Mathematical models for analysing diatonic modes

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Abstract: - The study herein aims to present two mathematical models for analysing heptachordic diatonic modes. We chose to approach the issue through a structuralist method, the first mathematical model being based on permuting some structures inside the octavian structures as this fact in itself allows a better visualisation of the characteristics of the modes; while the second model is based on juxtaposing tetrachords so that the final result is analogous to the one that is achieved if the first analytical model is applied.

Key-Words: - modality, structure, mathematical model, non-retrogradable, recurrence, mathematical formula

1 Diatonic modes
The ideas we are going to point out are somewhat correspondent to those defining real music, case in which the intervals of every mode bear a function and are hierachized in terms of certain criteria and are perceived as manifesting certain attractions and affinities among them. Therefore, we will talk only about models analysis. The tones and semitones from any melodic structure have never been equal. We set the limit at diatonic tones only, be them Greek, Gregorian, Folk or Acoustic as they are composed of two elements, tones and semitones that we may represent in letters or two numbers: (T, st) or (2, 1).

2 Two models for analysis
Any mode may be reduced to a scheme. This was possible only in European music, once the octave pitch has been mathematized. As the octave has no more than fine tones and two semitones, we may define a set made of seven elements as follows: \( A = \{st_1, st_2, T_1, T_2, T_3, T_4, T_5\} \). By permuting the elements within this set there may result a multiple number of permutations incurred by the formula:

\[
(1) \quad P_n = n!,
\]

which means a total number of 5040 structures. The result is irrelevant as in modes the way that tone are arrangement among them is not important. However, if one takes into account the distribution of tones and semitones within the limits of an octave, one may notice several situations determined by the number of tones in between semitones as well as by the way that this substructure is translated inside the larger one (we are in fact dealing with a cyclic structure). In fact, the mere positioning of the semitones if compared to that of the tones, simplifies a to a great extend the way such an issue needs to be approached.

The second analytical model considers the tetrachordic structures, their typology, the combinations as well as the way they are juxtaposed.

3 Solutions
We need to find the solutions offered by each analytical model one at the time.

3.1 The first mathematical model
The first mathematical model is based on the idea of not adjoining semitones but on creating some structures made of three, four, five and six elements, and on the cyclic permutation of there structures. Thus, the following substructures may result: \((st \ T st), (st \ T T st), (st \ T T T st)\) and \((st \ T T T T st)\).
3.1.1 Substructure \((st\ T\ st)\)

We therefore take into account the first case in which we separate the semitones from a single tone. Thus, we obtain the structure \((st\ T\ st)\) whose movement from left to right generates five corresponding structures to some diatonic modes that have already been theorized in music practice:

Fig.1

\[(st\ T\ st)\ T\ T\ T\ T\ \rightarrow\ \text{Acoustic 4}\]

\[T\ (st\ T\ st)\ T\ T\ T\ \rightarrow\ \text{Acoustic 3 (Hystrian)}\]

\[T\ T\ (st\ T\ st)\ T\ T\ \rightarrow\ \text{acoustic 2 (Major Melodic)}\]

\[T\ T\ T\ (st\ T\ st)\ T\ \rightarrow\ \text{Acoustic 1 (Lydian - Mixolydian)}\]

\[T\ T\ T\ (st\ T\ st)\ \rightarrow\ \text{Acoustic 7}\]

We may note the fact that the first with the last structure, the second with the fourth, are of recurrent type while the third in nonOretrogradable. We point out this fact as if sung descending it is identical, structurally speaking, to the recurrent mode that is sung ascending.

3.1.2 Substructure \((st\ T\ T\ st)\)

We hereby present the following substructure that generates, through the same procedure, four modal structures and which is characterized by the fact that the great number of elements from its composition determines a lesser number of transpositions:

Fig.2

\[(st\ T\ T\ st)\ T\ T\ \rightarrow\ \text{Locrian}\]

\[T\ (st\ T\ T\ st)\ T\ T\ \rightarrow\ \text{Aeolian (Minor)}\]

\[T\ T\ (st\ T\ T\ st)\ T\ \rightarrow\ \text{Mixolydian}\]

\[T\ T\ T\ (st\ T\ T\ st)\ \rightarrow\ \text{Lydian}\]

The first structure with the last and the second with the third are recurrent structures and the parity frame does not allow non-retrogradable structures.

3.1.3 Substructure \((st\ T\ T\ T\ st)\)

As noted in the previous case, the number of resulted structures is reduced to three:

Fig.3

\[(st\ T\ T\ T\ st)\ T\ T\ \rightarrow\ \text{Phrygian}\]

\[T\ (st\ T\ T\ T\ st)\ T\ \rightarrow\ \text{Dorian}\]

\[T\ T\ (st\ T\ T\ T\ st)\ \rightarrow\ \text{Ionian (major)}\]

The Phrygian as well as the Ionian and recurrent structures while the Dorian is a non-retrogradable one.

3.1.4 Substructure \((st\ T\ T\ T\ T\ st)\)

The last substructure that we intend to present generates only two structures, both recurrent:

Fig.4

\[(st\ T\ T\ T\ T\ st)\ T\ \rightarrow\ \text{Acoustic 6 (Dorian - Phrygian)}\]

\[T\ (st\ T\ T\ T\ T\ st)\ \rightarrow\ \text{Acoustic 5 (Minor melodic)}\]

We thus note the presence of a total number of heptachordic structures of diatonic nature.

3.2 The second mathematical model

Musical practice is familiar with nine types of tetrachords that will be merely pointed out without theorizing them: Major, Major modified, Harmonic, Minor, Phrygic, Lydic, Lydic modified, Exotic and Diminished. From these, those that are made up of augmented intervals are no longer taken into account. We therefore intend to analyze five tetrachords: Major, Minor, Phrygic, Lydic, and Diminished. Each tetrachord is to be noted with the greek letters \(\alpha, \beta, \gamma, \delta, \epsilon\).

The first three are based on a structure made up of two tones and one semitone. They result from the cyclic permutation of the three elements. Their diagram shall be described as following:

Fig.5

\[\alpha = (T\ T\ st)\]

\[\beta = (T\ st\ T)\]
\[ \gamma = (st \ T \ T) \]

The following structures either do not contain any semitones in their composition, or contain two semitones without their being adjoined. It therefore results:

Fig.6
\[ \delta = (T \ T \ T) \]

\[ \epsilon = (st \ T \ st) \]

### 3.2.1 Juxtaposing tetrachords through tone

In case tetrachords are juxtaposed through tone, the formula that needs to be applied in order to find out the total number of results is:

\[ (3) \quad A_n^k = \frac{n!}{(n-k)!} \]

to which one may also add number five which constitutes the number of arrangements presupposed by the repetition of elements, a number that is equal to the cardinal number of the set. The result is as follows:

\[ (4) \quad A_5^2 = \frac{5!}{(5-2)!} + 5 = 25 \]

The total of the results are presented in the table depicted above emphasizing on the arrangements presupposing the repetition of elements:

<table>
<thead>
<tr>
<th>( \alpha T\alpha )</th>
<th>( \beta T\alpha )</th>
<th>( \gamma T\alpha )</th>
<th>( \delta T\alpha )</th>
<th>( \epsilon T\alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha T\beta )</td>
<td>( \beta T\beta )</td>
<td>( \gamma T\beta )</td>
<td>( \delta T\beta )</td>
<td>( \epsilon T\beta )</td>
</tr>
<tr>
<td>( \alpha T\gamma )</td>
<td>( \beta T\gamma )</td>
<td>( \gamma T\gamma )</td>
<td>( \delta T\gamma )</td>
<td>( \epsilon T\gamma )</td>
</tr>
<tr>
<td>( \alpha T\delta )</td>
<td>( \beta T\delta )</td>
<td>( \gamma T\delta )</td>
<td>( \delta T\delta )</td>
<td>( \epsilon T\delta )</td>
</tr>
<tr>
<td>( \alpha T\varepsilon )</td>
<td>( \beta T\varepsilon )</td>
<td>( \gamma T\varepsilon )</td>
<td>( \delta T\varepsilon )</td>
<td>( \epsilon T\varepsilon )</td>
</tr>
</tbody>
</table>

For the total of results we would need to subtract those combinations exceeding the limits of the perfect octave by modifying the established number of tones and semitones or by using scales that are of no practical use.

The first column brings three octavian modes as the others go past the limits of the perfect octave:

Fig.7
\( \alpha T\alpha \) - Ionian (Major)
\( \alpha T\beta \) - Mixolydian
\( \alpha T\gamma \) - Acoustic 2 (Major Melodic)
\( \alpha T\delta \) - 8+
\( \alpha T\varepsilon \) - 8-

The second column is similar:

Fig.8
\( \beta T\alpha \) - Acoustic 5 (Minor Melodic)
\( \beta T\beta \) - Dorian
\( \beta T\gamma \) - Aeolian (Minor Natural)
\( \beta T\delta \) - 8+
\( \beta T\varepsilon \) - 8-

The third column brings two familiar structures, the second and the third. The first is of no practical use while the fourth and the fifth go past the limits of the perfect octave:

Fig.9
\( \gamma T\alpha \) - of no practical use
\( \gamma T\beta \) – Acoustic 6 (Dorian - Phrygian)
\( \gamma T\gamma \) - Phrygian
\( \gamma T\delta \) - 8+
\( \gamma T\varepsilon \) - 8-

The fourth column brings only one familiar structure, three of augmented octave and even one of double augmentation:

Fig.10
\( \delta T\alpha \) - 8+
\( \delta T\beta \) - 8+
\( \delta T\gamma \) - 8+
\( \delta T \delta - 8++ \)

\( \delta T \epsilon - \text{Acoustic 7} \)

Finally, the last column brings a result that is similar but is reversed:

Fig.11
\( \varepsilon T \alpha - 8- \)

\( \varepsilon T \beta - 8- \)

\( \varepsilon T \gamma - 8- \)

\( \varepsilon T \delta - \text{Acoustic 4} \)

\( \varepsilon T \epsilon - 8-- \)

A general outlook on the issue would lead us to the conclusion that from those twenty five structures presented initially, only ten are standing. The other fourteen are unravelled by the situation in which tetrachords are juxtaposed through semitone.

3.2.2. Juxtaposing tetrachords through semitone
In case tetrachords are juxtaposed through semitone, the results are similar to the previous one but only to a certain extend. The calculus formula is identical and the table is configured in the same way, the only different lying in the fact the the tone is substituted by semitone:

Table 2

<table>
<thead>
<tr>
<th>asta</th>
<th>bsta</th>
<th>ystā</th>
<th>dsta</th>
<th>esta</th>
</tr>
</thead>
<tbody>
<tr>
<td>astb</td>
<td>bstb</td>
<td>ystb</td>
<td>dstb</td>
<td>estb</td>
</tr>
<tr>
<td>astγ</td>
<td>bstγ</td>
<td>ystγ</td>
<td>dstγ</td>
<td>estγ</td>
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<tr>
<td>astδ</td>
<td>bstδ</td>
<td>ystδ</td>
<td>dstδ</td>
<td>estδ</td>
</tr>
<tr>
<td>astε</td>
<td>bstε</td>
<td>ystε</td>
<td>dstε</td>
<td>estε</td>
</tr>
</tbody>
</table>

The results may be displayed much quicker when first viewing the table. Thus, one may note that the first column and the fifth disappear from our analysis due to semitone adjoining. We note only four valid structures, two from the fourth row and second and third column and two from the fourth column. All the other structures are not valid as they neither fit into the limits imposed by the perfect octave, nor do they bring adjoining semitones. The new structures are presented below in corresponding order:

Fig.12
\( \beta st \delta - \text{Acoustic 3 (hystrian)} \)

\( \gamma st \delta - \text{Locrian} \)

\( \delta st \alpha - \text{Lydian} \)

\( \delta st \beta - \text{Acoustic 1} \)

Thus, the second mathematical model also detects a total of fourteen diatonic heptachordic structures.

4 Conclusions
When referring to the first mathematical model, we note the fact that we avoided two structures: \((st\ st)\) and \((st\ T\ T\ T\ T\ st)\), meaning that in which we avoid the juxtaposition of semitones and that in which the semitones were exposed to extremities being therefore divided in five tones. Taking all these into account, not to mention the fact that permuting the tones, as well as semitones, among them, is not a relevant fact for music, the formula for obtaining all possible modes becomes:

\[
(5) \quad P_7 = \sum_{k=1}^{6} k(2P_5 - 1), \quad \text{where} \quad k \in N
\]

the result is 21, much closer to what we all know. Yet, if we subtract the six structures obtained as a result of the cyclic permutation of the structure \((st\ st)\), as well as the only structure with semitones in extremities, we get exactly the result we need. We get the very same result by applying the second model and by taking into account the very same assumptions.

References: