Abstract: In this paper we present some properties concerning non-atomicity of regular null-additive fuzzy set multifunctions.

Key-words: fuzzy, null-additive, regular, atom, non-atomic.

1 Introduction

Since Zadeh [28] introduced fuzzy sets in 1965, the theory of fuzzyness began to develop thanks to its applications in probabilities (e.g. Dempster [3], Shafer [25]), computer and systems sciences, artificial intelligence (e.g. Mastorakis [17]), physics, biology, medicine (e.g. Pham, Brandl, Nguyen N.D. and Nguyen T.V. [23]).

In the last decades, many authors (e.g. Denneberg [4], Dinculeanu [5], Drewnowski [6], Narukawa [19], Olejček [20], Pap [21,22], Precupanu [24], Suzuki [26], Wu and Bo [27]) investigated the non-additive measure theory due to its applications in mathematical economics, statistics or theory of games (e.g. Aumann and Shapley [1]). In some of our previous articles ([2],[7-15],[18]) we introduced and studied different concepts (such as atom, pseudo-atom, regularity, Darboux property) in set-valued case.

In this paper we present some properties concerning non-atomicity of regular null-additive fuzzy set multifunctions defined on the Baire (or Borel) δ-ring of a Hausdorff locally compact space and taking values in \( \mathcal{P}_f(X) \), the family of nonvoid closed subsets of a real normed space \( X \).

2 Preliminary definitions and remarks

We now introduce the notations and several definitions used throughout the paper.

Let \( T \) be a Hausdorff locally compact space, \( C \) a ring of subsets of \( T \), \( B_0 \) the Baire \( \delta \)-ring generated by the \( G_\delta \)-compact subsets of \( T \) (that is, compact sets which are countable intersections of open sets) and \( B \) the Borel \( \delta \)-ring generated by the compact subsets of \( T \). It is well known that \( B_0 \subseteq B \) (see [5]).

\( X \) will be a real normed space, \( \mathcal{P}_0(X) \) the family of all nonvoid subsets of \( X \), \( \mathcal{P}_f(X) \) the family of all nonvoid closed subsets of \( X \), \( \mathcal{P}_{bf}(X) \) the family of all nonvoid closed bounded subsets of \( X \) and \( h \) the Hausdorff pseudometric on \( \mathcal{P}_0(X) \):

\[
h(M,N) = \max\{e(M,N), e(N,M)\},
\]

where \( e(M,N) = \sup_{x \in M} d(x,N) \), for every \( M,N \in \mathcal{P}_0(X) \) and \( d(x,N) \) is the distance from \( x \) to \( N \) with respect to the distance induced by the norm of \( X \).

Obviously, \( e(M,N) = 0 \) for every \( M,N \in \mathcal{P}_f(X) \) with \( M \subseteq N \). It is known that \( h \) becomes an extended metric on \( \mathcal{P}_f(X) \) (i.e. a metric that can also take the value \( +\infty \)) and \( h \) becomes a metric on \( \mathcal{P}_{bf}(X) \) ([16]).

We define \( |M| = h(M,\{0\}) \), for every \( M \in \mathcal{P}_0(X) \), where 0 is the origin of \( X \).

On \( \mathcal{P}_0(X) \) we consider the Minkowski addition
**Definition 2.1.** [2] Let \( \mu : C \to \mathcal{P}_0(X) \) be a set multifunction. \( \mu \) is said to be:

I) **monotone** if \( \mu(A) \subseteq \mu(B) \), for every \( A, B \in C \), with \( A \subseteq B \).

II) **fuzzy** if \( \mu \) is monotone and \( \mu(\emptyset) = \{0\} \).

III) **null-additive** if \( \mu(A \cup B) = \mu(A) \), for every \( A, B \in C \), with \( \mu(B) = \{0\} \).

IV) **null-null-additive** if \( \mu(A \cup B) = \{0\} \), for every \( A, B \in C \), with \( \mu(A) = \mu(B) = \{0\} \).

**Remark 2.2.** I) Any null-additive set multifunction is null-null-additive.

The converse is not valid. Indeed, let \( T = \{a, b\}, C = \mathcal{P}(T) \) and \( \mu : C \to \mathcal{P}_f(\mathbb{R}) \), defined by \( \mu(T) = [0, 2], \mu(\{a\}) = \mu(\emptyset) = \{0\} \) and \( \mu(\{b\}) = [0, 1] \).

Then it is null-null-additive, but it is not null-additive.

II) Let \( T = [0, +\infty), C = \mathcal{P}(T) \) and \( \mu : C \to \mathcal{P}_0(\mathbb{R}) \), defined by \( \mu(\emptyset) = \{0\}, \mu(A) = A \) if card\(A = 1 \), \( \mu(A) = [0, \delta(A)] \) if \( A \) is bounded with card\(A \geq 2 \) and \( \mu(A) = [0, +\infty] \) if \( A \) is not bounded (here, card\(A \) is the cardinal of \( A \) and \( \delta(A) = \sup\{|t-s|; t, s \in A\} \) is the diameter of \( A \)). Then \( \mu \) is null-additive and non-fuzzy.

**Definition 2.3.** [18] Let \( \mu : C \to \mathcal{P}_0(X) \) be a set multifunction.

I) A set \( A \in C \) is said to be an atom of \( \mu \) if \( \mu(A) \nsubseteq \{0\} \) and for every \( B \in C \), with \( B \subseteq A \), we have \( \mu(B) = \{0\} \) or \( \mu(A \setminus B) = \{0\} \).

II) \( \mu \) is said to be non-atomic if it has no atoms.

III) \( \mu \) is said to be semi-convex if for every \( A \in C \), there is a \( B \in C \), with \( B \subseteq A \), so that \( \mu(B) = \frac{1}{2} \mu(A) \).

**Example 2.4.** I) Let \( T = \{2n|n \in \mathbb{N}\}, C = \mathcal{P}(T) \) and the fuzzy set multifunction \( \mu : C \to \mathcal{P}_f(\mathbb{R}) \) defined by:

\[
\mu(A) = \begin{cases} 
\{0\}, & \text{if } A = \emptyset \\
\frac{1}{2} A \cup \{0\}, & \text{if } A \neq \emptyset 
\end{cases}
\]

were we denoted \( \frac{1}{2} A = \{\frac{1}{2} x; x \in A\} \).

If \( A \in C, A \neq \{0\} \) and card\(A = 1 \) or \( A = \{0, 2n\}, n \in \mathbb{N}^* \), then \( A \) is an atom of \( \mu \). If \( A \in C \), \( A \) has card\(A \geq 2 \) and there exist \( a, b, a \neq b \) in \( A \) such that \( a \neq b \) and \( ab \neq 0 \), then \( A \) is not an atom of \( \mu \).

II) Let \( T = [0, 3], C \) the Borel \( \sigma \)-algebra of \( T, \lambda : C \to [0, +\infty) \) the Lebesgue measure and the fuzzy set multifunction \( \mu : C \to \mathcal{P}_f(\mathbb{R}) \) defined, for every \( A \in C \), by:

\[
\mu(A) = \begin{cases} 
[-\lambda(A), \lambda(A)], & \text{if } \lambda(A) \leq 1 \\
[-\lambda(A), 1], & \text{if } \lambda(A) > 1.
\end{cases}
\]

Then \( \mu \) is non-atomic.

III) Let \( T = [0, +\infty), C \) the Borel \( \sigma \)-algebra of \( T, \lambda : C \to [0, +\infty) \) the Lebesgue measure and the fuzzy set multifunction \( \mu : C \to \mathcal{P}_f(\mathbb{R}) \) defined by \( \mu(A) = [0, \lambda(A)], \) for every \( A \in C \). Then \( \mu \) is semi-convex.

**Definition 2.5.** [10] Let \( A \in C \) be an arbitrary set and \( \mu : C \to \mathcal{P}_f(X) \) a set multifunction, with \( \mu(\emptyset) = \{0\} \).

I) \( A \) is said to be:

(i) **\( R \)-regular** with respect to \( \mu \) if for every \( \varepsilon > 0 \), there exist a compact set \( K \subseteq A, \ K \in C \) and an open set \( D \supseteq A, D \in C \) such that \( h(\mu(A), \mu(B)) < \varepsilon \), for every \( B \in C, K \subseteq B \subseteq D \).

(ii) **\( R_t \)-regular** with respect to \( \mu \) if for every \( \varepsilon > 0 \), there exists a compact set \( K \subseteq A, \ K \in C \) such that \( h(\mu(A), \mu(B)) < \varepsilon \), for every \( B \in C, K \subseteq B \subseteq A \).

(iii) **\( R_r \)-regular** with respect to \( \mu \) if for every \( \varepsilon > 0 \), there exists an open set \( D \supseteq A, D \in C \) such that \( h(\mu(A), \mu(B)) < \varepsilon \), for every \( B \in C, A \subseteq B \subseteq D \).

(iv) **\( R_t^r \)-regular** if for every \( \varepsilon > 0 \), there are a compact set \( K \subseteq C, K \subseteq A \) and an open set \( D \subseteq C, A \subseteq D \) so that \( |\mu(B)| < \varepsilon \), for every \( B \in C, B \subseteq D \setminus K \).

II) \( \mu \) is said to be **\( R \)-regular (\( R_t \)-regular, \( R_r \)-regular, \( R_t^r \)-regular, respectively)** if every \( A \in C \) is a **\( R \)-regular (\( R_t \)-regular, \( R_r \)-regular, \( R_t^r \)-regular, respectively)** set with respect to \( \mu \).

The reader is referred to [7], [10] for different implications among these types of regularities in the set-valued case.

**Definition 2.6.** Suppose \( C_1, C_2 \) are two rings so that \( C_1 \subseteq C_2 \) and let \( \mu : C_2 \to \mathcal{P}_f(X) \) be an arbitrary set multifunction.

We say that \( C_1 \) is dense in \( C_2 \) with respect to \( \mu \) if for every \( \varepsilon > 0 \) and every \( A \in C_2 \), there exists \( B \in C_1 \) so that \( B \subseteq A \) and \( |\mu(A \setminus B)| < \varepsilon \).

**Remark 2.7.** I) \( B_0 \) is dense in \( B \) with respect to a \( R_t^r \)-regular set multifunction \( \mu : B \to \mathcal{P}_f(X) \).

Indeed, for every \( \varepsilon > 0 \) and every \( A \in B \), there are a compact set \( K \subseteq B, K \subseteq A \) and an open set \( D \subseteq B, A \subseteq D \) so that \( |\mu(B)| < \varepsilon \), for every \( B \in B, B \subseteq D \setminus K \).
3 Null-additive fuzzy set multifunctions

In this section, we present some properties of regular null-additive fuzzy set multifunctions. By [9], one can obtain:

**Theorem 3.1.** Let $A \in \mathcal{B}$ with $\mu(A) \supseteq \{0\}$ and $\mu : \mathcal{B} \to \mathcal{P}(X)$ a $R_l$-regular null-additive fuzzy set multifunction. Then:

(i) $A$ is an atom of $\mu$ if and only if

$$\exists a \in A \text{ so that } \mu(A \setminus \{a\}) = \{0\};$$

(ii) $\mu$ is non-atomic if and only if for every $t \in T$, $\mu(\{t\}) = \{0\}$.

**Proof.** Suppose that, on the contrary, there exists an atom $B \in \mathcal{C}$ of $\mu$.

Because $\mu$ is $R_l$-regular, then, according to [10], it is $R_l$-regular.

Consequently, for $\varepsilon = |\mu(B)|$, there exists a compact set $K \subseteq B$ and $h(\mu(B), \mu(K)) < |\mu(B)|$.

We observe that $|\mu(K)| \supseteq \{0\}$.

Indeed, if $\mu(K) = \{0\}$, then $|\mu(B)| < |\mu(K)|$, which is false.

According to property (2), we have that for every $t \in K$, there exists $A_t \subseteq \mathcal{C}$ so that $t \in A_t$ and $e(\mu(B), \mu(A_t)) > 0$.

Because $\mu$ is $R_l$-regular, then, according to [10], it is $R_l$-regular.

In consequence, for every $t \in K$, for $A_t$ and $\varepsilon = e(\mu(B), \mu(A_t))$, there exists an open set $D_t \subseteq C$ so that $A_t \subseteq D_t$ and

$$e(\mu(D_t), \mu(A_t)) \leq h(\mu(D_t), \mu(A_t)) < e(\mu(B), \mu(A_t)).$$

Since $t \in A_t$ and $A_t \subseteq D_t$, then $K \subseteq \bigcup_{t \in K} D_t$.

Consequently, there exists $p \in \mathbb{N}^*$ such that $K \subseteq \bigcup_{i=1}^p D_{t_i}$, with $t_i \in K$, for every $i \in \{1, \ldots, p\}$.

This implies

$$\{0\} \subset \mu(K) = \left( \bigcup_{i=1}^p (D_{t_i} \cap K) \right).$$

One can easily check that there is $j \in \{1, \ldots, p\}$ such that $\mu(D_{t_j} \cap K) \supseteq \{0\}$.

Indeed, if $\mu(D_{t_i} \cap K) = \{0\}$, for every $i \in \{1, \ldots, p\}$, then, by the null-null-additivity of $\mu$, we have $\mu(K) = \left( \bigcup_{i=1}^p (D_{t_i} \cap K) \right) = \{0\}$, which is a contradiction.

Consequently, there is $j \in \{1, \ldots, p\}$ such that

$$\{0\} \subset \mu(D_{t_j} \cap K) \subseteq \mu(D_{t_j} \cap B).$$

Since $B$ is an atom of $\mu$, $\mu(B) \supseteq \{0\}$ and $\mu(D_{t_j} \cap B) \supseteq \{0\}$, then $\mu(B \setminus D_{t_j}) = \{0\}$, hence, the null-additivity of $\mu$ implies that $\mu(B) = \mu(D_{t_j} \cap B)$.

On the other hand, because $e(\mu(D_{t_j}), \mu(A_{t_j})) < e(\mu(B), \mu(A_{t_j}))$ and $e(\mu(D_{t_j} \cap B), \mu(D_{t_j})) = 0$, we have

$$e(\mu(B), \mu(A_{t_j})) = e(\mu(D_{t_j} \cap B), \mu(A_{t_j})) \leq e(\mu(D_{t_j} \cap B), \mu(D_{t_j})) + e(\mu(D_{t_j}), \mu(A_{t_j})) = e(\mu(D_{t_j}), \mu(A_{t_j})) < e(\mu(B), \mu(A_{t_j})),$$

which is a contradiction, so $\mu$ is non-atomic. □

**Theorem 3.3.** Let $\mu : \mathcal{B} \to \mathcal{P}(X)$ be a $R_l$-regular null-additive fuzzy set multifunction. Then $\mu$ is non-atomic if and only if

$$\forall B \in \mathcal{B}, \text{ with } \mu(B) \supseteq \{0\} \text{ and } \forall t \in T, \quad e(\mu(B), \mu(\{t\})) > 0.$$  

**Proof.** Obviously, for every $t \in T$, $\{t\} \in \mathcal{B}$.

The *Only if part* is a consequence of Theorem 3.2.

For the *If part*, suppose $\mu$ is non-atomic. Then, by Theorem 3.1, for every $t \in T$, $\mu(\{t\}) = \{0\}$, so $e(\mu(B), \mu(\{t\})) = |\mu(B)| > 0$.

**Proposition 3.4.** [12-18] Let $\mu : \mathcal{C} \to \mathcal{P}(X)$ be a semi-convex null-additive fuzzy set multifunction. Then $\mu$ is non-atomic.

**Proposition 3.5.** Suppose $\mu : \mathcal{B} \to \mathcal{P}(X)$ is a $R_l$-regular null-additive fuzzy set multifunction. If $\mu$ is non-atomic on $\mathcal{B}$, then $\mu$ is non-atomic on $\mathcal{B}_0$.

**Proof.** Suppose that, on the contrary, there exists an atom $A \in \mathcal{B}_0$ for $\mu/\mathcal{B}_0$. 

By [5], there is $K_0 \subseteq B_0$ so that $K \subseteq K_0 \subseteq D_0$.

Then $|\mu(A \setminus K_0)| < \varepsilon$, that is, $B_0$ is dense in $\mathcal{B}$ with respect to $\mu$.

II) If $\mu : \mathcal{C} \to \mathcal{P}(X)$ is a fuzzy set multifunction, then $\mu$ is non-atomic if and only if for every $A \in \mathcal{C}$, with $\mu(A) \supseteq \{0\}$, there exists $B \in \mathcal{C}$, with $B \subseteq A$, $\mu(B) \supseteq \{0\}$ and $\mu(A \setminus B) \supseteq \{0\}$.
Then \( \mu(A) \supseteq \{0\} \) and for every \( B \in \mathcal{B}_0 \) with \( B \subseteq A \) we have that \( \mu(B) = \{0\} \) or \( \mu(A \setminus B) = \{0\} \).

Because \( A \in \mathcal{B} \), \( \mu(A) \supseteq \{0\} \) and \( \mu \) is non-atomic on \( \mathcal{B} \), there is \( B_0 \in \mathcal{B} \) so that \( B_0 \subseteq A \), \( \mu(B_0) \supseteq \{0\} \) and \( \mu(A \setminus B_0) \supseteq \{0\} \).

Then \( |\mu(B_0)| > 0 \) and, since \( B_0 \) is dense in \( \mathcal{B} \), then for \( \epsilon = |\mu(B_0)| \), there exists \( C_0 \in \mathcal{B}_0 \) so that \( C_0 \subseteq B_0 \) and \( |\mu(B_0 \setminus C_0)| < \epsilon \).

Now, because \( C_0 \in \mathcal{B}_0 \) and \( C_0 \subseteq A \), by the assumption we made we get that \( \mu(C_0) = \{0\} \) or \( \mu(A \setminus C_0) = \{0\} \).

If \( \mu(C_0) = \{0\} \), then
\[
|\mu(B_0)| = |\mu(B_0 \setminus C_0)| < |\mu(B_0)|,
\]
which is false.

Consequently, \( \mu(A \setminus C_0) = \{0\} \), so
\[
|\mu(A \setminus B_0)| \leq |\mu(A \setminus C_0)| = 0,
\]
which implies \( \mu(A \setminus B_0) = \{0\} \), which is again false. So, \( \mu \) is non-atomic on \( B_0 \).

\( \square \)

**Concluding remarks**

In this paper we have presented some properties regarding non-atomicity of fuzzy set multifunctions defined on the Baire (or Borel) \( \delta \)-ring \( \mathcal{B} \) of a Hausdorff locally compact space and taking values in \( \mathcal{P}_f(X) \), the family of nonvoid closed subsets of a real normed space \( X \).

Thus, if \( \mu : \mathcal{B} \rightarrow \mathcal{P}_f(X) \) is a \( R \)-regular null-additive fuzzy set multifunction, then non-atomicity of \( \mu \) is equivalent to the following condition:

for every \( B \in \mathcal{B} \) so that \( \mu(B) \supseteq \{0\} \) and every \( t \in T \), it holds \( e(\mu(B), \mu(\{t\})) > 0 \).

**References:**


