An efficient and precise numerical-time-integration scheme for dynamic analysis

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Abstract: In this paper, an efficient and accurate step-by-step time integration algorithm with unconditional stability for dynamic analysis is presented. For a dynamic system under impulsive loadings with load discontinuities, a very small time step is usually required to provide acceptable levels of accuracy in performing time history analysis. To improve this drawback in time integration, the principle of momentum is available to integrate an impulsive loading with respect to time so that the load discontinuity in a time interval can be smoothed out through another expression of external momentum. Besides, time finite element methods feature a larger time step to represent a continuous response over a time step as two nodal responses of the time step. Combining both computational advantages described above, this study presents an improved time integration scheme with the properties of larger time step size and accuracy. From theoretical analysis of stability and accuracy, the proposed algorithm possesses the efficient features of unconditional stability, accuracy, and fast convergence in numerical computation. For simplicity of demonstration, a closed form solution of an excited single-degree-of-freedom (SDOF) system is employed to verify the feasibility and reliability of the new time integration algorithm, with which the numerical results obtained from the Newmark method are compared.

Key-Words: accuracy; time integration method; momentum; time element method; unconditionally stability.

1 Introduction

Step-by-step time integration methods are a widely useful approach to compute the time history response of structural dynamics for their simple and efficient nature. However, most of time integration techniques in dealing with the time history response of a structure subject to an impulsive load or a discontinuous loading, such as a rectangular impulse or sequential triangular impulses, very small time step is needed to solve the shock response of the structure. The reason for this is that the dynamic loading in the time domain appears discontinuity or rapid changes at the end of the impulse-type force [1]. Because of this, Chang [2-4] adopted the principle of momentum to smooth out the rapid changes of dynamic loadings and further convert the conventional second-order equation of motion in force equilibrium into another first-order momentum equation. He concluded that the discontinuity problem in an impulse loading would disappear in the external momentum after the time integration process of the force.

From the viewpoint of computation, finite element approximation is also a useful tool to discretize the time domain for solving structural dynamic problems. Over the past one decades, numerous researchers usually adopted the cubic Hermitian shape functions to construct time finite elements [5-7], from which most of the proposed algorithms are either conditionally stable or second-order accurate. Permitting the temporal field to be discontinuous at the discrete time levels, Chien and Wu [8] presented an improved predictor and multi-corrector solution algorithm based on the concept of time discontinuous Galerkin (TDG) finite element method to compute the dynamic response of linear structures using Gauss-Seidel method in an iterative way. In their studies, the computational efficiency of the TDG method within each time step would be considerably higher even a larger time step size is used, meanwhile, its accuracy is comparable to the work done by Li and Wiberg [9].

As the synthesis stated above, two points are worthy of discussion: (1) The discontinuity problem or rapid changes of a dynamic loading can be smoothed out through the equation transformation from conventional second-order equations in force equilibrium into first-order momentum equations of motion after time integration of dynamic forces; (2) The application of time finite element approximations to solve structural dynamic problems may provide a larger time step size [2,9] for the use of direct integration methods in performing time-history analysis.
Combining both computational advantages described above, this study presents an improved time integration scheme with the properties of larger time step size and accuracy. From theoretical analysis of stability and accuracy, the proposed algorithm possesses the efficient features of unconditional stability, accuracy, and fast convergence in numerical computation. For simplicity of demonstration, a closed form solution of an excited single-degree-of-freedom (SDOF) system is employed to verify the feasibility and reliability of the new time integration algorithm, with which the numerical results obtained from the Newmark method are compared.

2 Problem Formulation

For dynamic analysis of a linear system, the equations of motion for an n-DOF system is usually decomposed into n uncoupled equations by mode superposition method with the characteristics of normal modes [10]. For this reason, Fig. 1 shows an SDOF system under the action of an external force \( p(t) \) acting on its mass center, the equation of motion of the SODF system can be described as

\[
mx(t) + cx(t) + f(x,t) = p(t)
\]

where \( \dot{x} = \frac{\partial x}{\partial t} \), \( m = \) lumped mass, \( x = \) displacement, \( c = \) damping coefficient, \( k = \) elastic stiffness, and \( f(x,t) = \) elastic restoring force due to deformation. Considering the features of time elements with a large time step and the characteristics of momentum equation in smoothing out the rapid changes of external forces, a step-by-step time integration algorithm with unconditional stability and higher order accuracy will be presented in the following.

### 2.1 Approximations by time shape functions

With the approximation of time finite element, the displacement \( x(t) \) of the vibrating SDOF system over the time step \( \Delta t \) can be expressed as follows

\[
x(t) = N_1(t)x_1 + N_2(t)x_2 + N_3(t)v_1 + N_4(t)v_2
\]

where \( (x_1, v_1) \) stand for the nodal responses of displacement and velocity at time \( t_i \). As shown in Fig. 2, the cubic time shape functions \( N_i(t) \) are defined as

\[
N_1(t) = \left(1 - 3\tau^2 + 2\tau^3\right), N_2(t) = \left(3\tau^2 - 2\tau^3\right), \quad N_3(t) = \tau(1-\tau)^2, N_4(t) = \tau^2(\tau-1)
\]

and the nondimensional time parameter \( \tau \) is given as

\[
\tau = \frac{t - t_1}{t_2 - t_1} = \frac{t}{\Delta t}
\]

Here, \( t_1 \) is denoted as the beginning time (initial state) and \( t_2 = t_1 + \Delta t \) as the ending time (current state) over the time step \( \Delta t \). Then the elastic force \( f(x,t) \) proportional to the relative deformation \( x(t) \) is given by:

\[
f(x,t) = kx(t) = k\left[N_1(t)x_1 + N_2(t)x_2 + N_3(t)v_1 + N_4(t)v_2\right]
\]

where \( k = \) means the elastic stiffness of the SDOF system.

### 2.2 Application of momentum principle

Meanwhile, the loading function in the time domain is generally simulated or approximated by a series of linear segment forces over a small time interval \( \Delta t \) (see Fig. 3). Thus the external force \( p(t) \) within the time interval of \( t_1 \leq t \leq t_2 \) is approximated as follows:

\[
p(t) = \left\{\begin{array}{ll}
1 - \frac{t}{\Delta t} & p_1 + \frac{t}{\Delta t} p_2 \\
1 & p_2
\end{array}\right.
\]

where \( p_1 = \) nodal force at the initial time of \( t_1 \) and \( p_2 = \) nodal force at the ending time of \( t_2 \).
From the previous studies by Chang [1-4], using the principle of momentum to transform a second-order equation of motion into a first-order momentum equation of motion is beneficial to smooth out the rapid changes of dynamic loadings. Based on this concept, let us multiply Eq. (1) by a weighted momentum function $w(t)$ and use the principle of momentum to integrate the equation over the time step $\Delta t$ from $t_1$ to $t_2$. Based on this concept, multiplying Eq. (1) by a weight function $w_i(t)$, and integrating the equation over the time step $t_\Delta$ from $t_1$ to $t_2$, one can obtain the following integral equation

$$\int_{t_1}^{t_2} [m\ddot{x}(t) + c\dot{x}(t) + f(x(t), t) - p(t)]w(t)dt = 0$$  \hspace{1cm} (7)

Here, two weight functions of $w_i(t)$ for $i = 1, 2$ within the time interval of $t_1 \leq t \leq t_2$ in Eq. (7) are selected as follows: $w_1(t) = 1, w_2(t) = (t - t_1)/\Delta t - 1$.

$$W_1(t)$$

$$W_2(t)$$

Fig. 4 Weight functions.

It is noted that the weight function $w_2$ can be regarded as a linear weight function with respect to time of $t_1 \leq t \leq t_2$. The substitution of $w_1$ and $w_2$ into Eq. (7) yields a set of first order momentum equations and they can be further expressed as the following recurrence matrix equation in state space

$$\begin{bmatrix} \frac{c + k}{2} & \frac{m - \Delta t^2}{12}k \\ \frac{m - \Delta t^2}{12}k & \frac{m}{6} + \frac{c}{6} \frac{\Delta t}{2} \end{bmatrix}\begin{bmatrix} x_2 \\ v_2 \end{bmatrix} + \frac{\Delta t}{2}\begin{bmatrix} p_1 + p_2 \\ (p_2 - p_1)\Delta t \end{bmatrix}$$


\hspace{1cm} (8)

from which in addition to the current nodal forces of $p_1$ and $p_2$, the solution of state vector $<x_2, v_2>$ at current time $t_2$ just requires another knowledge of previous responses ($x_1, v_1$) at initial time $t_1$ of the current time step even though the acceleration terms of $(\ddot{x}_1, \ddot{x}_2)$ are absent. This feature goes forwards overcoming the load discontinuity or rapid changes of dynamic forces using step-by-step time integration schemes. As shown in Eqs. (8), the original second-order equation of motion (see Eq. (1)) of an SDOF system has been transformed into a set of equivalent first-order state space equation with recurrence characteristics.

3 Analysis of stability and accuracy

A step-by-step time integration method is unconditionally stable if the response of a dynamic system due to any initial conditions should not grow without bound for any time step [11], especially for a large time step ratio of $\Delta t/T_0$ used. Here, $T_0$ represents the fundamental period of an excited structure. To examine this property, let us consider the undamped SDOF system and define the following nondimensional parameters

$$\theta^2 = \frac{k}{m} \Delta t^2 = \omega^2 \Delta t^2, \ \alpha = 1 - \frac{\theta^2}{12}, \ \omega = \sqrt{\frac{k}{m}}$$  \hspace{1cm} (9)

where $\omega$ represents the natural frequency of the dynamic system. Then the recurrence matrix equation in Eq. (9) can be simplified as follows:

$$\begin{bmatrix} \theta/2 & \alpha \\ \alpha & \theta/2 \end{bmatrix} \begin{bmatrix} \omega x_2 \\ \omega v_2 \end{bmatrix} = \begin{bmatrix} -\theta/2 & \alpha \\ \alpha & \theta/2 \end{bmatrix} \begin{bmatrix} \omega x_1 \\ \omega v_1 \end{bmatrix}$$

\hspace{1cm} (10)

The solution of $<\omega x_2, \omega v_2>$ of Eq. (10) is given as

$$\begin{bmatrix} \omega x_2 \\ \omega v_2 \end{bmatrix} = \begin{bmatrix} \alpha^2 - (\theta/2)^2 & \alpha \theta \\ -\alpha \theta & \alpha^2 - (\theta/2)^2 \end{bmatrix} \begin{bmatrix} \omega x_1 \\ \omega v_1 \end{bmatrix}$$

\hspace{1cm} (11)

where $[A]$ is denoted as amplification matrix [11]. The characteristic values of $[A]$ in Eq. (11) are equal to

$$\lambda_{\alpha, \beta} = \left(1 - \frac{72\theta^2}{\theta^4 + 12\theta^2 + 144}\right) \pm i \frac{12\theta(12 - \theta^2)}{\theta^4 + 12\theta^2 + 144}$$

\hspace{1cm} (12)

$$= \cos \psi \pm i \sin \psi$$
From the condition of \( |\lambda_{1,2}| = 1 \) in Eqs. (12), the spectral radius \( \rho(A) \) of the amplification matrix \( A \) becomes unity and independent upon the size of time step. It means that the proposed time integration algorithm is unconditionally stable and of no numerical damping [11].

Next, let us consider the case of \( \theta = \omega \Delta t < 1 \). The application of power series expansion to the following approximations with respect to \( \theta \) yields

\[
\theta = \cos^{-1} \left( 1 - \frac{72 \theta^2}{\theta^4 + 12 \theta^2 + 144} \right) \approx 1 + \frac{\theta^4}{720} + o(\theta^6)
\]

The expression of \( \theta / \psi \) in Eq. (13) represents the period elongation due to numerical solution of using direct integration methods, which can be redefined as \( \theta / \psi = \omega \Delta t / \bar{\omega} \Delta t = \bar{T} / T_0 \). Here, \( \bar{\omega} \) and \( \bar{T} \) means the simulated frequency and period of the dynamic system using time integration schemes, respectively.

Fig. 5 illustrates the relative period error of \( (\bar{T} - T_0)/T_0 \) against various \( \Delta t / T_0 \) for the present scheme and the Newmark method with constant average acceleration [12], from which the proposed method possesses much fewer period elongation rate than the Newmark method. Such a feature in saving computational efforts will be demonstrated in the numerical examples.

4 Illustrative example

For the purpose of demonstration, the closed form solution of steady-state response for an undamped SDOF system subjected to a series of impulse-type forces is first derived. Then, the dynamic response predicted from the present algorithm as well as the Newmark method is used to compare their accuracy with the exact results.

4.1 Closed form solution

Exact solution is a better way to verify the computed results of numerical schemes. As shown in Fig. 6, this example is attempted to present a closed form solution of dynamic response of the undamped SDOF system subject to a series of rising triangle impulses with identical acting time \( T \). From Eq. (1), the governing equation of the undamped SDOF system becomes

\[
m\ddot{x} + kx = p(t)
\]

\[
p(t) = \begin{cases} 
  p_0 (2t/T - 1), & 0 \leq t/T \leq 1 \\
  p_0 (2(t/T - 1) - 1), & 1 \leq t/T \leq 2 \\
  \vdots \\
  p_0 (2(t/T - N) - 1), & N \leq t/T \leq N + 1 
\end{cases}
\]

with the initial conditions: \( x(0) = \Delta \eta, \dot{x}(0) = v_0 \). For the purpose of demonstration, the periodic oscillating behavior for the SDOF system is of interest in this example. Because of this consideration, let us introduce the following constraints into the initial conditions of the next time interval \( nT \leq t \leq 2nT \), that is

\[
x(0) = \Delta \eta = x(T), \dot{x}(0) = v_0 = \ddot{x}(T)
\]

Then the periodic response of the excited SDOF system at any time within the \( (n+1) \)th time interval of \( nT \leq t \leq n + n + 1 \) has been derived as

\[
x(t) \big|_{\tau = nT} = p_0 \left[ \frac{2}{\omega T} \cot \left( \frac{\omega T}{2} \right) \sin \omega T \right. + \left. 2 \frac{\omega T \sin \omega T}{\omega T} + \cos \omega T - 1 \right]
\]

\[
x(t) \big|_{\tau = nT} = \omega p_0 \left[ \frac{2}{\omega T} \cos \left( \frac{\omega (T - T/2)}{2} \right) \right]
\]

where \( n \) is a non-negative integer. As for the acceleration response, it can be obtained from twice differentiation of the displacement response in Eq. (29) with respect to time, that is,

\[
\ddot{x}(t) \big|_{\tau = nT} = \frac{p_0}{m} \sin \left( \frac{\omega (T - T/2)}{2} \right), \quad nT \leq t \leq n + 1
\]

It is noted that the discontinuity nature of acceleration response induced by the periodic rising triangle impulsive forces at \( t = nT \) can be observed from the response of Eq.(17) as

\[
\ddot{x}(nT^-) = -\frac{p_0}{m} \neq \ddot{x}(nT^+) = \frac{p_0}{m}
\]
4.2 Numerical verification

For the purpose of numerical comparison, the lumped mass \( m \) and stiffness \( k \) for the system are assumed as 1kg and 1N/m, respectively. The periodic duration \( T \) of rising triangular impulse is taken as 2s with an absolute maximum load of \( p_0 = 1N \). Thus, the SDOF system has a natural period of \( T_0 = \frac{2\pi}{\omega} = 6.28s \). In addition, three time steps smaller than \( 0.1T_0 \) (=0.628s), which are \( \Delta t = 0.333s, 0.1s, \) and \( 0.01s \), are selected for the Newmark method to compute the dynamic responses of the undamped SDOF system shown Fig. 5. The corresponding time history responses of displacement, velocity, and acceleration of the lumped mass \( (m) \) have been drawn in Figs. 7-9. It is found that except the smallest time step \( \Delta t = 0.01s \), the accuracy of responses computed from the Newmark method [13] cannot give reliable results. The reason for this is attributed to the fact that the discontinuity at each end of rising triangular impulses will result in an extra impulse [4], which would produce an extra displacement in the time history response analysis. On the other hand, the use of large time steps and non-zero initial conditions may result in overshooting phenomena in the first few cycles of response [14,15]. As can be seen from Fig. 7, the response curve of displacement computed by the Newmark method with the time step \( \Delta t = 0.333s \) appears a significant overshooting behavior because of the large time step size used and the non-zero initial conditions at each of the series of rising triangular impulse forces.

Next, let us use the present method to solve the same dynamic problem. With the large size of time step \( \Delta t = 0.333s \), the time history responses of displacement, velocity, and acceleration computed have been plotted in Figs. 7-9, respectively. As can be seen, the present numerical results marked with ‘circular bold’ symbol (●) agree very well with the exact solutions obtained from Eqs. (17) and (18) even though the time step used is rather large compared with those used by the Newmark method. On the other hand, the present numerical results indicate that the discontinuity characteristics of acceleration response at \( t = nT \) has been taken into consideration from the time history response of acceleration shown Fig. 9 because of the discontinuity nature at each termination of the sequence of rising triangular impulses depicted in Fig. 6. From the present example, the proposed approach exhibits its superiority over the Newmark method with regard to accuracy, no overshoot, stability, and time step size.

5 Conclusion

This study combined both advantages of time element approximation and weighted momentum principle to produce an improved step-by-step time integration technique with the features of fourth-order accuracy and larger time step in a rapid convergent way. The use of the present scheme can convert a conventional second-order force equation of motion in force equilibrium into a set of first-order momentum equations of motion in terms of state
space, from which the knowledge of acceleration response is allowed to be absent. This natur is beneficial to overcome the numerical inaccuracy caused by load discontinuity or rapid changes of exciting forces acting on a structure even a larger time step is selected.

From the computed responses of an SDOF system under periodic riding triangular impulses, the present method exhibits its superior computational efficiency in accuracy, stability, convergence, and time step size even though a rather large time step over ten percentage of structure period is selected. This feature is evident to save time-consuming and attain accurate results in numerical computation, particularly true for a long-term response analysis using time-step integration schemes.

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