Convergence of Markov Process for Fuzzy Autocatalytic Set of Fuzzy Graph Type-3 of An Incineration Process

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Abstract: - Fuzzy Autocatalytic Set (FACS) of Fuzzy Graph of Type-3 incorporates the concept of fuzzy, graph and autocatalytic set. It was initially defined and used in the modeling of a clinical waste incineration process which has produced more accurate results than using crisp graph. As it is a newly developed theory, FACS seems to have great potentials in generating new mathematical theories and structures that would enrich the theory itself. With this in mind, we have embarked the study into the Markovian approach. This paper applies the theory of classical Markov chain to the evaluation of steady state of an incineration process. New definition for transition probability matrix of FACS is presented. Two methods for obtaining steady state vector of Markov chain for incineration process are implemented. Our study on FACS with regard to the system led to some relation of a Markov process and Perron-Frobenius Theorem for stochastic matrix.

Key-Words: - Fuzzy Autocatalytic Set, Fuzzy Graph, Incineration Process, Perron-Frobenius Theorem, Transition Probability Matrix, Markov process.

1 Introduction
The emergence of fuzzy graph to autocatalytic sets has instigated a new concept named Fuzzy Autocatalytic Set (FACS) [11], [14]. A clinical waste incineration process in Malacca [11] is one of the applications in real-life situation that has been modeled using this new concept specifically using fuzzy graph of type-3 (see [2] for more detail about fuzzy graph). Several new results of FACS which linked to Perron Frobenius Theorem have been discussed in previous studies [11],[12],[14],[17],[18]. In this paper, we focus on the study of FACS of an incineration process from a new perspective by exploring the theory of classical Markov chain in its relation to the evaluation of steady state vector. We begin with some theoretical background of FACS of clinical waste incineration process followed by preliminaries of random walk and Markov chain for directed walk. The transformation of transition probability matrix of FACS is then presented followed by the analysis of its basic characteristics and the evaluation of steady state vector using both standard matrix multiplication and limit.

2 FACS of Fuzzy Graph Type-3
The main idea of the definition of FACS is the merger of fuzzy graph of type-3 to autocatalytic set [11]. The formal definition of FACS is given below.

Definition 2.1
Fuzzy autocatalytic set (FACS) was defined as a subgraph where each of whose nodes has at least one incoming link with membership value $\mu(e_i) \in (0,1], \forall e_i \in E$ (see [11],[14])

\[ \mu(e_i) \in (0,1], \forall e_i \in E \] (1)

The membership values for fuzzy edge connectivity for fuzzy graph are in the interval (0,1]. These values constitute the entries of the adjacency matrix for FACS as follows:

\[ C_{F_{ij}} = \begin{cases} 0 & \text{for } i = j \text{ and } e_i \notin E \\ \mu(e_i) \in (0,1] & \text{for } i \neq j \end{cases} \] (2)

As for incineration process, the membership values are determined through the chemical reaction taken place between six variables that play vital roles in the clinical waste incinerator, namely waste, fuel, oxygen, carbon dioxide, carbon monoxide and other gases including water. The set of vertices in the graph of FACS of the incineration process, $V = \{v_1, v_2, ..., v_6\}$ is represented by these six variables.
When a fuzzy graph of type -3 is specifically considered in the construction of FACS, the description of its fuzzy head, fuzzy tail and fuzzy edges connectivity of the edges are given in [14].

From the explanation given in [14] pertaining to the construction of FACS of Fuzzy Graph of Type-3 for the incineration process, the graph and its adjacency matrix using (2) is represented as in Fig. 1. The different color signifies the different range of membership value for the fuzzy edge connectivity. The greater the value of connectivity between the vertices, the thicker is the link between them. From the graph, one can easily see that it is strongly connected where each node in the graph has access to every other node.

\[
P(u,v) = \begin{cases} 
    \frac{w_{uv}}{d_u} & \text{if } u \to v \text{ is an edge in } G \\
    0 & \text{otherwise}
\end{cases}
\]

where \( d_u \) denotes the out-degree of vertex \( u \) and \( w_{uv} \) denotes the weight of edge from vertex \( u \) to \( v \).

The probability of moving from a node \( u \) to a node \( v \) is proportional to the weight of the edge \( u \to v \). According to Rubinfeld [10], the concept of a random walk with weighted edges is equivalent to the concept of finite Markov chain. Here, a random process is defined by a finite set of states, \( N = \{1,\ldots,n\} \) and a sequence \( X_0, X_1, X_2, \ldots \) of random variables. This process is a (finite) Markov chain if the transition probabilities at step \((k+1)\) depend only on the state at step \( k \), that is

\[
P(X_{k+1} = j | X_k = i_0, \ldots, X_0 = i_0) = P(X_{k+1} = j | X_k = i_k)
\]

Markov chain is also often described by transition probability matrix \( P = [p_{ij}]_{i,j \in N} \) defined as

\[
p_{ij} = P(X_{k+1} = j | X_k = i_k) \text{ for all } i, j \in N
\]

Next, the transition probability matrix is a (row) stochastic matrix, that is, a square nonnegative matrix with all row sums equal to 1 and no value of its entries is negative. Note that every stochastic matrix represent a Markov chain. Since it is a nonnegative matrix, Perron-Frobenius theorem gives useful information about the eigenvalues of such matrices as illustrated below.

Let \( P \in \mathbb{R}_{\geq 0}^{n \times n} \) be a stochastic matrix. Then

(a) \( \rho(P) = 1 \) and is an eigenvalue of \( P \);
(b) \( P \) has at least one invariant measure;
If \( P \) is irreducible, then
(c) \( P \) has exactly one invariant measure \( x^T \), and \( x^T \) is positive;
Moreover, if \( P \) is primitive, then
(d) \( \lim_{k \to \infty} P^k = 1x^T \).

Note that invariant measure of a stochastic matrix correspond exactly to its left Perron vector.

### 4 Transformation of FACS to Transition Probability Matrix

Since the concept of random walk with weighted edges is equivalent to the concept of finite Markov
chain, thus it is a special case of Markov chain [10]. We can then apply this concepts (3), (4), (5) for the incineration process with the assumption that population size and the possible state are both constants. Here, population size denotes the variables in the process whereas the possible states denote the possible condition or form of the population in the system at a particular time. We also assume that it takes the same amount of time to move from one state to another. Thus, Markov transition probability matrix of FACS can be constructed. In this case, the Markov chain is homogeneous and its dynamics are described by the transition probability matrix, $P$ which is given in the following definition.

**Definition of Transition Matrix for FACS**

Suppose $G_{FTF}(V,E)$ is a no loop FACS of Fuzzy Graph Type-3. The probability transition matrix of FACS of Fuzzy Graph Type-3 is $P$, with $P(u,v)$ is the probability of moving from $u$ to $v$ as

$$P(u,v) = \begin{cases} \frac{\mu(u,v)}{d_{out}(u)} & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

(6)

where for a vertex $u$, the out-degree of $u$ is $d_{out}(u) = \sum \mu(u,t)$ and $\mu(u,v)$ is the ordinary membership value of an edge from $u$ to $v$. Hence, from (6), we will have the transition probability matrix for FACS of incineration process as below:

$$P = \begin{bmatrix}
0 & 0.00001 & 0.15615 & 0.51632 & 0.00001 & 0.32752 \\
0.06529 & 0 & 0.680040 & 0.000010 & 0.319950 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}$$

Each of the entry of the matrix is the probability of a transition from a state to another.

**5 Characteristics of Transition Probability Matrix of FACS of Fuzzy Graph of Type-3.**

The above transition probability matrix $P$ provides some basic facts or characteristics of FACS. There are:

**Fact 1:** For a strongly connected graph of FACS, the transition matrix is (row) stochastic matrix since the sum of the entries on each and every row is 1 and no value of its entries is negative.

**Fact 2:** For a strongly connected graph of FACS, the transition probability matrix is irreducible.

Eventhough transition matrix is not symmetric, we still can deduced some useful properties of the matrix since all of its entries are nonnegative. We can look into their eigenvalues and eigenvectors in order to relate it to Perron-Frobenius theorem for nonnegative matrix.

The eigenvalues, $\lambda$ and their respective eigenvectors, $x$ of $P$ using MATLAB version 7.0 is given as

$$\lambda = 1.00000000000000,-0.19686879108908,-0.40154987138365+0.58558458293596i$$

For any nonnegative irreducible matrix, Perron-Frobenius (PF) theorem guarantees that there exists an eigenvalue which is real and larger than or equal to all other eigenvalues in magnitude. Here,

$$\lambda_1 = \rho(P) = 1$$

**Fact 3:** The largest eigenvalue of transition probability matrix of FACS is real and equal to 1 with algebraic multiplicity one.

**Fact 4:** Since $\lambda_1 = \rho(P) = 1$, then the strongly connected graph of FACS is aperiodic.

This fact follows from Lemma 2 by Costello [4]: A strongly connected graph $G$ is periodic if the transition matrix $P$ has an eigenvalue $\rho \neq 1$ such that $|\rho| = 1$. Further, one can see that this matrix has a truly dominant eigenvalue and by definition of primitive matrix [13], we confirmed that matrix $P$ above is primitive since $P^k > 0$ for $k = 6$. Moreover, it is also called regular transition matrix since for $k = 6$, $P^k > 0$, $\forall i,j = 1,...,n$.

**Fact 5:** The regular transition probability matrix of FACS is primitive. Perron-Frobenius theorem also states that there exists a positive eigenvector
corresponding to the largest eigenvalue. Here, we have
\[
x = \begin{pmatrix} 0.40824829046386 \\ 0.40824829046386 \\ 0.40824829046386 \\ 0.40824829046386 \\ 0.40824829046386 \\ 0.40824829046386 \end{pmatrix} = 0.40824829046386 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
\]

Since \( Px = \lambda x = 1 \times x = x \), thus the ‘all 1’ vector is a right eigenvector of \( P \) with eigenvalue equal to 1.

**Fact 6:** The right eigenvector of transition probability matrix of FACS is column vector with all entries equal to 1.

We can also calculate the unique left Peron vector for matrix \( P \). We have
\[
x^T = \begin{pmatrix} 0.36211364 \\ 0.362110019 \times 10^{-6} \\ 0.056543479 \\ 0.222912591 \\ 4.75200599 \times 10^{-6} \\ 0.358421916 \end{pmatrix}
\]

Note that \( \sum_{j=1}^{6} x_{1j} = 1 \).

**Fact 7:** The left eigenvector \( x^T = (x_1, ..., x_6) \) corresponds to the largest eigenvalue of transition probability matrix of FACS is row vector which is unique and \( \sum_{j=1}^{6} x^T_{1j} = 1 \).

### 6. Steady State Vector for FACS of Fuzzy Graph Type-3.

The term steady state vector or stationary distribution vector used in this paper is equivalent. The fundamental Ergodic Theorem for Markov chains in [5] states that the Markov chain represented by transition matrix, \( P \) has a stationary distribution vector, \( X \) under three conditions:

i. \( P \) is stochastic
ii. \( P \) is irreducible
iii. \( P \) is aperiodic

Scrutinizing the characteristics of transition matrix of FACS of fuzzy graph type-3 of incineration process, \( P \) as discussed in Section 5.0, we can deduce that this matrix is an Ergodic Markov chain. Ergodic Markov chain ensures the existence of steady state vector which in turn can describe the long term probabilities of the system.

Two well-known methods are used in finding steady state vector for FACS.

1. Using standard matrix multiplication in which \( P^n \rightarrow P^{n-1} \).

For example \( P^3 = P^2 P^1 \) for \( n = 1,2,3, ... \). As \( n = 109 \) we obtained the entries of the matrix as shown below:

\[
\begin{pmatrix}
0.36211369969258 & 0.000003621100189 & 0.056543479446405 \\
0.36211369969258 & 0.000003621100189 & 0.056543479446405 \\
0.36211369969258 & 0.000003621100189 & 0.056543479446405 \\
0.36211369969258 & 0.000003621100189 & 0.056543479446405 \\
0.36211369969258 & 0.000003621100189 & 0.056543479446405 \\
0.36211369969258 & 0.000003621100189 & 0.056543479446405
\end{pmatrix}
\]

Elements in each and every column in this matrix is identical showing that the numbers in each column converges to a particular number. Numbers in the row represent the steady state vector of \( P \), that is
\[
X = \begin{pmatrix} 0.36211364 \\ 0.362110019 \times 10^{-6} \\ 0.056543479 \\ 0.222912591 \\ 4.75200599 \times 10^{-6} \\ 0.358421916 \end{pmatrix}
\]

We interpret the \( i \)th entries to be the probabilities of the relative concentration of the variables at the end of the process as: waste (\( v_1 \)) has relative concentration with the highest probability of 0.36211364 and water and other pollutions (\( v_6 \)) with second highest probability of 0.358421916. This result is consistent with Sabariah’s finding [11] where finally only two variables that play important role in the clinical waste incineration process namely waste (\( v_1 \)) and other pollutions (\( v_6 \)).

2. Using Theorem 7.3 (Anton & Rorres, [1]) stated that: If \( P \) is a regular transition matrix and \( X \) is any probability vector, then as \( n \to \infty \lim_{n \to \infty} XP^n = Q \) where \( Q \) is a fixed probability vector independent of \( n \) all of whose entries are positive.

For example, \( XP^n = Q^n, XP^n = Q^2, ..., XP^n = Q^{109} \)

The state vectors \( Q^n \) for \( n = 0, 1, 2, ..., 109 \) can be calculated using initial relative concentration, of \( i_{th} \) variable at initial time \( t = 0 \) where
\( X = (0.269, 0.031, 0.645, 0, 0, 0.055) \) as given in Sabariah [11] and Theorem 7.3 in [1]. The result is shown in Table 6.1. For all values of \( n \) greater than 109, all state vectors are equal to \( Q^{109} \) to eleven decimal places.

The result in Table 1 shows that regular markov chain of FACS of Fuzzy Graph Type-3 for incineration process converges to a steady state vector

\[
Q^{109} = \begin{pmatrix}
0.36211364 & 0.362110019 \times 10^{-6} & 0.056543479 \\
0.22912591 & 4.75200599 \times 10^{-6} & 0.358421916
\end{pmatrix}
\]

These fixed probability vector or steady state vector of \( P \) is actually equivalent to the unique left Perron vector as discussed in Section 5.0. Thus, we conclude that the steady state vector, \( X \) or \( Q \) and the left Perron vector, \( \chi \) is the same, and so finding the steady state vector of \( P \) of FACS is equivalent to finding the dominant left eigenvector (left Perron vector) of \( P \) of FACS.

### Table 1: Convergence of a state vectors for clinical waste incineration process

<table>
<thead>
<tr>
<th>( n )</th>
<th>( Q^{(n)} )</th>
<th>( n=0 )</th>
<th>( n=1 )</th>
<th>( n=60 )</th>
<th>( n=100 )</th>
<th>( n=109 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_1 )</td>
<td>0.269</td>
<td>0.0971120500000000</td>
<td>0.362113640171713</td>
<td>0.362113639969257</td>
<td>0.362113639969258</td>
<td></td>
</tr>
<tr>
<td>( q_2 )</td>
<td>0.031</td>
<td>0.000002689973100</td>
<td>0.000003621100176</td>
<td>0.000003621100189</td>
<td>0.000003621100189</td>
<td></td>
</tr>
<tr>
<td>( q_3 )</td>
<td>0.645</td>
<td>0.042003929960700</td>
<td>0.056543479254075</td>
<td>0.056543479446405</td>
<td>0.056543479446405</td>
<td></td>
</tr>
<tr>
<td>( q_4 )</td>
<td>0</td>
<td>0.569951281113089</td>
<td>0.22291259080916</td>
<td>0.222912591281958</td>
<td>0.222912591281957</td>
<td></td>
</tr>
<tr>
<td>( q_5 )</td>
<td>0</td>
<td>0.00015899973100</td>
<td>0.00004752005981</td>
<td>0.00004752005989</td>
<td>0.00004752005989</td>
<td></td>
</tr>
<tr>
<td>( q_6 )</td>
<td>0.055</td>
<td>0.290914148980010</td>
<td>0.358421916667139</td>
<td>0.358421916196203</td>
<td>0.358421916196202</td>
<td></td>
</tr>
</tbody>
</table>

#### 7. Conclusion

In this paper, Markov chain of FACS of fuzzy graph type-3 represented by transition probability matrix, \( P \) is defined. This definition leads to the construction of transition probability matrix of FACS for the clinical waste incineration process. This development gives interesting result both on the matrix itself and also the process. Basic characteristics of the matrix are highlighted including its dominant eigenvalue and its right and left eigenvector (Perron vector). The Markov chain of FACS of fuzzy graph type-3 is found to converge to a steady state vector which is equivalent to the left Perron vector. The transition matrix of FACS also provide sound interpretation of the outcomes.

**Acknowledgements:** This research was partly supported by FRGS Grant, UTM with Vote No 78483.

**References:**


