Vertical Non-linear Vibrations of the Automobiles

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Abstract: - It is known that the vibrations are an important source of noise and discomfort. In our paper we purpose a non-linear model with two degrees of freedom for the vertical vibrations of the automobiles. The influence of the road is considered to be a harmonic one. For this model we obtain the equations of motion, we study the equilibrium positions and their stability and, also, the stability of the motion. Finally, we complete discuss a numerical application.

Key-Words: - non-linear, vibrations, equilibrium, motion, stability, viscous damping

1 Introduction
The automobiles’ vibrations have a very great importance for the passengers’ comfort, for the durability of the car and for the degree of the noise felt by the automobile’s occupants.

In our paper we purpose to study a non-linear two degrees of freedom model for the automobile’s vibrations in a vertical plane.

2 Mathematical model
Let us consider an automobile schematized as two bodies of masses $m_1$ and $m_2$ which oscillate in a vertical plan. Between the bodies 1 and 2 there exist a non-linear spring and a visco-elastic damper, and the body 1 is linked to the ground by a non-linear spring and a visco-elastic damper (Fig. 1).

For the non-linear springs we assume that the elastic force that appears in such a spring is given by

$$ F_e = ky + ey^3, $$

where $y$ is the elongation of the spring.

Because of the non-uniformity of the road, the ground acted on the automobile with the excitation $z$ which is considered to be a harmonic-type one,

$$ z = E \cos \omega t, $$

where $E$ is its amplitude.

3 The equations of motion
Denoting by $z_1$ and $z_2$ the displacements of the two masses, by $\dot{z}_1$ and $\dot{z}_2$ their velocities, then the elastic and damping forces that appear in the system (Fig. 2) have the expressions:

$$ F_{e01} = k_1(z_1 - z) + e_1(z_1 - z)^3, $$
$$ F_{e02} = k_2(z_2 - z_1) + e_2(z_2 - z_1)^3, $$
$$ F_{d01} = c_1(\dot{z}_1 - \dot{z}), $$
$$ F_{d12} = c_2(\dot{z}_2 - \dot{z}_1). $$

Isolating the two bodies, one obtains the equations of motion.
4 The equilibrium positions in the absence of the external excitation

In this case \( z = 0 \) and the system (8) transforms in

\[
\begin{align*}
\ddot{\xi}_1 &= -k_1(\xi_1 - z) - e_1(\xi_1 - z)^3 - c_1(\dot{z}_1 - \dot{z}) - \frac{m_1 g + k_2(z_2 - z_1)^3 + e_2(z_2 - z_1)^3 + c_2(\dot{z}_2 - \dot{z}_1)}{m_2}, \\
\ddot{\xi}_2 &= -k_2(z_2 - z_1)^3 - e_2(z_2 - z_1)^3 - c_2(\dot{z}_2 - \dot{z}_1) - m_2 g.
\end{align*}
\] (7a)

Denoting \( \xi_1 = z_1, \xi_2 = z_2, \xi_3 = \dot{z}_1, \xi_4 = \dot{z}_2 \), it results the following system of four ordinary non-linear first order differential equations

\[
\begin{align*}
\dot{\xi}_1 &= \xi_3, \\
\dot{\xi}_2 &= \xi_4, \\
\dot{\xi}_3 &= -k_1(\xi_1 - z) + e_1(\xi_1 - z)^3 - \frac{c_1(\dot{z}_1 - \dot{z})}{m_1} + k_2(\xi_2 - \xi_1)^3 + e_2(\xi_2 - \xi_1)^3 - \frac{c_2(\dot{z}_2 - \dot{z}_1)}{m_1}, \\
\dot{\xi}_4 &= -k_2(\xi_2 - \xi_1)^3 + e_2(\xi_2 - \xi_1)^3 - \frac{c_2(\dot{z}_2 - \dot{z}_1)}{m_2} - \frac{m_1}{m_2}(\xi_4 - \xi_3) - g.
\end{align*}
\] (7b)

The equilibrium positions are at the intersections of the nullclines; therefore we shall equate to zero the right-hand terms in the relations (9). It follows immediately that \( \xi_3^{eq} = 0, \xi_4^{eq} = 0 \) and, multiplying the third and the fourth expression (9) and summing, one gets the equation

\[
e_1\xi_3^{eq} + k_1\xi_1 + (m_1 + m_2)g = 0. \tag{10}
\]

Making now \( \xi_1 \mapsto -\xi_1 \), it results the equation

\[
e_1\xi_3^{eq} + k_1\xi_1 - (m_1 + m_2)g = 0. \tag{11}
\]

If \( e_1 > 0 \) then the equation (10) has exactly one negative real root and therefore there exists only one equilibrium position \( \xi_3^{eq} \). If \( e_1 < 0 \) then the equation (10) has a positive real root and two negative real roots. The equilibrium position for \( \xi_1 \) is unique determined if and only if \( e_1 > 0 \).

Recalling the fourth equation (9) and denoting \( \xi_2 - \xi_1 = u \), one obtains the equation

\[
e_2u^3 + k_2u + m_2g = 0 \tag{12}
\]

and making \( u \mapsto -u \), it results the equation

\[
e_2u^3 + k_2u - m_2g = 0. \tag{13}
\]

Proceeding in an analogous way, we deduce that the equilibrium position for \( \xi_2 \) is unique if and only if \( e_2 > 0 \).
Obviously, the equilibrium position is also unique for \( e_1 = 0 \) and \( e_2 = 0 \), in the linear case.

Further on we shall consider that \( e_1 > 0 \) and \( e_2 > 0 \).

5 The stability of the equilibrium

Let us denote by \( f_k(\xi_1, \xi_2, \xi_3, \xi_4) \) the expressions of the right-hand terms in the relations (9) and by \( j_{kl} = \frac{\partial f_k}{\partial \xi_l} \), \( k, l = 1, 4 \), the corresponding partial derivatives. Keeping into account that the all partial derivatives \( j_{kl} \), \( k, l = 1, 4 \), are zero, excepting \( j_{13}, j_{24}, j_{31}, j_{32}, j_{33}, j_{34}, j_{41}, j_{42}, j_{43}, \) and \( j_{44} \), it results the characteristic equation

\[
\left| \begin{array}{cccc}
-\lambda & 0 & 1 & 0 \\
0 & -\lambda & 0 & 1 \\
j_{31} & j_{32} & j_{33} - \lambda & j_{34} \\
j_{41} & j_{42} & j_{43} & j_{44} - \lambda \\
\end{array} \right| = 0 
\]

or, equivalently,

\[
(\lambda^4 - (j_{33} + j_{44})\lambda^2 - (j_{31}j_{42} - j_{33}j_{44})\lambda + j_{31}j_{42} - j_{32}j_{41}) = 0, 
\]

wherefrom

\[
\lambda^4 - (j_{33} + j_{44})\lambda^2 - (j_{31}j_{42} - j_{33}j_{44})\lambda + j_{31}j_{42} - j_{32}j_{41} = 0, 
\]

Denoting

\[
b_0 = 1 > 0, 
\]

\[
b_1 = -(j_{33} + j_{44}) = \frac{c_1}{m_1} + \frac{c_2}{m_2} > 0, 
\]

\[
b_2 = -(j_{31} + j_{42} - j_{33}j_{44}) = \frac{k_1 + 3e_1^{2}\xi_1^2}{m_1} + \frac{k_2 + 3e_2(\xi_2 - \xi_1)^2}{m_1} + \frac{c_1c_2}{m_1m_2} > 0, 
\]

\[
b_3 = j_{33}j_{42} + j_{44}j_{31} - j_{41}j_{34} = \frac{c_1}{m_1} \frac{k_2 + 3e_2(\xi_2 - \xi_1)^2}{m_2} + \frac{c_2}{m_2} \frac{k_1 + 3e_1^{2}\xi_1^2}{m_1} > 0, 
\]

\[
b_4 = j_{31}j_{42} - j_{42}j_{31} = \frac{k_1 + 3e_1^{2}\xi_1^2}{m_1} \times \frac{k_2 + 3e_2(\xi_2 - \xi_1)^2}{m_2} > 0, 
\]

it results that the stability can be studied with the aid of the Routh–Hurwitz criterion, respectively if are fulfilled the following conditions

\[
b_i > 0, \ i = 1, 4, 
\]

\[
|b_0 b_2| = b_0b_2 - b_0b_3 > 0, 
\]

\[
|b_1 b_0 0| = b_1b_2b_3 - b_0b_3^2 > 0, 
\]

\[
|b_3 b_2 b_1 b_0| = b_4(b_1b_2b_3 - b_0b_3^2) > 0. 
\]

It is easy to verify that these relations hold true and therefore the equilibrium is stable.

6 The stability of the motion

Let us recall back the system (8) and let us consider that we give the deviations \( \xi_i \rightarrow \xi_i + \eta_i, \ i = 1, 4 \), to the system’s solution \( \xi_i, \ i = 1, 4 \). It results the system in deviations

\[
\begin{align*}
\dot{\eta}_1 & = \eta_3, \\
\dot{\eta}_2 & = \eta_4, \\
\dot{\xi}_3 + \dot{\xi}_3 & = -\frac{k_1(\xi_1 + \eta_1 - z)}{m_1} + \frac{e_1(\xi_1 + \eta_1 - z)^3}{m_1} + \\
& + \frac{k_2(\xi_2 - \xi_1)^3}{m_1} + \frac{c_1c_2}{m_1m_2} - \frac{c_1}{m_1} (\xi_3 + \eta_3 - z) + \\
& + \frac{c_2}{m_2} (\xi_4 - \xi_3 + \eta_4 - \eta_3 - g), \\
\end{align*}
\]
\[ \ddot{\xi}_4 + \dot{\eta}_4 = -\frac{k_2(\ddot{\xi}_2 - \ddot{\xi}_1 + \ddot{\eta}_2 - \ddot{\eta}_1)}{m_2} - \frac{\varepsilon_2(\ddot{\xi}_2 - \ddot{\xi}_1 + \ddot{\eta}_2 - \ddot{\eta}_1)^3}{m_2} - \frac{c_2(\ddot{\xi}_3 - \ddot{\eta}_3 + \ddot{\eta}_4 - \ddot{\eta}_3) - g}{m_2}. \] (26d)

Keeping into account that \( \ddot{\xi}_i, \ i = 1, 4 \), are the solutions of the system (8) and keeping only the linear terms in \( \ddot{\eta}_i, \ i = 1, 4 \), the system (26) becomes

\[ \ddot{\eta}_1 = \ddot{\eta}_3, \] (26a)
\[ \ddot{\eta}_2 = \ddot{\eta}_4, \] (26b)
\[ \ddot{\eta}_3 = -\frac{k_1 \ddot{\eta}_1}{m_1} - \frac{3 \varepsilon_1 (\ddot{\eta}_3 - z)^2 \ddot{\eta}_1}{m_1} + \frac{k_2(\ddot{\eta}_2 - \ddot{\eta}_1)}{m_1} + \frac{3 \varepsilon_2(\ddot{\xi}_2 - \ddot{\xi}_1)^2(\ddot{\eta}_2 - \ddot{\eta}_1)}{m_1} - \frac{c_1 \ddot{\eta}_3 + c_2 (\ddot{\eta}_4 - \ddot{\eta}_3)}{m_1}, \] (26c)
\[ \ddot{\eta}_4 = -\frac{k_2(\ddot{\eta}_2 - \ddot{\eta}_1)}{m_2} - \frac{3 \varepsilon_2(\ddot{\xi}_2 - \ddot{\xi}_1)^2(\ddot{\eta}_2 - \ddot{\eta}_1)}{m_2} - \frac{c_2 (\ddot{\eta}_4 - \ddot{\eta}_3)}{m_2}. \] (26d)

From this point the discussion is similar to that from the paragraph 5.

## 7 Numerical application

Let us consider the case defined by the following numerical parameters \( m_1 = 100 \text{ kg}, \ m_2 = 1200 \text{ kg}, \ k_1 = 1.5 \times 10^4 \text{ N/m}, \ k_2 = 7.5 \times 10^4 \text{ N/m}, \ \varepsilon_1 = 20 \text{ N/m}^3, \ \varepsilon_2 = 200 \text{ N/m}^3, \ c_1 = 10^3 \text{ Ns/m}^3, \ c_2 = 10^5 \text{ Ns/m}^3, \ \omega = 2 \text{ s}^{-1}, \ g = 10 \text{ m/s}, \ E = 0.05 \text{ m}.

The equilibrium position is obtained from the equations
\[ 20z_1^3 + 1.5 \times 10^6 z_1 + 13000, \] (27)
\[ 200(z_2 - z_1)^3 + 7.5 \times 10^4 \times (z_2 - z_1) + 12000 = 0. \] (28)

Applying the Lobacevski–Graeffe method, result the values \( z_1 = -0.00866 \text{ m}, \ z_2 = -0.16865 \text{ m} \).

Choosing as initial values \( v_1^0 = -0.0025 \text{ m}, \ v_2^0 = -0.05 \text{ m}, \ \dot{v}_1^0 = 0 \text{ m/s}, \ \dot{v}_2^0 = 0 \text{ m/s}, \) and considering that the excitation has non-zero values only for \( 0 \leq t \leq 2 \text{ s} \), we captured the time history for different variables in the next figures.

![Fig. 3. Time history of \( z_1 \) for \( 0 \leq t \leq 5 \text{ s} \).](image3)

![Fig. 4. Time history of \( z_2 \) for \( 0 \leq t \leq 5 \text{ s} \).](image4)

![Fig. 5. Time history of \( \dot{z}_1 \) for \( 0 \leq t \leq 5 \text{ s} \).](image5)

![Fig. 6. Time history of \( \dot{z}_2 \) for \( 0 \leq t \leq 5 \text{ s} \).](image6)

## 8 Conclusions

In our paper we described a non-linear model for the study of the vertical vibrations of an automobile. We
obtained the differential equations of motions, we studied the equilibrium positions and we proved that the equilibrium position is unique if and only if $\varepsilon_1$ and $\varepsilon_2$ are positive values. The stability of the motion can be studied analytical only in very particular cases, the general case being a numerical study. Finally, we consider a numerical application with the non-zero excitation only for a period of time and for this application we obtained that the motion is stable.

References: