DESIGNING ROBUST AW COMPENSATION FOR UNCERTAIN DISCRETE-TIME LINEAR SYSTEMS

ADDISON RÍOS-BOLÍVAR, FRANCKLIN RIVAS-ECHEVERRIA

Mérida, 5101, Venezuela
ilich@ula.ve, rivas@ula.ve

ABSTRACT
In this contribution an approach for the synthesis of controller with compensation of saturation in the actuators is presented. The technique is applied in discrete-time linear systems with politopical uncertainties, taking as bases a performance index on a transfer function that involves difference between the output signal of the actuator and the output of the controller and its effect on the control signal. Thus, the saturation in the actuators is considered like a disturbance in the performance of the action of the control signal, which allows to establish performance indices for the design of the control law (static output feedback), and for the synthesis of a gain of compensation for the problem of windup. Thus, the method exhibits certain properties of robustness in front of changes in the operation of the actuators. For the synthesis of the gains of static output feedback and the gain of compensation, a method derived from the techniques of robust optimal control in $H_2/H_\infty$ for linear systems with politopical uncertainties, is proposed, which is based on linear matrix inequalities (LMIs). In order to verify the properties of the propose technique a numerical example is also presented.

Keywords: Discrete-time Linear Systems. Static Output Feedback. Robust Control. Polytopical Uncertainty. Linear Matrix Inequalities (LMI). $H_2/H_\infty$ Norms.

1. INTRODUCTION
Actually, it is common to find control schemes for complex industrial processes that must satisfy multiple objectives and therefore must operate in different control modes. Each specific situation defines the mode of operation and each of the modes has a controller that is designed to meet performance objectives for that mode. Switching between operating systems is achieved by selecting the plant input from the outputs of a number of controllers in parallel, each of which corresponds to a particular mode. Then the switching of modes is the replacement of the plant’s input due to the output of a controller is replaced by another.

Because of the limitations (saturation) and controller’s switching, the control input to the processes will be different from the output of these controllers. When this happens, the controllers’s output is not properly leading the plant and the states of the plant are heavily updated. This situation is called windup. Overall, the windup is an inconsistency between the control input to the process and the internal states of the controller. Adverse effects of rebound are a significant performance deterioration, over-shooting and in some cases instability (Campo and Morari, 1990).

The control systems analysis and synthesis problem for linear, time-invariant systems (LTI) subject to limitations and substitutions of the plant inputs has two ways to be addressed: one is incorporating the limitations and substitutions in the control system’s design. The other way is attacking the problem through compensation, which consists of two stages: a first step concerns the design of the control system without considering the physical constraints of the actuators, so that they can be imposed different performance objectives. The second step considers the design of some compensation through an additional feedback to minimize the negative effects of actuator’s saturation.

In general, this technique is studied as the problem of anti-windup bumpless transfer (AWBT) or of anti-windup compensation (AW) problem. For the purpose of obtaining global compensators in (Kothare et al, 1994) provides a general framework for AWBT problem, which is based on the paradigm of designing a linear controller for ignoring the non-linear inputs and incorporate compensation in order to minimize the adverse effects of windup. The main disadvantage of these methods is not considering the effects of actuator’s saturation on the performance indices that are imposed to the controller design. In addition, changes in the actuator’s saturation limits are not taken into account which, from the robustness point of view, can be very inconvenient. Consequently, this measurement may be a limiting factor for the purposes of implementing the compensation in
In order to consider the robustness in the AWBT compensation design (Mulder et al., 2001) presents a general formulation of the multivariable AWBT synthesis problem. The resulting compensation is obtained by minimizing a weighted gain in $L_2$, using linear matrix inequalities, but action changes of the actuators are not considered and it is not evaluated the effects on performance indices of closed loop system when it is in saturation. Moreover, (Wada and Saeki, 2003) presents a method based on $L_2$ performance, where is obtained an anti-windup static compensator. In this case, both nominal performance and robust performance are reduced to a generalized eigenvalue problem. (Hu and Lin, 2001) presents the design of AW compensation for disturbance rejection of LTI discrete time systems, where, only considering state feedback provides conditions for an invariant set in the presence of persistent disturbance in the design of compensation.

In (Mulder and Kothare, 2001) it is shown the design of a dynamic controller and a static gain compensation simultaneously, which consider the effect of saturation on performance in closed loop, but does not take into consideration possible changes in the operating limits of the actuators, so there is not robustness. One approach used for the synthesis of AW compensation gain under robustness criteria, based on LMIs is presented in (Gomes da Silva Jr. and Tarbouriech, 2005). While (Cao and Lin, 2006) presents a design of anti-windup compensation for polytopic systems. There are used parameters dependent Lyapunov functions for reducing conservatism. Robustness is considered for the dynamic system uncertain. Moreover, in (Rios-Bolivar and Godoy, 2006) a technique for AWBT compensation for multivariable controllers, based on $H_\infty$ and $H_\infty$ norms characterization as linear matrix inequalities is provided. Robustness is considered for ensuring closed-loop performance in spite of unknown changes in the actuators saturation limits. In (Rios-Bolivar et al., 2010) a method for the synthesis of control systems with AW compensation for LTI uncertain systems is presented, based on a performance index as described in (Rios-Bolivar and Godoy, 2006). This paper presents a method for designing static feedback with robust compensation for DTLS based on $H_\infty$ norms. The synthesis of the feedback gain and the gain compensation are performed simultaneously, taking into account the closed loop performance and changes in the actuators saturation limits. The robust performance property is assessed on the closed loop transfer function between the actuator output and the controller output, which is considered as a disturbance signal. Thus, it can be consider the minimization of $H_\infty$-$H_\infty$ norms as a convex optimization problem for that transfer function, in conjunction with the closed loop performance specifications in the absence of saturation.

For the performance in closed-loop, the norms-based criteria are applied. Thus, there are improved versions of the bounded real lemma in order to minimize the conservatism (Oliveira et al., 2002, Xu et al., 2007, He et al., 2005), or performance in $H_\infty$ (Oliveira et al., 2002, Pipeleers et al., 2009), this in order to reduce conservatism.

2. INTRODUCTORY PROBLEM

For introducing the robustness problem in the design of anti-windup compensation, consider the following linear system:

$$
\begin{align*}
    x(k+1) &= A x(k) + B_1 \omega(k) + B_2 \sigma(u) \\
    z(k) &= C_1 x(k) + D_{11} \omega(k), \\
    y(k) &= C_2 x(k) + D_{21} \omega(k),
\end{align*}
$$

where $x \in \mathbb{R}^n$ are the states, $\omega \in \mathbb{R}^r$ are the disturbances, $u \in \mathbb{R}^p$ are the controls, $z \in \mathbb{R}^m$ are the controlled outputs, and $y \in \mathbb{R}^q$ are the measured outputs. $A$, $B_1$, $B_2$, $C_1$, $C_2$, $D_{11}$, $D_{21}$ are known matrices of appropriate dimensions. The nonlinear function $\sigma(\cdot)$ denotes the saturation of the actuator, which is defined by

$$
\sigma(u) = \begin{cases} 
    u_{\text{min}} & \text{if } u(k) < u_{\text{min}} \\
    u(k) & \text{if } u_{\text{min}} \leq u(k) \leq u_{\text{max}} \\
    u_{\text{max}} & \text{if } u(k) > u_{\text{max}},
\end{cases}
$$

This nonlinear function describes the constraints and/or replacements due to the operation of the actuators. In order to meet the control requirements, a dynamic controller with compensation can be considered:

$$
\begin{align*}
    \zeta(k+1) &= A_c \zeta(k) + B_c y(k) + E_c [\sigma(u) - u] \\
    u(k) &= C_c \zeta(k) + D_c y(k),
\end{align*}
$$

where $A_c$, $B_c$, $C_c$, $D_c$ correspond to the controller's dynamic matrices, which are design parameters. $E_c$ represents the compensation gain, which is also a design parameter. When saturation occurs, the signal $\vartheta = \sigma(u) - u$ is
nonzero and adverse effects are present. Considering the minimization of such effects by compensation, controller’s dynamic matrices $A$, $B$, $C$, and $D$, are designed without taking into account saturation while synthesizing the compensation gain $E$, for improving the closed loop performance. Let’s consider $T_{w}$, such a transfer function:

$$T_{w}(z) = \frac{A}{C_1 + D_1} = C_1(zI - A)^{-1}B_1 + D_1 \tag{1}$$

where

$$A = \begin{pmatrix} A + B_2D_1C_2 & B_2C_1 \\ B_2C_2 & A_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} B_1 + B_2D_1D_2 \\ B_2D_{21} \end{pmatrix}, \quad C_1 = (C_1 \quad 0), \quad D_1 = D_{11}. \tag{2}$$

**Problem 1:** According to performance requirements, without saturation it is necessary to design a dynamic controller (3) such that

1. The closed-loop system is asymptotically stable.
2. $\|T_w\|_2 < \mu$ or $\|T_w\|_\infty < \gamma$ where $\mu > 0$, $\gamma > 0$.

This is a typical problem in the robust optimal control literature. When saturation is activated it’s necessary to preserve stability properties. Then the synthesis problem consists in obtaining $E$, for ensuring effective compensation under any condition of robust stability for perturbed systems. The second condition, linked to the stability of bounded input systems, allows the compensation under robust stability conditions. Furthermore, it is necessary to consider robustness issues with respect to the actuators operation. In this sense, it must be designed the gain compensation for ensuring closed loop stability and minimizing the effect of disturbance signal $\vartheta$ over the controlled output signal depending on the control signal $u$ under saturation. For example, consider $z_u(k)$ as a controlled output defined by

$$z_u = D_xC_2x + C_\omega \zeta \tag{3}$$

corresponding to control $u(k)$ without considering the disturbance $\omega(k)$ effect. Then, the design problem of the compensation gain can be targeted from the $H_2$ and $H_\infty$ norms.

Consider the transfer function of disturbance $\vartheta$ and the controlled output $z_u$:

$$T_{w \vartheta}(z) = \frac{A}{C_2 \quad D_2} = C_2(zI - A)^{-1}B_2 + D_2 \tag{4}$$

where

$$B_2 = \begin{pmatrix} B_2 \\ E_\omega \end{pmatrix}, \quad C_2 = (D_xC_2 \quad C_\omega), \quad D_2 = 0.$$ 

**Problem 2:** Given the dynamic system (1), it’s desired to design the compensation gain for the controller (3), such that:

1. The closed-loop system is asymptotically stable.
2. The effect of $\vartheta$ over $z_u$ will be minimal, in some sense.

Therefore, it’s needed to design compensation mechanisms with robustness features.

In summary, it has been raised two problems: a typical robust optimal control problem and a gain compensation synthesis for AWBT robust problem. Both problems can be considered under the same optics as a multiobjective control problem. Under this formulation, the synthesis goals may be a mixture of $H_2$ performance, $H_\infty$ performance, disturbance rejection, which can be described using linear matrix inequalities, as has been established within the multi-objective controller’s synthesis. In (Rios-Bolivar, 2008), it has been solved this problem for systems with known dynamic matrices. The result there obtained, from a very conservative point of view, solves the problem of handling disturbances and certain variations in the actuators limits of action, but does not consider uncertainties in the dynamic model of the system to be controlled. Thus, for ensuring robustness in both the processes dynamics and actuators deterioration effect, it has to be considered the control systems design for handling disturbances and uncertainties, as will be presented below.

3. ROBUST CONTROL AND AW COMPENSATION

In this moment, we are interested in designing robust controllers and compensators for processes where polytopic uncertainties. Consider the uncertain system

$$x(k + 1) = A(x(k) + B_\omega(\omega(k) + B_\sigma(\sigma(k)))$$

$$z(k) = C_\omega x(k) + D_\omega(\omega(k)) \tag{5}$$

$$y(k) = C_y x(k),$$

where $x \in \mathbb{R}^n$ are the states, $\omega \in \mathbb{R}^r$ are the disturbances, $u \in \mathbb{R}^p$ are the controls, $z \in \mathbb{R}^m$ are the controlled outputs, and $y \in \mathbb{R}^q$ are the measured outputs. Additionally, the unknown matrices of the system form an uncertain polytope, ie belong to a convex hull, where

$$\Omega_N = \left\{ (A, B_\omega, B_\sigma, C, D_\omega) : (\alpha) \right\} = \sum_{i=1}^{N} \alpha_i (A_i, B_\omega i, B_\sigma, C, D_\omega), \tag{6}$$

$$i = \mathbb{N}$$

Consider the transfer function of disturbance $\vartheta$ and the controlled output $z_u$:

$$T_{w \vartheta}(z) = \frac{A}{C_2 \quad D_2} = C_2(zI - A)^{-1}B_2 + D_2 \tag{4}$$

where

$$B_2 = \begin{pmatrix} B_2 \\ E_\omega \end{pmatrix}, \quad C_2 = (D_xC_2 \quad C_\omega), \quad D_2 = 0.$$ 

$$A = \begin{pmatrix} A + B_2D_1C_2 & B_2C_1 \\ B_2C_2 & A_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} B_1 + B_2D_1D_2 \\ B_2D_{21} \end{pmatrix}, \quad C_1 = (C_1 \quad 0), \quad D_1 = D_{11}.$$
where \( \alpha_i \geq 0, y \sum_{i=1}^{N} \alpha_i = 1 \).

The matrices \((A, B_\alpha, C_\alpha, D_\alpha)(\alpha)\) are related functions of the uncertain parameter vector \( \alpha \in \mathbb{R}^N \) described by the convex combination of the vertices of the matrices \( A_i, B_{\alpha_i}, C_{\alpha_i}, D_{\alpha_i}, i = 1, 2, \ldots, N \).

As can be seen, the \( C_y \) matrix is considered known. This corresponds with the fact that it represents, from the practical standpoint, a fundamental and necessary issue for the control systems implementation, corresponding to the sensors, which allow us building controllers with the measured output feedback. Now, applying an static output feedback control according to the introductory problem, then

\[
u(k) = K_y(k) + E_c \vartheta(k), \tag{7}\]

where \( K \) is the output feedback static gain, to be designed. Thus, the closed loop matrices for \( T_{z\alpha} (z) \) are

\[
\begin{align*}
A &= A(\alpha) + B_u(\alpha) KC_y, \\
B &= [B_\alpha(\alpha) B_\alpha(\alpha) + B_u(\alpha) E_c], \\
C &= C(\alpha), \quad D = [D_\alpha(\alpha) 0].
\end{align*}
\]

Similarly, if, according to (7),

\[
\begin{align*}
z_u(k) &= K_y(k) + E_c \vartheta(k) \text{ then the closed loop matrices for } T_{z\omega} (z) \text{ are} \\
A &= A(\alpha) + B_u(\alpha) KC_y, \\
B &= [B_\alpha(\alpha) B_\alpha(\alpha) + B_u(\alpha) E_c], \\
C &= KC_y, \quad D = [0 0 E_c].
\end{align*}
\]

Obviously, for state feedback \( C_y = I \). Therefore, the gain can be obtained on the basis of our approach in two ways: 1) designing first the output feedback static gain, taking into account only the transfer function \( T_{z\alpha} (z) \) and then determine the compensation from the transfer function \( T_{z\omega} (z) \). 2) Designing, simultaneously, both gains. This is the case we will follow below.

3.1. Control and AW Compensation in \( \mathcal{H}_2 \)

**Theorem 3.1** Consider the system (5) on the polytope (6) with \( z_u(k) = K_y(i) + E_c \vartheta(k) \). It’s obtained a control law by static output feedback and AW compensation, as in (7), ensuring a sub-optimal performance in \( \mathcal{H}_2 \) for \( T_{z\omega} (z) \) and \( T_{z\vartheta} (z) \), from the following optimization problem:

\[
\min_{G, Y, R, X, x_i, \vartheta(i), i = 1, \ldots, N} \text{tr}(Z_j): \quad \text{such that}
\]

The following set of LMIs are equivalent:

\[
\begin{align*}
&\begin{bmatrix} X & * & * & * \\
X A^T + Y^T B_u^T & X & * & * \\
B_u^T & 0 & I & * \\
B_u^T & R^T B_u^T & 0 & 0 & I \\
\end{bmatrix} > 0, \\
&\begin{bmatrix} Z_2 & * & * & * \\
Z_1 & Y^T & X & * \\
0 & 0 & I & * \\
0 & 0 & 0 & I \\
\end{bmatrix} > 0
\end{align*}
\]

from which, the static output feedback and compensation gains are given by

\[
K = Y X^{-1} C_y (C_y C_y^T)^{-1}, \\
E_c = R.\tag{8}
\]

**Demonstration**: The proof is obtained by congruent transformation and variable changes.

3.2. Control and AW Compensation in \( \mathcal{H}_\infty \)

**Theorem 3.2** Consider the system (5) on the polytope (6) with \( z_u(k) = K_y(i) + E_c \vartheta(k) \). A control law is obtained using feedback control and static output for AW compensation in the form (7), ensuring a sub-optimal performance in \( \mathcal{H}_\infty \) for \( T_{z\omega} (z) \) and \( T_{z\vartheta} (z) \), from the following optimization problem:

\[
\min_{G, Y, R, X, x_i, \vartheta(i), i = 1, \ldots, N, j = 1, 2} \gamma_j^2: \quad \text{such that}
\]

The following set of LMIs are equivalent:

\[
\begin{align*}
&\begin{bmatrix} X & * & * & * \\
0 & I & * & * \\
0 & 0 & I & * \\
A_i X + B_u Y & B_{z\vartheta} & B_{z\vartheta} + B_u R & X \\
C_{z\vartheta} & X & D_{z\vartheta} & 0 & 0 & \sqrt{2} I \\
0 & * & * & * \\
0 & 0 & I & * \\
A_i X + B_u Y & B_{z\omega} & B_{z\omega} + B_u R & X \\
Y & 0 & R & 0 & \sqrt{2} I \\
\end{bmatrix} > 0, \\
&\begin{bmatrix} Z_2 & * & * & * \\
Z_1 & Y^T & X & * \\
0 & 0 & I & * \\
0 & 0 & 0 & I \\
\end{bmatrix} > 0
\end{align*}\tag{9}
\]
The results obtained in $\mathcal{H}_\infty$ and $\mathcal{H}_2$ can be combined in order to obtain multiobjective controllers.

4. NUMERICAL EXAMPLE

For an illustrative example, consider the variant unstable dynamic system shown in (Lee et al., 2006):

\[
\begin{align*}
\dot{x}(k+1) &= \begin{pmatrix} \alpha & 0.3 & 2 \\ 1 & 0 & 1 \\ 0.3 & 0.6 & 0.6 \end{pmatrix} x(k) + \\
& \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \omega(k) + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \sigma(u) \\
\omega(k) &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sigma(u) \\
y(k) &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} x(k),
\end{align*}
\]

where the parameter $\alpha \in [2, 2.7]$. For different values of $\alpha$, the open loop system is unstable.

In this example, we consider the saturation of the actuators in the sense that $\sigma(u) \in [-6.2, 6.2]$. First, for the purposes of assessing stability under uncertainty, we consider the design of a static gain output feedback without taking into account the saturation of the actuators. Then, applying Theorem 3.2 yields the following gain:

\[
K = \begin{pmatrix} -1.1198 \\ -0.3840 \end{pmatrix},
\]

\[
E_c = \begin{pmatrix} -0.9954 & -0.0148 \\ 0.0165 & -0.8736 \end{pmatrix},
\]

obtaining also the performance indices $\|T_{z_2}(z)\|_\infty < 0.9255$ and $\|T_{x_3}(z)\|_\infty < 0.2700$. Consequently, considering the saturation of the actuators in the design of the controller, it is improved the performance index of the disturbances on the controlled output effect. All these poles belong to the stable region; therefore the closed loop system is robustly stable.

For evaluating the compensation effect, as in the Figure 3 is shown the timing behavior with respect to changes without compensation is improved. The saturation effect is minimized in the actuator saturation is present and there is not compensation, the temporal behavior deteriorates, as shown in Figure 2. This suggests the gain compensation design. Thus, applying again Theorem 3.2, the static output feedback gain and compensation gain are obtained:

\[
K = \begin{pmatrix} -1.1186 \\ -0.7739 \end{pmatrix},
\]

\[
E_c = \begin{pmatrix} -0.9954 & -0.0148 \\ 0.0165 & -0.8736 \end{pmatrix},
\]

showing the temporal behavior of the controlled outputs when there is not saturation in the actuators. It shows the stabilization of the closed loop system.
the sense that steady state is reached in less than
about the behavior without compensation,
approximating the time evolution in the absence
of saturation in the actuators.

Figure 3: Controlled outputs under compensation.

5. CONCLUSIONS
From the output, it has been proposed and solved
the static output feedback control and AW
compensation problem in discrete time linear
systems with polytopic uncertainties. The
synthesis method consists of designing two gains,
which corresponds to the static output feedback
and the other is the AW compensation gain.
Therefore, under the robust stability premises and
disturbance handling, design arises in the context
of optimal robust control with multi-objective
criteria, from which gains are obtained under the
premises of the definition for $H_2$ and $H_\infty$
norms as linear matrix inequalities. Consequently, the
extended output feedback and AW compensation
gains are obtained by solving sets of LMIs
according to the multi-objective criteria imposed.
The results, as well as formal verification, have
been evaluated numerically, and indicate the
effectiveness of the robust AW compensation.

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