\quad - c_s(e_z + z_m) + x_m^2 + y_m + u_3
\end{align*}
\]

(11)

in which errors are taken as \( e_x = x_m - x_m \), \( e_y = y_m - y_m \) and \( e_z = z_m - z_m \). Let \( a_s, b_s, c_s \) and \( a_m, b_m, c_m \) are unknown uncertain parameters, we can choose Lyapunov function for (11) as follows.
The controller is taken as

\[ u_i = -k_i e_i + 0.5(x_m - \hat{a}_m y_m)z_m - z_m e_y \]
\[ -\hat{a}_m(e_y - e_\gamma + y_m - x_m) + y_m e_z + \hat{b}_m \]
\[ u_z = -k_z e_z + e_z - x_m z_m + y_m \]
\[ +0.5\hat{a}_m x_m z_m - 0.5\hat{c}_m y_m z_m \]
\[ u_z = -k_z e_z + 1 + x_m y_m - x_m^2 - y_m^2 \]
\[ -\hat{b}_z + \hat{c}_z (e_z + z_m) \]

and the parameter estimate update law as follows

\[ \dot{\hat{a}}_m = 0.5z_m (e_x y_m - e_x x_m) \]
\[ \dot{\hat{b}}_m = -e_x \]
\[ \dot{\hat{c}}_m = 0.5y_m z_m e_y \]
\[ \dot{\hat{a}}_z = e_z (e_y + y_m) - e_z^2 - x_m e_z \]
\[ \dot{\hat{b}}_z = e_z \]
\[ \dot{\hat{c}}_z = -e_z^2 - e_z z_m \]

Because (15) is negative, then according to the Lyapunov stability theorem, the synchronization of chaotic systems (9) and (10) is achieved [6, 7].

\[ \frac{dV(e, \hat{P}, \hat{Q})}{dt} = \frac{1}{2}(e^T e + \hat{P}^2 + \hat{Q}^2) = \frac{1}{2}(e^T e + \hat{a}_z^2 + \hat{b}_z^2 + \hat{c}_z^2 + \hat{a}_m^2 + \hat{b}_m^2 + \hat{c}_m^2) \]

\[ (12) \]

V. FIXED ESTIMATION PARAMETERS SIMULATION RESULT

In this section, numerical simulations are given to verify the method proposed. The numerical simulations are carried out using MATLAB. Fourth order Runge-Kutta integration method is used to solve the systems of differential equations. In addition, a time step of 0.001 is employed.
We select the fixed parameters of master system as $a_m = 16; b_m = 1; c_m = -1$ and initial condition \([1 1 1]\) and fixed parameters of slave system as $a_s = 4; b_s = 16; c_s = 1$ and initial condition \([1 1 1]\). The phase portrait of synchronized slave system is shown in Fig. 3 and time evolution of state space variables of master and slave are displayed in Fig. 4, 5 and 6. In Fig. 4 the time evolution of synchronization errors $e_x, e_y$ and $e_z$ between systems (9) and (10) are shown. The initial values for estimated parameters $(\hat{\alpha}_m, \hat{\beta}_m, \hat{\gamma}_m, \hat{\delta}_s, \hat{\epsilon}_s, \hat{\zeta}_s)$ were set to 10 for all parameters. The time evolutions of estimated parameters of master and slave system are shown in Fig. 8 and 9. From the identification curves of the unknown parameters of the master and slave we can see that parameters identification of slave system is more than 4 times faster than master.

VI. TIME VARYING PARAMETERS ESIMATION

In this part the time one varying parameter of master, slave and booth, master and slave were tested. In first simulation only the one master parameter is periodically changing. This parameter is given by equation

$$a_m = 15 - 5\text{sign}(\sin(2\pi(t - 200)/300))$$  \hspace{1cm} (16)

Until the time $t<200$ parameter $a_m = 16$ and after, the $a_m$ is given by (16). Simulation results (parameter estimation of master and slave system) are shown in Fig. 10 and 11.

In the second simulation only the one slave parameter is periodically changing. This parameter is given by equation

$$b_s = 12 - 4\text{sign}(\sin(2\pi(t - 200)/200))$$  \hspace{1cm} (17)

Until the time $t<200$ parameter $b_s = 16$ and after, the $b_s$ is given by (17). Simulation results (parameter estimation of master and slave system) are shown in Fig. 12 and 13.
In the third simulation, both master parameter $a_m$ and slave parameter $b_s$ are changing according (16) and (17). Until the time $t<200$ parameter $a_m = 16$ and $b_s = 16$ and after, the $a_m$ is given by (16) and $b_s$ by (17). The estimation results and synchronization errors are illustrated in Fig. 14, 15 and 16. From this results can be seen different speed of synchronization and parameters estimation, but synchronization possibility between 2 different systems.

VII. CONCLUSIONS

In this paper, the problem of adaptive synchronization for a different types of chaotic systems with uncertain parameters was presented. The synchronization controller was developed and systems were simulated. Results can be used for system identification and secure communication [8].

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REFERENCES


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