Abstract: Gödel maintains that with mathematical reason we perceive the most general concepts and their relations, which are separated from space-time reality insofar as the latter is completely determined by the totality of particularities without any reference to the formal concepts. Gödel holds that impredicativity presupposes that the totality of all properties exists somehow independently of our knowledge and our definitions, and that our definitions merely serve to pick out certain of these previously existing properties. Gödel asserts that the problem of giving a foundation for mathematics, i.e. the totality of methods of proof actually used by mathematicians, can be considered as falling into two different parts: these methods of proof have to be reduced to a minimum number of axioms and primitive rules of inference, and a justification in some sense or other has to be sought for these axioms.

Key-Words: Conceptual Realism, Mathematical Intuition, Abstract Concepts

1 Introduction
Gödel argues that mathematics describes a non-sensual reality, which exists independently both of the acts and the dispositions of the human mind and is only perceived incompletely by the human mind.

2 Problem Formulation and Solutions
Mathematical intuition can be replaced by conventions about the use of symbols and their application. Gödel maintains that with mathematical reason we perceive the most general concepts and their relations, which are separated from space-time reality insofar as the latter is completely determined by the totality of particularities without any reference to the formal concepts. The “given” underlying mathematics is related to the abstract elements contained in our empirical ideas. The concepts in a mathematical proposition form an objective reality of their own, which we cannot create or change, but only perceive and describe. The objects and theorems of mathematics are as objective and independent of our free choice and our creative acts as is the physical world. His first incompleteness theorem states the incompleteness of any formal system rich enough to express arithmetic (any consistent formal system adequate for the expression of arithmetic must leave out much of mathematical reality). According to the second incompleteness theorem, the consistency of a formal system that contains arithmetic cannot be formally proved within that system (no such formal system can even prove itself to be self-consistent).

Gödel emphasizes that a function is computable in any formal system containing arithmetic if and only if it is computable in arithmetic, where a function $f$ is called computable in $S$ if there is in $S$ a computable term representing $f$. Gödel holds that impredicativity presupposes that the totality of all properties exists somehow independently of our knowledge and our definitions, and that our definitions merely serve to pick out certain of these previously existing properties. Our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations. The question of the objective existence of the objects of mathematical intuition is an exact replica of the question of the objective existence of the outer world. Finitary mathematics is the
mathematics of concrete intuition (abstract concepts are needed for the proof of consistency of number theory). The primitive terms of intuitionistic logic lack the complete perspicuity and clarity which should be required for the primitive terms of an intuitionistic system. The notion of derivation or of proof must be taken in its intuitive meaning as something directly given by intuition. The concept of reductive proof is defined by the fact that the chain of definitions of the concept occurring in the theorem together with certain axioms about the primitive terms forms by itself a proof.

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An adequate proof-theoretic characterization of concrete intuition will include induction procedures which are not concretely intuitive and which could very well yield a proof of the inductive inference for $\varepsilon_0$ or larger ordinals. Besides mathematical intuition, there exists another criterion of the truth of mathematical axioms, namely their fruitfulness in mathematics. The simplest case of an application of the criterion arises when some axiom has number-theoretical consequences verifiable by computation up to any given integer. The solution of certain arithmetical problems requires the use of assumptions essentially transcending arithmetic.

Buzaglo investigates Gödel’s argument that attempts to use the process of forced expansion to deduce the independent existence of concepts as well as a capacity to perceive them. According to Gödel’s completeness theorem, if there is an object for a first-order theory, and this theory is consistent, then there must be a model for this theory that is made up of sets. Concepts are partially undefined as long as the boundaries between the defined and the undefined parts are clear. Of all possible classes, cardinality should be the criterion for determining which are legitimate sets. Paradoxes are singular points to which the truth predicate cannot be expanded. As Buzaglo puts it, Gödel claims that the very existence of forced expansions proves (a) the existence of concepts and (b) naturally leads to the possibility of perceiving them.

The reality of concepts is independent of the role of predicates in sentences, the existence of paradoxes is an argument for the existence of concepts, and the perception of concepts is different from the perception of objects. The fact that concepts force themselves on us allows us to deduce that they must exist. The system of basic laws that need to be conserved, which makes the expansion a forced one, is unique. We have a strong requirement that the concept of the number of a set should be independent of the properties of its elements. When we have only an intuitive concept, two different concepts can seem to be one, but when we sharpen our knowledge, we may discover that there are actually two different concepts involved. Buzaglo adds that whenever there are several forced expansions, one of them is more natural than the others. On Gödel’s reading, the uniqueness of an expansion can be explained only if we assume that the outcome of the expansion was defined even before the expansion took place. [1]

Parsons holds that Gödel aims at what other philosophers would call a theory of reason rather than a theory of intuition. Mathematical objects are independent of our “constructions” in much the same sense in which the physical world is independent of our sense-experience. Relativity theory supports the Kantian view that time and change are not to be attributed to things as they are in themselves. Gödel’s realism extends to concepts (objects signified in some way by predicates). The principles by which sets, or axioms about them, are generated naturally lead to further extensions of any system they give rise to. The existence of recognition-transcendent truth is meaningful (if the mathematical truths that the human mind can know can be generated by a Turing machine, then the proposition that this set is consistent would be a mathematical truth that we could not know). The independence of the continuum hypothesis should in no way imply that it does not have a determinate truth-value. Axiomatic set theory describes a transfinite iteration of the set-forming operations of the simple theory of types. Mathematical propositions are true by virtue of the meaning of the terms occurring in them. Parsons remarks that Gödel uses mathematical intuition in both object-relational and propositional attitude senses (in the latter it is not always clear what epistemic force the term is intended to have).

Mathematical intuition might be developed altogether without any commitment as to the extent to which it is a guide to truth. Mathematical axioms can have an a posteriori justification through their consequences. Parsons reasons that Gödel’s epistemology of set theory involves recognizing the fact of intuition concerning the concepts and axioms, and giving credence to it. [7]

According to Goldstein, Gödel mistrusts our ability to communicate: natural language is imprecise, and
we usually do not understand each other. Our knowledge of mathematics exceeds our formal proofs. We have access to an independent mathematical reality. All of our formal systems strong enough for arithmetic are either inconsistent or incomplete. Language is subordinate to reality. Goldstein points out that Gödel is committed to the possibility of reaching out beyond our experiences to describe the world “out yonder.” Mathematics is a means of unveiling the features of objective mathematical reality. Our minds, in knowing mathematics, are escaping the limitations of man-made systems, grasping the independent truths of abstract reality. Gödel shows how to construct a true but unprovable proposition for any formal system whatsoever containing arithmetic. For each formal system there will be truths expressible in that system that will not be provable. There is expressible knowledge which cannot be formalized. [4] Sieg focuses on Gödel’s multifaceted work concerning the analysis of mechanical procedures: the reduction of the concept of finite procedure to that of a finite machine is “the most satisfactory way” to arrive at a precise definition. Gödel defines general recursive functions as those functions whose values can be computed in an equational calculus. As Sieg puts it, Gödel does not think of general recursiveness as a rigorous explication of effective calculability. Gödel characterizes the computations of values of recursively specified number-theoretic functions by uniform and arithmetically meaningful steps. [8] Tait argues that, for Gödel, intuition provides evidence of truth of propositions to be taken as axioms (i.e. as the basis of proof): intuition is a means of discovering new truths (they are somehow already implicit in our concepts, but not logically derivable from the axioms we have already accepted concerning these concepts). [9] Gaifman contends that Gödel’s incompleteness result shows that self-reflection cannot encompass the whole of our reasoning. Its proof uses an encoding of sentences, sentence parts, and proofs by natural numbers. The fixed-point theorem says that, for any given formula, \( \phi(x) \), with one free variable, there is a sentence \( \alpha \) such that the following biconditional is provable in \( T: \alpha \leftrightarrow \phi(\alpha) \), where “\( \alpha \)” is the standard name of \( \alpha \)'s Gödel number. If \( T \) is \( \omega \)-consistent, then \( \neg \gamma \) is not provable in it. [3] On Nagel and Newman’s reading, Gödel shows that it is impossible to give a meta-mathematical proof of the consistency of a system comprehensive enough to contain the whole of arithmetic (Gödel’s argument makes it unlikely that a finitistic proof of the consistency of arithmetic can be given), and that Principia, or any other system within which arithmetic can be developed, is essentially incomplete. Meta-mathematical statements about a formalized arithmetical calculus can be represented by arithmetical formulas within the calculus. The meta-mathematical statement cannot be established unless rules of inference are used that cannot be represented within the calculus. All meta-mathematical statements about the structural properties of expressions in the calculus can be adequately mirrored within the calculus itself. No antecedent limits can be placed on the inventiveness of mathematicians in devising new rules of proof. There are innumerable problems in elementary number theory that fall outside the scope of a fixed axiomatic method. [6] Yasugi and Passell remark that Gödel uses a formal system as merely rules on sequences of meaningless symbols, and uses its mathematical meaning to carry meta-mathematics inside it. Gödel proves the consistency of the axiom of choice and the generalized continuum hypothesis with the axiom of set theory, and demonstrates the consistency of the assumption that all sets are constructible. The collection of all sets is existentially clear and real. [10] Chaitin argues that the endeavor to define and measure randomness has clarified the significance and the implications of Gödel’s incompleteness theorem. Virtually all formal systems are incomplete: in each of them there is at least one statement that is true but that cannot be proved. Gödel’s original proof constructs a paradoxical assertion that is true but not provable within the usual formalizations of number theory. Gödel’s original proof applies to a particular formalization of number theory. [2] According to Hintikka, Gödel shows that our basic working logic is complete in one natural sense of completeness. If an axiomatic system containing elementary arithmetic is consistent, then one can find a specific proposition \( G \) that is true but unprovable in that system. The basic elementary part of logic known as ordinary first-order logic is complete. Gödel’s actual completeness result had the effect of minimizing the distinction between model-theoretical and proof-theoretical concepts (the class of logical truths of the received first-order logic can be captured by purely synthetical proof methods).
3 Conclusions
The construction Gödel uses to find an unprovable proposition \( G \) is relative to some given axiom system of arithmetic. Gödel’s impossibility theorem shows that not all true arithmetical propositions can be proved mechanically, in one and the same formal system. Any axiomatization of elementary arithmetic is incomplete because there exists a sentence \( G \) such that neither \( G \) nor \( \neg G \) is provable on the basis of the axioms. Logical and set-theoretical truths are truths about the abstract region of the actual “world” (there is no sharp distinction between logical and factual truths). Gödel uses the notion of truth to emphasize the objective character of mathematical objects and of logical and mathematical truths. The notions of truth and provability have to be distinguished sharply in arithmetic, and arithmetical truths cannot be exhausted by any axiomatization. Computable functions might be identified with general recursive functions (i.e. functions definable by a finite number of recursive equations of the kind that can be used to define the basic arithmetical operations). [5]

References: