Multilinear completely bounded projective u-covariant maps extended on twisted crossed products

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Abstract: The theory of projective representations has applications in quantum chemistry, molecular and solid state physics, computer science and automatic control. It is used, for example, to form molecular orbitals and symmetry products. In this paper we construct projective covariant representations associated with projective coordinates of molecules. In this paper we construct projective covariant representations associated with projective physics, computer science and automatic control. It is used, for example, to form molecular orbitals and symmetry products.

Key–Words: Projective u-covariant representations, Completely bounded multilinear maps, Twisted crossed products.

1 Introduction

The theory of projective representations is an important mathematical tool in many branches of quantum chemistry, molecular and solid state physics, computer science and automatic control [3], [6], [7], [8].

Christensen and Sinclair were the first that introduced in [2] the notion of completely bounded (respectively, completely positive) multilinear operators from a C*-algebra into the algebra of continuous linear operators. Using the representation given by Stinespring’s Theorem for a completely positive linear operator from a C*-algebra into the algebra of continuous linear operators on a Hilbert space, together with Wittstock’s Theorem that decomposes a completely bounded linear operator as a finite linear combination of completely positive linear operators [9], it can be obtained a representation of a completely bounded linear operator from a C*-algebra into the algebra of continuous linear operators on a Hilbert space. Christensen and Sinclair gave a representation theorem for completely bounded symmetric multilinear operators from a C*-algebra into the algebra of continuous linear operators, which generalizes this representation of completely bounded linear operators.

Heo [4] introduced the notion of a covariant multilinear map from a C*-algebra to another, which we generalize to a projective u-covariant multilinear map in Definition 5. First, we prove Proposition 6 and Proposition 7 which are the projective covariant versions of Theorem 2.8 and Lemma 3.1 in [2] and Lemma 2.1 and Lemma 2.2 in [4] and then, using these results, we construct in Theorem 8 projective covariant representations associated with projective u-covariant completely bounded symmetric multilinear maps as a generalization of Theorem 3.1, [4].

Busby and Smith [1] introduced the twisted actions and constructed in Theorem 2.2, [1] a Banach ∗-algebra denoted by L¹(G, A, α, ω) as a generalization of the group algebra L¹(G) (the algebra of all integrable functions on G with scalar values), now the integrable functions on G taking values in a C*-algebra A. They studied the theory of covariant representations of the twisted dynamical system (G, A, α, ω) and proved that the representations of the twisted group algebra L¹(G, A, α, ω) are in one-to-one correspondence with the covariant representations of the twisted dynamical system (G, A, α, ω) (Theorem 3.3,[1]) and that the enveloping C*-algebra of L¹(G, A, α, ω) is the twisted crossed product of A by G under the action α relative to the multiplier ω denoted by A ×ω αG. Our main result is Theorem 9 in which we show that for a given unital C*-dynamical system (G, A, α) with G an amenable group, a projective u-covariant completely bounded multilinear map can be extended on the twisted crossed product (A ×ω αG)k to a completely bounded multilinear map.

2 Definitions and Notations

Definition 1 ([2]) Let A and B be C*-algebras, let Mn(A) (respectively Mn(B)) be the algebra of all n × n matrices over A (respectively B) and let
\[ \rho : A^k = A \times \ldots \times A \rightarrow B \text{ be a } k \text{-linear map. The } \\
k \text{-linear map } \rho_n : M_n(A)^k \rightarrow M_n(B) \text{ is defined by } \\
\rho_n(a_1, A_2, \ldots, A_k) = (\sum_{r,s,t} \rho(a_{1r}, a_{2rs}, \ldots, a_{kt})) \\
\text{for all } A_p = (a_{nij}) \in M_n(A), 1 \leq p \leq k \text{ and for all } \\
n \in N. \text{ We define the norm of } \rho_n \text{ by } \\
\|\rho_n\| = \sup \{\|\rho_n(a_1, A_2, \ldots, A_k)\|\}, \\
\text{where } A_p \in M_n(A) \text{ with } \|A_p\| \leq 1 \text{ for } 1 \leq p \leq k \\
\text{and define the completely bounded norm of } \rho_n \text{ by } \\
\|\rho\|_{cb} = \sup \{\|\rho_n\|; n \in N\}. \\
The k \text{-linear map } \rho \text{ is called completely bounded if } \\
\|\rho\|_{cb} < \infty. \\
\text{Definition 2 } ([2]) \text{ The } k \text{-linear map } \rho^* : A^k \rightarrow B \text{ is defined by } \\
\rho^*(a_1, a_2, \ldots, a_k) = \rho(a_1^*, \ldots, a_k^*, a_1^*) \\
\text{for all } a_1, a_2, \ldots, a_k \in A. \text{ The } k \text{-linear map } \rho \text{ is } \\
called symmetric if } \rho = \rho^*. \\
\text{Definition 3 } ([2]) \text{ If } (A_1, \ldots, A_k) \in M_n(A)^k. \text{ Let } \\
\|\rho_n\|_s = \sup \{\|\rho_n(a_1, A_2, \ldots, A_k)\|\} \\
\text{where } A_j \in M_n(A) \text{ with } \|A_j\| \leq 1 \text{ for } 1 \leq j \leq k \\
\text{and } (A_1, \ldots, A_k) = (A_1^*, \ldots, A_k^*) \text{ for all } n. \text{ Then the } \\
symmetrically completely bounded norm } \cdot \|\cdot\|_{cb} \text{ is defined by } \\
\|\rho\|_{cb} = \sup \{\|\rho_n\|; n \in N\}. \\
\text{Definition 4 } ([2]) \text{ A } k \text{-linear map } \rho : A^k \rightarrow B \text{ is } \\
called completely positive if } \rho(a_1, \ldots, A_k) \geq 0 \text{ for all } \\
(A_1, \ldots, A_k) = (A_1^*, \ldots, A_k^*) \in M_n(A)^k \text{ with } \\
A_m \geq 0 \text{ if } k \text{ is odd, where } m = \left\lfloor \frac{k+1}{2} \right\rfloor \\
\text{and for all } n \in N. \\
\text{Definition 5 } \text{ If } (G, A, \alpha) \text{ is a } C^* \text{-dynamical system, then the action } \\
\alpha : G \rightarrow \text{Aut}(A) \text{ induces the action } \\
\alpha : G \rightarrow \text{Aut}(A^k) \text{ by } \\
\alpha_g(a_1, a_2, \ldots, a_k) = (\alpha_g(a_1), \ldots, \alpha_g(a_k)) \\
\text{for all } a_1, a_2, \ldots, a_k \in A. \text{ Given a projective unitary } \\
\text{representation } u : G \rightarrow U(H) \text{ with the multiplier } \omega, \\
a k \text{-linear map } \rho : A^k \rightarrow U(H) \text{ is called projective } \\
u \text{-covariant if } \\
\rho(\alpha_g(a_1, a_2, \ldots, a_m)) = \rho(\alpha_g(a_1), \ldots, \alpha_g(a_m)) = \\
u_g \rho(a_1, a_2, \ldots, a_m) u_g^* \\
\text{for each } a_1, a_2, \ldots, a_m \in A \text{ and } g \in G. \\
3 \text{ The projective covariant representation associated with a projective } \\
u \text{-covariant completely bounded } \\
symmetric multilinear map } \\
\text{Proposition 6 } \text{ Let } (G, A, \alpha) \text{ be a } C^* \text{-dynamical system with } G \text{ amenable and let } u : G \rightarrow U(H) \\
\text{be a projective unitary representation of } G \text{ with the } \\
multiplier } \omega. \text{ If } \varphi : A^k \rightarrow B(H) \text{ is a } \text{projective } \\
u \text{-covariant completely bounded symmetric } k \text{-linear map with } k \geq 2, \text{ then there is a } \text{projective } u \text{-covariant } \\
\text{completely positive linear map } \rho : A \rightarrow B(H) \text{ such that } \\
- \rho_n(X^*X) \leq \varphi_n(X^*, A_2, \ldots, A_{k-1}, X) \leq \rho_n(X^*X) \\
\text{for all } X \in M_n(A) \text{ and } A^* = A = \\
(A_2, \ldots, A_{k-1}) \in M_n(A)^{k-2} \text{ with } |A| \leq 1 \text{ and for all } \\
n \text{and such that } \|\rho\| = \|\rho\|_{cb} \leq \|\varphi\|_{cb}. \\
\text{Proof: } \text{ By Theorem 2.8, [2] there is a completely bounded and completely positive } \\
k \text{-linear map } \psi : A \rightarrow B(H) \text{ such that } \\
- \psi_n(X^*X) \leq \varphi_n(X^*, A_2, \ldots, A_{k-1}, X) \leq \psi_n(X^*X) \\
\text{for all } X \in M_n(A) \text{ and } A^* = A = \\
(A_2, \ldots, A_{k-1}) \in M_n(A)^{k-2} \text{ with } |A| \leq 1 \text{ and for all } \\
n \text{and such that } \|\psi\| = \|\varphi\|_{cb}. \\
\text{Let } m \text{ be a right invariant mean on } G \text{ and define } \rho : A \rightarrow B(H) \text{ by } \\
\langle \rho(a) \xi, \eta \rangle = m(t \mapsto \langle u_t^* \psi(a_t) \xi, \eta \rangle) \\
\text{for } a \in A \text{ and } \xi, \eta \in H. \text{ Since } \psi \text{ is completely positive and } m \text{ is a positive linear functional, we see that } \rho \text{ is } \\
completely positive. \text{ Since } u \text{ is a unitary representation, we have } \\
\langle \rho(\alpha_s(a)) \xi, \eta \rangle = \\
m(t \mapsto \langle u_t^* \psi(\alpha_t(\alpha_s(a))) u_t \xi, \eta \rangle) = \\
m(t \mapsto \langle u_s u_s^* u_t^* \psi(\alpha_{ts}(a)) u_t \xi, \eta \rangle) (1) \\
\text{Since } \omega(t, s) u_t u_s = u_{ts}, \text{ where } \omega(t, s) u_{ts} u_{ts}^* = u_{ts}, \\
\text{so } u_{ts}^* = \omega(t, s)^{-1} u_{ts} = \omega(t, s) u_{ts}, \text{ because } \\
\omega(t, s) \omega(t, s) = |\omega(t, s)|^2 = 1, \text{ so the relation (1) becomes } \\
\langle \rho(\alpha_s(a)) \xi, \eta \rangle = \\
m(t \mapsto \langle u_s \omega(t, s) u_t^* \psi(\alpha_{ts}(a)) \omega(t, s) \omega(s) u_{ts} \xi, \eta \rangle) = \\
m(t \mapsto \langle \omega(t, s) \omega(t, s) u_t^* \psi(\alpha_{ts}(a)) u_s \xi, \xi \eta \rangle) = \\
m(t \mapsto \langle u_{ts} \psi(\alpha_{ts}(a)) u_{ts} \xi, \xi \eta \rangle = \

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\[ m(t \mapsto \langle u_s u_t^* \psi(\alpha_{x}(a)) u_s u_t^* \xi, \eta \rangle) = \]
\[ m(t \mapsto \langle u_t^* \psi(\alpha_{x}(a)) u_s u_t^* \xi, \eta \rangle) = \]
\[ m(t \mapsto \langle u_t^* \psi(\alpha_{x}(a)) u_s u_t^* \xi, u_t^* \eta \rangle) = \]
\[ \langle \rho(a) u_t^* \xi, u_t^* \eta \rangle = \langle u_s \rho(a) u_t^* \xi, \eta \rangle, \]
for every \( s \in G \), where the fifth equality follows from the right invariance of \( m \). Thus we have \( \rho(\alpha_x(a)) = u_s \rho(a) u_t^* \) for each \( s \in G \), which means that \( \rho \) is projective \( v \)-covariant.

For any \( x \in A, a^* = a = (a_2, \ldots, a_{k-1}) \in A^{k-2} \) and \( \xi, \xi \in H \), we have
\[
\langle (\rho(x^* x) - \varphi(x^*, a_2, \ldots, a_{k-1}, x)) \xi, \xi \rangle = \]
\[
m(t \mapsto \langle \varphi(x^*, a_2, \ldots, a_{k-1}, x) \xi, \xi \rangle) = \]
\[
m(t \mapsto \langle \varphi(x^*, a_2, \ldots, a_{k-1}, x) \xi, \xi \rangle - \]
\[\varphi(x^*, a_2, \ldots, a_{k-1}, x)) \xi, \xi \rangle = \]
\[
m(t \mapsto \langle u_t^* \varphi(\alpha_{x}(x^*)) u_t^* \xi, \xi \rangle - \]
\[\varphi(\alpha_{x}(x^*), \alpha_{x}(a_2), \ldots, \alpha_{x}(a_{k-1}), x) \rangle u_t^* \xi, \xi \rangle, \]
where the third equality follows from the \( v \)-covariance of \( \varphi \). Similarly, we have
\[
\langle (\rho(x^* x) - \varphi(x^*, a_2, \ldots, a_{k-1}, x)) \xi, \xi \rangle \geq 0.
\]
From the above two inequalities, we conclude that
\[
-\rho_n(X^* X) \leq \varphi_n(X^*, A_2, \ldots, A_{k-1}, X) \leq \rho_n(X^* X)
\]
for all \( X \in M_n(A) \) and \( A^* = A = (A_2, \ldots, A_{k-1}) \in M_n(A)^{k-2} \) with \( \| A \| \leq 1 \) and for all \( n \).

Since \( \| \varphi \| = \| \varphi \|_{scb} \), we can obtain \( \| \rho \| \leq \| \varphi \|_{scb} \) from \( \| \rho(A) \| = \| u_s^* \varphi(A) u_t^* \| \) and this completes the proof.

**Proposition 7** Let \( (G, A, \alpha) \) be a \( C^* \)-dynamical system with \( G \) amenable and let \( \psi : G \rightarrow U(H) \) be a projective unitary representation of \( G \) with the multiplier \( \omega \). Let \( \varphi : A^k \rightarrow B(H) \) be a projective \( v \)-covariant completely bounded \( k \)-linear map with \( k \geq 2 \). If \( \rho : A \rightarrow B(H) \) is a projective \( v \)-covariant completely positive \( k \)-linear map such that
\[
-\rho_n(X^* X) \leq \varphi_n(X^*, A_2, \ldots, A_{k-1}, X) \leq \rho_n(X^* X)
\]
for all \( X \in M_n(A) \) and \( A^* = A = (A_2, \ldots, A_{k-1}) \in M_n(A)^{k-2} \) with \( \| A \| \leq 1 \) and for all \( n \), then there are

(i) a projective covariant representation \((\Phi, v, K)\) of \((G, A, \alpha)\) into \( B(K) \), where \( v \) is a projective unitary representation with the multiplier \( \omega \);

(ii) a linear operator \( V \in B(H, K) \) with \( \| V \|^2 = \| \rho \| \);

(iii) a projective \( v \)-covariant completely bounded symmetric \((k - 2)\)-linear map \( \psi : A^{k-2} \rightarrow B(K) \) with \( \| \psi \|_{scb} \leq 1 \) (when \( k = 2 \), \( \psi \) is just a fixed self-adjoint element of \( B(K) \) commuting with \( v_g \)) such that:

1. \( \varphi(a_1, \ldots, a_k) \)
\[ V^* \Phi(a_1) \psi(a_2, \ldots, a_{k-1}) \Phi(a_k) V \]
for all \( a_1, \ldots, a_k \in A \);

2. \( V u_g = v_g V \) for all \( g \in G \).

**Proof:** By Lemma 3.1, [2] there is a Hilbert space \( K \), a \( * \)-representation \( \Phi \) of \( A \) on \( K \), a linear operator \( V \in B(H, K) \) with \( \| V \|^2 = \| \rho \| \) and a completely bounded symmetric \((k - 2)\)-linear map \( \psi : A^{k-2} \rightarrow B(K) \) with \( \| \psi \|_{scb} \leq 1 \) satisfying condition (iii) 1). Following the proof of Lemma 3.1, [2] we first form the algebraic tensor product \( A \otimes_{alg} H \) and endow it with the pre-inner product by setting
\[
\langle x \otimes \xi, y \otimes \eta \rangle_{A \otimes_{alg} H} = \langle \rho(y^* x) \xi | \eta \rangle_H
\]
and extending linearly. To obtain \( K \) we divide by the kernel of \( \langle \cdot, \cdot \rangle_{A \otimes_{alg} H} \) and completes. The representation \( \Phi \) of \( A \) is defined by
\[
\Phi(a) (x \otimes \xi) = ax \otimes \xi, \ a, x \in A, \ \xi \in H.
\]
If \( A \) is unital, the linear operator \( V : H \rightarrow K \) is defined by \( V \xi = 1_A \otimes \xi \). If \( A \) is nonunital, let \( \{ e_\lambda \} \) be a bounded approximate identity of positive elements of norm \( \leq 1 \) in \( A \) and define \( V \in B(H, K) \) by
\[
V \xi = w^* - \lim_\lambda (e_\lambda \otimes \xi)
\]
The completely bounded symmetric \((k - 2)\)-linear map \( \psi : A^{k-2} \rightarrow B(K) \) is given by
\[
\langle \psi(a_2, \ldots, a_{k-1}) (x \otimes \xi), y \otimes \eta \rangle_{A \otimes_{alg} H} = \langle \varphi(y^* x, a_2, \ldots, a_{k-1}, \xi | \eta \rangle_H
\]
We first define a map \( v : G \rightarrow B(K) \) by setting
\[
v_g (x \otimes \xi) = \alpha_g (x) \otimes u_g \xi, \ g \in G
\]
and extending linearly to \( A \otimes_{alg} H \). Since
\[
\langle \alpha_g (x) \otimes u_g \xi, \alpha_g (y) \otimes u_g \eta \rangle_{A \otimes_{alg} H} = \langle (\rho(\alpha_g (y))^* \alpha_g (x)) u_g \xi | u_g \eta \rangle_H
\]
\[
(\rho(\alpha_g(y^*x))u_\eta \xi | u_\eta \eta)_H = \\
(u_\eta^* \rho(\alpha_g(y^*x))u_\eta \xi | u_\eta \eta)_H = \\
(\rho(y^*x)\xi | \eta)_H = \langle x \otimes \xi, y \otimes \eta \rangle_{A \otimes_{alg} H},
\]
by the \(u\)-covariance of \(\rho\). \(v_\eta\) is a unitary representation because
\[
\langle v_\eta(x \otimes \xi), y \otimes \eta \rangle_{A \otimes_{alg} H} = \\
\langle \alpha_g(x) \otimes u_\eta \xi, y \otimes \eta \rangle_{A \otimes_{alg} H} = \\
(\rho(y^*\alpha_g(x))u_\eta \xi | u_\eta \eta)_H = \\
(\rho(\alpha_g(\alpha_g^{-1}(y^*)x)u_\eta \xi | \eta)_H = \\
(u_\eta \rho(\alpha_g^{-1}(y^*)x)u_\eta^* u_\eta \xi | \eta)_H = \\
(\rho(\alpha_g^{-1}(y^*)x)u_\eta^* \eta)_H = \\
\langle x \otimes \xi, \alpha_g^{-1}(y) \otimes u_\eta^* \eta \rangle_{A \otimes_{alg} H} = \\
\langle x \otimes \xi, v_{\eta^{-1}}(y \otimes \eta) \rangle_{A \otimes_{alg} H},
\]
where the forth equality results from the \(u\)-covariance of \(\rho\).

We show now that \(v\) is a projective representation with the multiplier \(\omega\). We have
\[
\langle \omega(g_1, g_2) v_{g_1} v_{g_2} (x \otimes \xi), y \otimes \eta \rangle_{A \otimes_{alg} H} = \\
\langle \omega(g_1, g_2) \alpha(g_2)(x) \otimes u_{g_2} \xi, y \otimes \eta \rangle_{A \otimes_{alg} H} = \\
\langle \omega(g_1, g_2) \alpha(g_2) \alpha(g_1)(x) \otimes u_{g_1} u_{g_2} \xi, y \otimes \eta \rangle_{A \otimes_{alg} H} = \\
\langle \omega(g_1, g_2) \alpha_{g_1 g_2}(x) \otimes u_{g_1} u_{g_2} \xi, y \otimes \eta \rangle_{A \otimes_{alg} H} = \\
\langle \alpha_{g_1 g_2}(x) \otimes u_{g_1} u_{g_2} \xi, y \otimes \eta \rangle_{A \otimes_{alg} H} = \\
\langle v_{g_1} v_{g_2} (x \otimes \xi), y \otimes \eta \rangle_{A \otimes_{alg} H}
\]
So, \(\omega(g_1, g_2) v_{g_1} v_{g_2} = v_{g_1 g_2}\) for all \(g_1, g_2 \in G\).

For each \(g \in G\) and \(a \in A\), we have
\[
v_g \Phi(a) v_g^* (x \otimes \xi) = v_g \Phi(a) (\alpha_g^{-1}(x) \otimes u_{g^{-1}} \xi) = \\
v_g (\alpha_g^{-1}(x) \otimes u_{g^{-1}} \xi) = \\
\alpha_g (a) \alpha_g^{-1}(x) \otimes u_{g^{-1}} \xi = \alpha_g(a) x \otimes \xi = \Phi(a)(x) \otimes \xi,
\]
which implies that \((\Phi, v, K)\) is a projective covariant representation of \((G, A, \alpha)\).

Since \(v_\xi V \xi = v_\xi (1_A \otimes \xi) = \alpha_g (1_A) \otimes u_\xi \xi = 1_A \otimes u_\xi \xi = V u_\xi \xi \) for each \(\xi \in H\), we get \(V u_\eta = v_\eta V\) for each \(g \in G\). Similarly, this equality is also obtained in the nonunital case.

To show the \(u\)-covariance of \(\psi\), let \(a_2, \ldots, a_{k-1} \in A\). Then we have
\[
\langle \psi(\alpha_g(a_2), \ldots, \alpha_g(a_{k-1})) (x \otimes \xi), y \otimes \eta \rangle_{A \otimes_{alg} H} = \\
\langle \varphi(y^*, \alpha_g(a_2), \ldots, \alpha_g(a_{k-1}), x) | \eta \rangle_H = \\
\langle u_\eta \varphi(\alpha_g^{-1}(y^*), a_2, \ldots, a_{k-1}, \alpha_g^{-1}(x)) u_\eta^* \xi | \eta \rangle_H = \\
\langle \varphi(\alpha_g^{-1}(y^*), a_2, \ldots, a_{k-1}, \alpha_g^{-1}(x)) u_\eta^* \xi | u_\eta^* \eta \rangle_H = \\
\langle \psi(a_2, \ldots, a_{k-1}) (\alpha_g^{-1}(x) \otimes u_\eta^* \xi), \alpha_g^{-1}(y) \otimes u_\eta^* \eta \rangle_{A \otimes_{alg} H} = \\
\langle \psi(a_2, \ldots, a_{k-1}) (v_g(\alpha_g^{-1}(x) \otimes \xi)), v_g(y \otimes \eta) \rangle_{A \otimes_{alg} H} = \\
\langle v_g \psi(a_2, \ldots, a_{k-1}) v_g^*(x \otimes \xi), y \otimes \eta \rangle_{A \otimes_{alg} H},
\]
where the third equality holds by the \(u\)-covariance of \(\varphi\).

When \(k = 2\), we get a self-adjoint operator \(S\) in \(B(K)\) such that
\[
\langle S(x \otimes \xi), y \otimes \eta \rangle_{A \otimes_{alg} H} = \langle \varphi(y^*, x) | \xi \rangle_H
\]
For each \(g \in G\), we have
\[
\langle v_g^* S v_g (x \otimes \xi), y \otimes \eta \rangle_{A \otimes_{alg} H} = \\
\langle S v_g (x \otimes \xi), v_g(y \otimes \eta) \rangle_{A \otimes_{alg} H} = \\
\langle S(\alpha_g(x) \otimes u_\eta \xi), \alpha_g(y) \otimes u_\eta \eta \rangle_{A \otimes_{alg} H} = \\
\langle \varphi(\alpha_g(y^*), \alpha_g(x)) u_\eta \xi | u_\eta \eta \rangle_H = \\
\langle u_\eta^* \varphi(\alpha_g(y^*), \alpha_g(x)) u_\eta \eta \rangle_H = \\
\langle \varphi(y^*, x) | \xi \rangle_H = \langle S(x \otimes \xi), y \otimes \eta \rangle_{A \otimes_{alg} H},
\]
where the fifth equality results from the \(u\)-covariance of \(\varphi\).

\(\square\)

**Theorem 8** Let \((G, A, \alpha)\) be a \(C^*\)-dynamical system with \(G\) amenable and let \(u : G \rightarrow U(H)\) be a projective unitary representation of \(G\) with the multiplier \(\omega\). Let \(\varphi : A^k \rightarrow B(H)\) \((k \geq 2)\) be a projective \(u\)-covariant completely bounded symmetric \(k\)-linear map and \(m = \lfloor k+1 \rfloor^2 \). Then

(a) if \(k\) is odd, there are projective covariant representations \((\Phi_i, v_i, K_i)\), \(1 \leq i \leq m-1\) and \((\Psi_j, w_j, K_j)\), \(j = 1, 2\) \(of(G, A, \alpha)\), where \(v_i, 1 \leq i \leq m-1\) and \(w_j, j = 1, 2\) are projective unitary representations with the multiplier \(\omega\) and linear operators \(V_i \in B(H_i, H_{i+1})\), \(0 \leq i \leq m-2\), where \(H_0 = H\).
with \( \|V_0\| \cdot \|V_1\| \ldots \|V_{m-2}\| = \|\varphi\|_{scb}^{\frac{1}{2}} \) and \( W_j \in \mathcal{B}(H_{m-1}, K_j) \), \( j = 1, 2 \) with \( \|W_1^* W_1 + W_2^* W_2\| = 1 \) such that

\[
\varphi(a_1, \ldots, a_k) = V_0^* \Phi_1(a_1) V_1^* \cdots V_{m-2}^* \Phi_{m-1}(a_{m-1}).
\]

By Proposition 7 and the induction it follows that for all \( a_1, \ldots, a_k \in A \) and each \( g \in G \) with \( v_0 u_g = v_1(g) V_0 \) for each \( g \in G \). From relations 2 and 3, we have \( \|V_0\| \leq \|\varphi\|_{scb} \). In the proof of Lemma 3.1, [2], the equality

\[
\varphi_n(a_1, \ldots, A_k) = (V_0)_n^* (\Phi_1)_n(a_1) \psi_n(a_2, \ldots, A_{k-1})(\Phi_1)_n(a_k)(V_0)_n
\]

holds for all \( A_1, \ldots, A_k \in M_n(A) \). Hence \( \|\varphi\|_{scb} \leq \|\psi\|_{scb} \leq \|V_0\|^{2} \). So \( \|\varphi\|_{scb} = \|V_0\|^{2} \). Then we obtain (a) by induction.

(b) \( k \) even. By Proposition 7 and the induction it is sufficient to consider only the case \( k = 2 \). Let \( \varphi \) be a projective \( u \)-covariant completely bounded symmetric \( 2 \)-linear map from \( A^2 \) into \( \mathcal{B}(H) \). By Proposition 6, there is a projective \( u \)-covariant completely positive linear map \( \rho : A \longrightarrow \mathcal{B}(H) \) dominating \( \varphi \). From Proposition 7, we conclude that there are projective covariant representations \( (\Phi, v, K) \) of \( (G, A, \alpha) \), where \( v \) is a projective unitary representation with the multiplier \( \omega \), a linear operator \( V \in \mathcal{B}(H, K) \) with \( \|V\|^{2} \leq \|\varphi\|_{scb} \) and a self-adjoint operator \( W \in \mathcal{B}(K) \) with \( \|W\| = 1 \) such that \( \varphi(n, b) = V^* \Phi(a) W \Phi(b) V \) and \( W = v^* g W v_g \) for all \( a, b \in A \) and \( g \in G \). Then we obtain (b) by induction. \( \square \)

4 The extension on twisted crossed products of a projective \( u \)-covariant completely bounded symmetric multilinear map

Theorem 9 Let \( (G, A, \alpha) \) be a \( C^* \)-dynamical system with \( G \) amenable and let \( u : G \longrightarrow \mathcal{U}(H) \) be a projective unitary representation with the multiplier \( \omega \). If \( \varphi : A^k \longrightarrow \mathcal{B}(H) \) is a projective \( u \)-covariant completely bounded \( k \)-linear map, then there is a completely bounded \( k \)-linear map

\[
\rho : (A \times_\alpha G)^k \longrightarrow \mathcal{B}(H)
\]
given by

\[
\rho(f_1, \ldots, f_k) = \int_{G^k} \varphi(f_1(s_1), \alpha(s_1) f_2(s_2)), \ldots, \alpha(s_{k-1}) f_k(s_k)) \times u_{s_1} u_{s_2} \ldots u_{s_k} d\mu(s_1) d\mu(s_2) \ldots d\mu(s_k)
\]

for all \( f_1, \ldots, f_k \in A \).
for all $f_1, \ldots, f_k \in C_c(G, A)$, where $C_c(G, A)$ is the $*$-algebra of continuous functions from $G$ into $A$ with compact support.

**Proof:** The proof is divided into two cases.

(a) $k$ odd. By Theorem 8, for any projective $\mathfrak{P}$-algebras $\mathcal{P}$ such that $\mathcal{P}(H_i, \mathcal{P}_i)$, $1 \leq i \leq m - 1$ and $(\mathcal{P}_j, \mathcal{P}_j)$, $j = 1, 2$ of $(G, A, \alpha)$, where $v_i$, $1 \leq i \leq m - 1$ and $w_j$, $j = 1, 2$ are projector unitary representations with the multiplier $\omega$ and linear operators $V_i \in \mathcal{B}(H_{m-1}, H_{m-1})$, $0 \leq i \leq m - 2$, where

$$H_0 = H$$

such that

$$\varepsilon(a_1, \ldots, a_k) = V^*_\Phi^0 \Phi_1(a_1) V^*_V \cdots V^*_{m-2} \Phi_{m-1}(a_{m-1}).$$

$$[W^*_V \Phi_1(a_m) W_1 - W^*_V \Phi_2(a_m) W_2].$$

$$\Phi_{m-1}(a_{m+1}) V_{m-2} \cdots \Phi_1(a_k) V_0$$

for all $a_1, \ldots, a_k \in A$ and

$$v_i(g) V_0 = V_0 u_i g, v_i+1(g) V_i = V_i v_i(g),$$

$$v_{m-1}(g) W_j = W_j^* w_j(g)$$

for each $g \in G$, $i = 1, \ldots, m - 1$ and $j = 1, 2$.

We define $\Phi^i \times v_i$ and $\psi_j \times w_j$ by

$$(\Phi^i \times v_i)(f) = \int_G \Phi^i(f(s)) v_i(s) d\mu(s), 1 \leq i \leq m - 1$$

$$(\psi_j \times w_j)(f) = \int_G \psi_j(f(s)) w_j(s) d\mu(s), j = 1, 2$$

for every $f \in C_c(G, A)$.

From Theorem 3.3, [1] we see that $\Phi^i \times v_i$ (respectively, $\psi_j \times w_j$) extends a representation of $A \times \alpha G$ into $\mathcal{B}(H_i)$ (respectively $B(K_j)$), denoted again by $\Phi^i \times v_i$ (respectively, $\psi_j \times w_j$).

We define a $k$-linear map $\rho: (A \times \alpha G)^k \to \mathcal{B}(H)$ by

$$\rho(f_1, \ldots, f_k) =$$

$$V^*_\Psi^0 \Phi_1(v_1) V^*_V \cdots V^*_{m-2} \Phi_{m-1}(v_{m-1})(f_{m-1}).$$

$$[W^*_V \psi_1 \psi_2(f_m) W_1 - W^*_V \psi_2 \psi_2(f_m) W_2].$$

$$\Phi_{m-1}(v_{m+1})(f_m) V_{m-2} \cdots \Phi_1(v_1)(f_1) V_0$$

for each $f_1, \ldots, f_k \in A \times \alpha G$. We consider only the case $k = 3$ because the general case is similar. For $f_1, f_2, f_3 \in C_c(G, A)$, we have

$$\rho(f_1, f_2, f_3) =$$

$$W^*_\psi \Phi \times v(f_1) [W^*_V (\Phi_1 \times v_1)(f_2) W_1 -$$

$$W^*_V (\Phi_2 \times w_2)(f_2) W_2](\Phi \times v)(f_3) V =$$

$$\int \Phi^* \Phi(f_1(s_1)) v(s_1) [W^*_V \Psi_1(f_2(s_2)) w_1(s_2) W_1 -$$

$$W^*_V (f_2(s_2)) w_2(s_2) W_2]$$

$$\Phi(f_3(s_3)) v(s_3) V d\mu(s_1) d\mu(s_2) d\mu(s_3) =$$

$$\int \Phi^* \Phi(f_1(s_1)) [W^*_V \Xi_1 s_1 \Psi_1(f_2(s_2)) w_1(s_2) W_1 -$$

$$W^*_V (f_2(s_2)) w_2(s_2) W_2]$$

$$\Phi(f_3(s_3)) v(s_3) V d\mu(s_1) d\mu(s_2) d\mu(s_3) =$$

$$\int \Phi^* \Phi(f_1(s_1)) [W^*_V \Xi_1 \Psi_1(a_1) \Xi_1(f_2(s_2)) w_1(s_2) W_1 -$$

$$W^*_V (f_2(s_2)) w_2(s_2) W_2]$$

$$\Phi(f_3(s_3)) v(s_3) V d\mu(s_1) d\mu(s_2) d\mu(s_3) =$$
for every $f \in \mathcal{C}(G, A)$. From Theorem 3.3, [1] we see that $\Phi_i \times v_i$ extends to a representation of $A \times_{\alpha} G$ into $B(H_i)$ denoted again by $\Phi_i \times v_i$.

We define a $k$-linear map $\rho : (A \times_{\alpha} G)^k \rightarrow B(H)$ by

$$
\rho(f_1, \ldots, f_k) = \int_{G^k} \Phi(f_1(s))V^{*v_1}(f_1)\Phi(f_2(s))V^{*v_2}(f_2)\cdots \Phi(f_k(s))V^{*v_k}(f_k)\Phi(s)\, d\mu(s)
$$

for each $f_1, \ldots, f_k \in A \times_{\alpha} G$.

We consider only the case $k = 4$ because the general case is similar. Let $f_i \in \mathcal{C}(G, A), 1 \leq i \leq 4$. Then we have

$$
\rho(f_1, f_2, f_3, f_4) =
$$

$$
\int_{G^4} \Phi(f_1(s_1))V^{*v_1}(f_1)\Phi(f_2(s_2))V^{*v_2}(f_2)\Phi(f_3(s_3))V^{*v_3}(f_3)\Phi(f_4(s_4))V^{*v_4}(f_4)\Phi(s)\, d\mu(s)
$$

(b) $k$ even. By Theorem 8, there are projective covariant representations $(\Phi_i, v_i, H_i)$, $1 \leq i \leq m$ of $(G, A, \alpha)$, where $v_i, 1 \leq i \leq m$ are projective unitary representations with the multiplier $\omega$ and linear operators $V_i \in B(H_i, H_{i+1}), 0 \leq i \leq m - 1$, $\|\omega\| \cdot \|V_i\| \cdots \|V_{m-1}\| = \|\varphi\|_{\text{sec}}$ where $H_0 = H$ and $W = W^* \in B(H_m)$ with $\|W\| = 1$ such that

$$
\varphi(a_1, \ldots, a_k) = V^{*\Phi_1(a_1)}V^{*\Phi_2(a_2)}\cdots V^{*\Phi_m(a_m)}
$$

$$
W^{*\Phi_m(a_{m+1})}V_{m-1} \cdots V_i^{*\Phi_1(a_k)}V_0
$$

for all $a_1, \ldots, a_k \in A$ and

$$
\Phi_i \times v_i
$$

by

$$
(\Phi_i \times v_i)(f) = \int_G \Phi_i(f(s))v_i(s)d\mu(s), 1 \leq i \leq m-1
$$

for every $f \in \mathcal{C}(G, A)$.
Φ_1 (α_1 s_1) v_1 (s_2 s_3 s_4) V_0 \mu (s_1)
\int_{G^4} \Omega (s_2, s_3) \omega (s_2 s_3, s_4) V_0^* \Phi_1 (f_1 (s_1)) V_1^* \Phi_2 (α_1 s_2) (f_2 (s_2)) W
\Phi_2 (α_1 s_2 (f_2 (s_2))) V_1 \Phi_1 (α_1 s_2 s_3) (f_4 (s_4)) v_1 (s_2 s_3 s_4) V_0 \mu (s_1) \mu (s_2) \mu (s_3) \mu (s_4)
\int_{G^4} \Omega (s_2, s_3) \omega (s_2 s_3, s_4) V_0^* \Phi_1 (f_1 (s_1)) V_1^* \Phi_2 (α_1 s_2 (f_2 (s_2))) W
\Phi_2 (α_1 s_2 (f_2 (s_2))) V_1 \Phi_1 (α_1 s_2 s_3) (f_4 (s_4)) V_0 \mu (s_1) \mu (s_2) \mu (s_3) \mu (s_4)
\int \varphi (f_1 (s_1), α_1 s_2 (f_2 (s_2)), α_1 s_2 s_3 (f_4 (s_4))) u_1 u_2 u_3 u_4 \mu (s_1) \mu (s_2) \mu (s_3) \mu (s_4)

References: