Abstract: Dynamic economic models generally consist in difference or differential behavioral equations. Several arguments are in favor of continuous time systems: the multiplicity of decisions overlapping in time, a more adequate formulation of market adjustments and distributed lag processes, the properties of estimators, etc. The type of dynamic equations also refers to historical and practical reasons. In some cases of the economic dynamics, delay differential equations (DDEs) may be more suitable to a wide range of economic models. The dynamics of the Kalecki’s business cycle model is represented by a linear first-order DDE with constant coefficients, in the capital stock. Such a DDE, with constant or flexible lags, also occurs in the continuous time Solow’s vintage capital growth model. This is due to the heterogeneity of goods and assets. In some qualitative study, the time delay is replaced by the Taylor series for a sufficiently small delay and a not too large higher-order derivative. DDEs with constant lags may be preferably solved by using Laplace transforms. Numerous techniques are also proposed for the solution of DDEs, like the inverse scattering method, the Jacobian elliptic function method, numerical techniques, the differential transform method, etc. This study uses a block diagram approach with application to reference economic models, with help of the software MATHEMATICA 6.0. and its specialized packages for signal processing, such as "Control System Professional".

Key–Words: delay differential equation, method of steps algorithm, differential transform method, Laplace transform solution, Bode diagrams, $(x, k)$-root plateau.

1 Introduction

The elementary theory of DDEs is introduced by solving and representing simple reference examples. MATHEMATICA plots will show how the parametrized solutions are generated. The technical aspects also concern the differential transform technique, for practical reasons. Two representative applications refer to economics: the earlier Kalecki’s business cycle continuous-time system with a fixed delay [1, 10], and the Solow vintage capital growth control system with a variable delay [3, 5].

2 Elementary functional differential equation theory

2.1 Differential-difference equations

The linear form of a DDE of differential order $m$ and difference order $n$ with constant coefficients takes the form $^{1}[4]$

$$
\sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij}y^{(j)}(t - \omega_i) = f(t).
$$

The subclass of the first-order DDE will then be written

$$
a_0y'(t) + a_1y'(t - \omega) + b_0y(t) + b_1y(t - \omega) = f(t),
$$

where $f(t)$ is assumed to be integrable and of boundary variation.

2.2 Method of Steps Algorithm (MSA)

Let a real-valued DDE with constant delay be

$$
y'(t) = f(t, y(t), y(t - \theta)), \quad t > \theta, \quad \theta > 0.
$$

$^{1}$The general form is

$$
F(t, y(t), y(t - \omega_1), \ldots, y(t - \omega_m),
y'(t), y'(t - \omega_1), \ldots, y'(t - \omega_m), \ldots,
y^{(n)}(t), y^{(n)}(t - \omega_1), \ldots, y^{(n)}(t - \omega_m)) = 0
$$

.
The method of steps simply consists in extending forward an initial solution in the direction of increasing $t$. The equation is solved for each meshpoint $\{0, \theta, 2\theta, \ldots, k\theta, \ldots\}$. Suppose that $y'(t) = y_0(t)$ is given on $[-\theta, 0]$. The computation process is:

At step $1$, a solution $y(t) = y_1(t)$ is determined on $[0, \theta]$ by solving, analytically or numerically, the resulting ordinary differential equation (ODE)

$$y'(t) = f(t, y(t), y_0(t - \theta)), \quad y(0) = y_0(\theta).$$

At step $2$, a solution $y(t) = y_2(t)$ is determined on $[\theta, 2\theta]$ by solving the resulting ODE

$$y'(t) = f(t, y(t), y_1(t - \theta)), \quad y(\theta) = y_1(\theta).$$

\ldots

At step $k$, a solution $y(t) = y_k(t)$ is determined on $[(k - 1)\theta, k\theta]$ by solving the resulting ODE

$$y'(t) = f(t, y(t), y_{k-1}(t - \theta)), \quad y((k-1)\theta) = y_{k-1}(\theta).$$

2.2.1 Dynamics of the Frisch-Holme DDE

Let the Frisch-Holme DDE be [7] $^	ext{2}$

$$y'(t) = -ay(t) - by(t - \theta).$$

With the parameter values $a = 0$, $b = -1$, $\theta = 1$, the DDE is now

$$y'(t) = y(t - 1).$$

Let the initial function be simply set to $y_0(t) = 1$, at period $0$. The ODE to solve at next period will be $y'(t) = y_0(t - 1), y(1) = y_0(1)$, at period $1$, and that of the next period will be $y'(t) = y_1(t - 1), y(2) = y_1(2)$, a.s.o. The solutions are represented in Fig.1. The solutions are given by

$$y(t) = \sum_{k=0}^{n} \frac{(t - k)^k}{k!}, \quad t \in [n, n + 1], \quad n \in \mathbb{N}_0.$$  

2.2.2 Stability of the Frisch-Holme DDE

The stability of all solutions is achieved asymptotically, provided that all the characteristic roots lie in the negative complex half-space [4, 8, 11]. Searching for solutions of the form

$$y(t) = ce^{\rho t}, \quad \rho = \alpha + j\beta, \quad j = \sqrt{-1} \quad (1)$$

leads to the characteristic equation

$$D(\rho) \equiv \rho + a + be^{-\rho} = 0, \quad (2)$$

which possesses many solutions for $b \neq 0$. We are searching for $(a, b)$-values for which $\rho = j\theta$, for $\theta$ real. Inserting Eq.(2) into Eq.(1), and separating the real and imaginary parts, we obtain a parametric function in $\theta$ for $a$ and $b$, defined by

$$a = -\theta \cot \theta, \quad b = \theta \sec \theta.$$  

The spectrum of roots of $D(\rho)$ in the complex plane is plotted $^3$ in the left Fig.2 for $(a, b) = (-.5, 1)$. The figure to the right shows the stability regions in the $(a,b)$-plane. The solutions are either monotonic or oscillatory. The extension to a linear (linearized) system

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Solutions for $y'(t)=y(t-1)$.
}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Stability domain for $y'(t) + ay(t) + by(t - \theta) = 0$.
}
\end{figure}

$^2$Kalecki (1935)[10] early retained a similar DDE for the macroeconomic system

$$y'(t) = \frac{m}{\theta} y(t) - \frac{m + n\theta}{\theta} y(t - \theta),$$

for which a main cyclical solution is obtained for the parameter values: $m = .95, n = .121$ and $\theta = .6$ (7 months).

$^3$The characteristic equation $D(\rho) = 0$ with $\rho = \alpha + j\beta, \quad j = \sqrt{-1}$ is numerically solved for $(a, b) = (-.5, 1)$. The set of equations to be solved is $\{a + \alpha + be^{-\alpha} \cos \beta = 0, \quad \beta - be^{-\alpha} \sin \beta = 0\}$. The roots can be computed by using a Newton-Raphson method, with adequate starting values.
of DDEs introduces the matrix equation [6]
\[ y'(t) = A_0 y(t) + \sum_{i=1}^{m} A_i y(t-\theta_i), \quad A_i \in \mathbb{R}^{n \times m}, \quad i \in \mathbb{N}_m, \]
whose characteristic equation reads
\[ \det(D(\rho)) = \rho I - A_0 - \sum_{i=1}^{m} A_i e^{-\rho \theta_i} = 0. \]

### 2.3 Differential Transform Method (DTM)

The approximate solution of DDE by using the Zhou’s differential transform method was extended by Arikoglu and Ozkol (2006)[2]. The method is used to solve linear and nonlinear DDE with variable coefficients.

#### 2.3.1 Introduction to the method

The transformation of the kth derivative of a function \( f(t) \) is
\[ F(k) = \frac{1}{k!} f^{(k)}(t)|_{t=t_0}, \]
and the inverse transformation is defined by
\[ f(t) = \sum_{k=0}^{\infty} F(k)(t - t_0)^k. \]

The transformation of usual functions is presented in Appendix B. Let a DDE take the general form
\[ f\left( g^{(n_1)}(t+\theta_1), g^{(n_2)}(t+\theta_2), \ldots, g^{(n_p)}(t+\theta_p) \right) = 0, \]
with the boundary conditions (BCs)
\[ g^{(a)}(t)|_{t=t_i} = c_i, \quad i \in \mathbb{N}_m. \]

The differential transformation \( \mathcal{D}[\cdot] \) of the BCs is given by
\[ \sum_{k=0}^{T} \frac{k!}{(k-a)!!} G(k)(b_i - t_0)^{k-a_i} = c_i, \quad i \in \mathbb{N}_m. \]

**Theorem 1 [2]** Let \( G(k) = \mathcal{D}[g(t)] \), the solution of Eq.(3) depends on the solution of the unknown coefficients \( G(0), G(1), \ldots, G(T) \). We have
\[ F_k\left( G(0), G(1), \ldots, G(T) \right) = 0, \quad k \in \mathbb{N}_{0,T-m}. \]

Eq.(4) consists in \( m \) equations, the remaining \( T - m + 1 \) standing from the transformation of Eq.(3).

**Proof.** See Arikoglu and Ozkol (2006)[2].

### 2.3.2 Application to the Kalecki’s system

The continuous-time Kalecki’s business cycle system (1935)[10] is
\[ \begin{align*}
Y(t) &= C(t) + I(t), \\
C(t) &= cY(t), \quad c \in (0,1), \\
I(t) &= \frac{1}{\theta} \int_{t-\theta}^{t} B(\tau)d\tau, \\
B(t) &= \lambda (vY(t) - K(t)), \quad \lambda \in (0,1), \\
K'(t) &= B(t - \theta).
\end{align*} \]

The aggregate demand in Eq.(5) of consumption \( C \) and investment outlays \( I \) equals the total revenue \( Y \). In Eq.(6), consumption is proportional to the total revenue. In Eq.(7), the investment orders depend on the past investment decisions \( B \) with a fixed delay of \( \theta \). The determination of the investment decisions, in Eq.(8), is proportional to the gap between the desired level of equipment \( vY(t) \) and the existing capital stock \( K \). According to Eq.(9), a fixed delay separates the deliveries of capital goods from the orders. After some algebraic manipulations \(^4\), the system may be reduced to a linear first-order DDE with constant coefficients, in the variable \( K(t) \)
\[ K'(t) = aK(t) - bK(t - \theta), \quad K(0) = 1, \]
where \( a = \frac{\lambda v}{\theta(1-c)} \) and \( b = \lambda \left( 1 + \frac{v}{\theta(1-c)} \right) \). Using the Table B.1, we obtain the transformation of Eq.(10)
\[ (k+1)\bar{K}(k+1) - a\bar{K}(k) \]
\[ + b \sum_{h_1=k}^{T} \binom{h_1}{k} (-1)^{h_1-k} \bar{K}(h_1) = 0. \]

The unique BC is also transformed to \( \bar{K}(0) = 1 \). Taking for numerical values \( \lambda = 2/5, \quad \theta = 1, \quad c = 3/4, \quad v = 1/2, \) the coefficients are \( a = .8 \) and \( b = 1.2 \). Choosing \( T = 4 \), with the BC and \( k = 0, 1, 2, 3 \) and taking for numerical values, a linear system in the variables \( \bar{K}(1), \bar{K}(2), \bar{K}(3), \bar{K}(4) \) is solved. We have the system
\[ \begin{align*}
.2\bar{K}(1) - 1.2\bar{K}(2) + 1.2\bar{K}(3) - 1.2\bar{K}(4) &= .4, \\
.4\bar{K}(1) - .4\bar{K}(2) + 3.6\bar{K}(3) - 4.8\bar{K}(4) &= 0, \\
.4\bar{K}(2) - .6\bar{K}(3) + 7.2\bar{K}(4) &= 0, \\
.4\bar{K}(2) - .8\bar{K}(4) &= 0.
\end{align*} \]

\(^4\)The integral Eq.(7) is also written as
\[ I(t) = \theta^{-1} \int_{t-\theta}^{t+\theta} K'(x)dx = \theta^{-1}(K(t+\theta) - K(t)), \]
where \( x = \tau + \theta \).
We obtain the solution
\[ \bar{K}(1) = -0.56, \bar{K}(2) = -0.4, \bar{K}(3) = 0.0533, \]
\[ \bar{K}(4) = 0.0266. \]

Using the inverse transformations from Table B.1, the series solution for the problem is divergent with
\[ K(t) = 1 - 1.56t - 0.4t^2 + 0.0533t^3 + 0.0266t^4 + O(t^5). \]

3 Generalized DDE for economic systems

3.1 Kalecki’s business cycle model with discrete delay

The Laplace transform of the system Eq.(5-9) is
\[ \bar{Y}(s) = \frac{1}{(1-c)} \bar{I}(s) - \frac{1}{(1-c)} \bar{u}(s), \]
\[ \bar{I}(s) = \frac{1}{\theta} (e^{\theta s} - 1) \bar{K}(s), \]
\[ \bar{B} = \lambda v \bar{Y}(s) - \lambda \bar{K}(s), \]
\[ s \bar{K}(s) = e^{-\theta s} \bar{B}(s), \]

where the initial conditions are zero and where \( u(t) \) denotes the input of the system. The Laplace transformed variables \( X(t) \) are written \( \mathcal{L}[X(t)] = \bar{X}(s) \).

Solving the system w.r.t. \( \bar{K}(s) \), the transfer function is
\[ \frac{\bar{K}(s)}{\bar{u}(s)} = \frac{\phi(s, \theta) \lambda v}{(1-c) \left( s + \phi(s, \theta) \lambda - \frac{(1-\phi(s, \theta)) \lambda v}{\theta (1-c)} \right)}, \]

where \( \phi(s, \theta) = e^{-\theta s} \) is approximated by the Taylor’s series \( 1 - \theta s + \frac{\theta^2 s^2}{2} + O(s^3) \). Taking the parameter values \( \lambda = 2/5, \theta = 1, c = 3/4, v = 1/2 \) and a unit delay \( \theta = 1 \), the transfer function is
\[ 2 \frac{s^2}{2} - s + 1 \]
\[ \frac{2}{3s^2} - \frac{s}{2} + 1. \]

The constant is 2 (about 6 dB), the zeros are the complex conjugates \( 1 \pm j \), so as the poles \( \frac{1}{6}(1 \pm j \sqrt{23}) \).

The Bode diagrams in Fig.3 of the transfer function show the response of the system to a sine signal. The impact of a shorter/longer time delay is illustrated in Fig.4.

3.2 Solow vintage capital growth model with variable delay

The purpose of a growth model with heterogeneous productive capital, is to determine the optimal age structure of machines by taking adequate investment decisions.
3.2.1 Equations of the system

The Solow vintage capital growth system consists of four equations. The first two equations describe the vintage technology and the market good equilibrium. The next two equations concern the labor market (labor demand and equilibrium). We have

\[
y(t) = \int_{t-T(t)}^{t} i(\tau) d\tau, \quad (11)
\]

\[
l(t) = \int_{t-T(t)}^{t} i(\tau)e^{-a(\tau)} d\tau, \quad (12)
\]

\[
i(t) = (1-c)y(t), \quad (13)
\]

\[
l(t) = 1. \quad (14)
\]

Eq.(11) represents an AK technology \((A = 1)\) in the vintage productive capital, where all existing machines are supposed to be in use. The variables are defined by: production \(y\), investment \(i\), age \(T\) of the oldest machine, and \(\tau\) a generation of machines. Eq.(12) is the labor demand \(l(t)\). Each machine at \(t\) requires \(e^{-a(t)}t\) workers, and new machines are more productive, since the element \(a(t)\) with \((a(t)\)' \(> 0\) denotes the technological progress. According to the equilibrium condition Eq.(13) of the good market, a fixed proportion \(1 - c\) of income is saved and totally invested. Since the labor supply is assumed to be constant over time, the equilibrium condition on the labor market is defined by Eq.(14).

3.2.2 System of DDEs with flexible delays

The differentiation of the system Eqs.(11 – 14) leads to a system of two DDEs in \(y'(t)\) and \(T'(t)\) with flexible delays. We have

\[
y'(t) = (1-c)y(t)(1-\Psi(t)), \quad T'(t) = 1 - \frac{y(t)}{y(t-T(t))}\Psi(t),
\]

where

\[
\Psi(t) = e^{-a(t)}t / e^{-a(t-T(t))} \times (t-T(t)).
\]

The function \(\Psi(t)\) is a ratio between the labor requirement of the new machines at \(t\) to the replaced ones at \(t - T(t)\) [5].

3.2.3 Modified Method of Steps Algorithm

The MSA cannot be extended directly to the time-varying state-dependent DDE

\[
y'(t) = f\left(t, y(t), l(t - \theta(t, y(t)))\right),
\]

where \(\theta(t, y(t))\) is the lag function of time \(t\) and state variable \(y(t)\). In fact, the successive meshpoints \(\{0, \sigma_1, \sigma_2, \ldots, \sigma_k, \ldots\}\) will differ from each other and are unknown. At each step \(k\), the meshpoint \(\sigma_{k+1}\) must be determined given the computed solution \(y(t) = y_{i+1}(t)\) on \([\sigma_i, \sigma_{i+1}]\) for \(i = 0, \ldots, k\). The equation

\[
\sigma_{k+1} - \theta(\sigma_{k+1}, y(\sigma_{k+1})) = \sigma_i
\]

is solved for the meshpoint \(\sigma_i\) corresponding to the smallest \(\sigma_{k+1} > \sigma_k\). One illustration of the process is given by Boucekkine et al. [5]. Suppose that the generalized DDE is given by

\[
y'(t) = y(t - \theta(t)),
\]

where the lag function is defined by \(\theta(t) = t + \sin t\) and the initial function \(y_0(t) = 1\).

At the step \(i\), for \(i = 0, 1 \in \{0, 1\}\) we have to solve \(y'(t) = y_0(t) = 1\) on \([0, \sigma_1]\). We find the solution \(y(t) = y_1(t) = 1 + t\). At this step, for \(i = 1\), we have to solve \(y'(t) = y_1(t - \sin t)\), which gives contradictory results on \([0, 2\pi]\). Indeed, the solution of \(y'(t) = -\sin t + 1, y(0) = 1\) is \(t + \cos t\) on \([0, 2\pi]\), whereas the solution of \(y'(t) = -\sin t + 1, y(\pi) = 1 + \pi\) is \(2 + t + \cos t\) on \([\pi, 2\pi]\). Since, the smallest value may be chosen in practice, we retain \(\sigma_1 = \pi\). The modified algorithm then introduces two substeps for each step of the MSA [5]:

(i) Replacing the solution \(y_i(t)\) on \([\sigma_{i-1}, \sigma_i]\) for \(i = 0, \ldots, k\) in the DDE, the resulting ODE is solved and gives \(y_{k+1}(t)\).

(ii) The largest meshpoint value \(\sigma_{k+1}\) is such that \(y'_{k+1}(t)\) is consistent with the DDE over \([\sigma_{k}, \sigma_{k+1}]\). The solution is \(y(t) = y_{k+1}(t)\) on \([\sigma_{k}, \sigma_{k+1}]\). The numerical code used by Boucekkine et al. [5] is based on the 5th-order Runge-Kutta method.

A Laplace transform solution of equation \(y'(t) = -by(t - 1)\)

The Laplace transform (denoted by \(L[\cdot]\)) method is used to solve a first-order linear DDE:

\[
y'(t) + by(t - 1) = 0, \quad t > 0, \quad (A.1)
\]
which boundary conditions are \( y(t) = 0, t \in [-1, 0] \), and where \( b \) is a constant. We know that\(^8\) \( \mathcal{L}[y(t)] = \int_{0}^{\infty} y(t_1)e^{-st_1} dt_1 = Y(s), s \in \mathbb{C}, \) and \( \mathcal{L}[y'(t)] = sY(s) - y_0. \) Multiplying Eq.(A.1) by \( e^{-st} \), and integrating from 1 to infinity, we have also
\[
\int_{1}^{\infty} y'(t)e^{-st} dt + b \int_{1}^{\infty} y(t-1)e^{-st} dt = 0. \tag{A.2}
\]
Let us examine the two integrals of Eq.(A.2). Integrating by parts the first integral and assuming \( y(t)e^{-st} \rightarrow 0 \) as \( t \rightarrow \infty, \) we obtain
\[
\int_{1}^{\infty} y'(t)e^{-st} dt = -y(1)e^{-s} + s \int_{1}^{\infty} y(t)e^{-st} dt. \tag{A.3}
\]
Using a change of variable for the second integral yields
\[
b \int_{1}^{\infty} y(t-1)e^{-st} dt = b \int_{-1}^{\infty} y(t_1)e^{-s(t_1+1)} dt_1, \]
\[
= by_0e^{-s} \int_{-1}^{\infty} e^{-st} dt + be^{-s} \int_{-1}^{\infty} y(t_1)e^{-s(t_1+1)} dt_1, \]
\[
= by_0e^{-s}\left[\frac{e^{-st}}{-s}\right]_0^\infty + be^{-s}Y(s), \]
\[
= \frac{by_0(1-e^{-s})}{s} + be^{-s}Y(s). \tag{A.4}
\]
Placing Eqs.(A.3) and (A.4) into Eq.(A.2) yields
\[
sY(s) - y_0 + \frac{by_0(1-e^{-s})}{s} + be^{-s}Y(s) = 0. \tag{A.5}
\]
Solving Eq.(A.5) for \( Y(s) \) and assuming that \( s - e^{-s} \neq 0, \) we get
\[
Y(s) = \frac{y_0}{s} - \frac{by_0}{s(s+be^{-s})}. \tag{A.6}
\]

**Theorem 2 [12].** Let \( f(t) \) be integrable over any finite interval such that (i) \( \int_{0}^{\infty} f(t)e^{-st} dt \) converges absolutely on the real line \( \text{Re } s = c \) and that (ii) \( f(t) \) is of bounded variation in the neighborhood of \( t, \) then
\[
\int_{(c)} F(s)e^{st} ds = \frac{1}{2}\left( f(t+0) - f(t-0) \right),
\]
where the LHS is a contour integral taken along the line from \( c - jT \) to \( c + jT \) in the complex plane\(^9\).

From Eq.(A.6), \( Y(s) \) may also be expressed as
\[
Y(s) = \frac{y_0}{s} - \frac{by_0}{s^2(1+b e^{-s})},
\]
\[
= y_0\left( \frac{1}{s} - \sum_{p=0}^{\infty} (-1)^p b^{p+1} e^{-ps}s^{-p-2} \right).
\]
Applying the theorem 2, we have
\[
y(t) = \int_{(c)} y_0 \left( e^{st} - \sum_{p=0}^{\infty} (-1)^p b^{p+1} e^{s(t-p)} / s^{p+2} \right) ds,
\]
\[
= y_0 \left( \int_{(c)} e^{st} ds - \sum_{p=0}^{\infty} (-1)^p b^{p+1} \int_{(c)} e^{s(t-p)} / s^{p+2} ds \right).
\]

Giving that (see [12], p.8)
\[
\int_{(c)} e^{hk} ds = \begin{cases} \frac{h^n}{n!}, & \text{Re } h > 0, \\ 0, & \text{Re } h < 0, \end{cases}
\]
we obtain the solution
\[
y(t) = y_0 \left(1 - \sum_{p=0}^{[t]} (-1)^p b^{p+1} (t-p)^{p+1} / (p+1)! \right),
\]
where \([t]\) denotes the largest integer less or equal to \( t.\)

**B DTM Performances**

Let a nonlinear DDE with backward-forward delays, and variable coefficients\(^{10}\)
\[
y''(t) - e^{-t}y'(t-1)y(t+1) = 0, \tag{B.1}
\]
where the BCs are: \( y(0) = y'(0) = 1, t \in [0, 1]. \)
The exact solution is clearly \( e^t. \) Using the Table B.1\(^{11}\) knowing that \( \mathcal{D}[e^{-x}] = (-1)^k / k!, \) we obtain the transformation of Eq.(B.1)
\[
(k+1)(k+2)Y(k+2)
\]
\[
- \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} \sum_{h_1=0}^{T} \sum_{h_2=0}^{T} (k_1+1) \left( \begin{array}{c} h_1 \\ k_1 + 1 \end{array} \right) \left( \begin{array}{c} h_2 \\ k_2 - k_1 \end{array} \right) (-1)^{h_1-k_1-1} Y(h_1)Y(h_2)
\]
\[
\times \left( \begin{array}{c} h_1 \\ k_2 - k_1 \end{array} \right) \left( \begin{array}{c} h_2 \\ k_2 - k_1 \end{array} \right) (-1)^{k-k_2} = 0.
\]

\(^8\) The inverse Laplace transform is given for a suitable \( c \) by
\[
\mathcal{L}^{-1}[Y(s)] = (2\pi i)^{-1} \int_{c-j\infty}^{c+j\infty} Y(s)e^{st} ds = y(t).
\]
\(^9\) The contour integral is represented by
\[
\int_{(c)} F(s)e^{st} ds = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{c-jT}^{c+jT} F(s)e^{st} ds.
\]
\(^{10}\) This example is drawn from [2] with adapted notations, and
using the package MATHEMATICA.

\(^{11}\) The Table B.1 of usual differential transforms is issued from
the theorems by Arikoglu (2006) [2].
The BCs are also transformed to \( Y(0) = Y(1) = 1 \). Choosing \( T = 8 \), with the BCs and \( k = 0, 1, 2 \), a nonlinear system in \( Y(1), Y(2), \ldots, Y(8) \) is solved. Using the inverse transformations from Table B.1, the series solution for the problem is obtained

\[
g(t) = 1 + t + 0.499973t^2 + 0.166615t^3 + 0.041599t^4 \]
\[
+ 0.008289t^5 + 0.001373t^6 + 0.000187t^7 + 0.000016t^8 + \mathcal{O}(t^9).
\]

The errors increase significantly with time in Fig. B.1.

![Figure B.1: DTM performances](image)

**C Tinbergen’s shipbuilding cycle**

The Tinbergen’s equation [13] is of the form

\[
g'(t) = -by(t - 1), \quad b > 0, \quad t > \theta. \quad \text{(C.1)}
\]

It is also assumed that \( g(t) = h(t), \quad t \in [0, \theta) \), where \( h(t) \) is some given function. In this application to the shipbuilding industry, \( y \) denotes the deviation of the tonnage from a mean value and \( \theta \) a given constant construction time. In this equation, Tinbergen assumes the rate of new shipbuilding to be proportional to a delayed tonnage deviation, with a one to two years delay \( \theta \) and a ranged intensity reaction \( b \in \left[ \frac{1}{2}, 1 \right] \). An endogenous cycle is found for the shipbuilding industry, with a period of 7 years 6 months for \( \theta = 1 \) and 8 years 9 months for \( \theta = 2 \).

### C.1 Characteristic equation

Let the form of the unknown function be \( g(t) = e^{\rho t} \), the characteristic equation from Eq. (C.1) is

\[
D(\rho) = \rho + be^{-\rho \theta} = 0, \quad \rho \in \mathbb{C}, \quad \text{(C.2)}
\]

\[12\]Solving the nonlinear system by using the primitives of *MATHEMATICA*, 128 solutions are obtained. The 20 real solutions are shown in Fig. B.1 to the left.

\[13\]A nonlinear DDE version is given by [12]

\[
y'(t) = -by(t - 1) - \varepsilon y^3(t - 1), \quad \varepsilon, b > 0.
\]

<table>
<thead>
<tr>
<th>(g(t) \pm h(t))</th>
<th>(G(k) \pm H(k))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c \ g(t))</td>
<td>(c \ G(k_1))</td>
</tr>
<tr>
<td>(t^n)</td>
<td>(\delta(k - n) = \begin{cases} 1, &amp; k = n \ 0, &amp; k \neq n \end{cases})</td>
</tr>
<tr>
<td>(g(t) \ h(t))</td>
<td>(\sum_{k_1=0}^{k} G(k_1) H(k - k_1))</td>
</tr>
<tr>
<td>(g^{(n)}(t))</td>
<td>(\frac{(k+n)!}{k!} G(k + n))</td>
</tr>
<tr>
<td>(p(t) \ g^{(n)}(t))</td>
<td>(\sum_{h_1=0}^{k} P(k_1) \frac{(k-k_1+n)!}{(k-k_1)!} \times G(k - k_1 + n))</td>
</tr>
<tr>
<td>(g(t + a))</td>
<td>(\lim_{T \to \infty} \sum_{h_1=0}^{T} \binom{h_1}{k} a^{h_1-k} G(h_1))</td>
</tr>
<tr>
<td>(g^{(n)}(t + a))</td>
<td>(\lim_{T \to \infty} (\frac{(k+n)!}{k!}) \sum_{h_1=0}^{T} \binom{h_1}{k+n} a^{h_1-k-n} G(h_1))</td>
</tr>
<tr>
<td>(p(t) \ g^{(n)}(t + a))</td>
<td>(\lim_{T \to \infty} \sum_{k_1=0}^{k} \sum_{h_1=0}^{T} \binom{h_1}{k} P(k_1) \frac{(k-k_1+n)!}{(k-k_1)!} \times a^{h_1-k+k_1-n} P(k_1) G(h_1))</td>
</tr>
</tbody>
</table>

Table B.1: Differential transforms \( F(k) = D[f(t)] \)
where \( \rho = \beta + \alpha j, j = \sqrt{-1} \). Inserting Eq.(C.2) into Eq.(C.1) and separating the real and imaginary parts, we get the system

\[
\begin{align*}
\cos u &= -\frac{v}{\theta b} e^v, \\
\sin u &= \frac{1}{\theta b} e^v,
\end{align*}
\]

where \( u \equiv \alpha \theta \) and \( v \equiv \beta \theta \). Eliminating \( v \), we obtain an even function \( f(u) \) in which the structural coefficients \( \theta, b \) are not explicit. We have

\[
f(u) = \frac{u}{\tan u} + \ln \frac{\sin u}{u} = C, \quad (C.3)
\]

where \( C \equiv -\ln(\theta b) \). A further analysis of the characteristic equation is given by Pinney [12] by means of the \((x,k)\)-root plateau in the parameter space\(^{14}\). The properties of the characteristic equation are summarized in Fig.C.1 (see also [9]).

<table>
<thead>
<tr>
<th>existence of roots</th>
<th>kind of roots</th>
<th>range of ( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>no root</td>
<td>pseudo-positive</td>
<td>([0, \frac{1}{2}])</td>
</tr>
<tr>
<td>one root</td>
<td>pseudo-positive</td>
<td>([-\frac{1}{2}, 0])</td>
</tr>
<tr>
<td>2n roots</td>
<td>pseudo-positive</td>
<td>((2n - \frac{1}{2})\pi, (2n + \frac{1}{2})\pi, n \in \mathbb{N})</td>
</tr>
<tr>
<td>2n+1 roots</td>
<td>pseudo-positive</td>
<td>((- (2n + \frac{1}{2})\pi, - (2n - \frac{1}{2})\pi, n \in \mathbb{N})</td>
</tr>
</tbody>
</table>

Figure C.1: Characteristic equation properties

C.2 Existence of exponential components

Let \( z \equiv \rho \theta \), Eq.(C.2) may be expressed as

\[
-\frac{z}{\theta b} = e^{-z}. \quad (C.4)
\]

The two parts of Eq.(C.2) are plotted \(^{15}\) in Fig.C.2. The condition for tangency of the two curves \( 1/(\theta b) = e^{-z} \) is \( z = \ln(\theta b) \). Inserting in Eq.(C.3), we get \( C = 1 \). The solution of the DDE Eq.(C.1) is a pure exponential \(^{16}\) of the type

\[
y(t) = (C_1 + C_2 t) e^{\rho t}.
\]

For \( C > 1 \), the solutions are composed of two exponentials in the period ranges

\[
T \in \left( \frac{\theta}{k}, \frac{\theta}{k - \frac{1}{2}} \right), \quad k \in \mathbb{N}.
\]

C.3 Existence of cyclical components

A cycle corresponds to each real solution of Eq.(C.4) when \( C < 1 \). The two sides of this equation are represented in Fig.C.3. Real branches of \( f(u) \) decrease monotonically in all the intervals \([k2\pi, (2k + 1)\pi]\), \( k \in \mathbb{N}_0 \). According to \( u = \alpha \theta \) and \( \alpha = \frac{2\pi}{T} \), the corresponding period ranges are

\[
T \in \left( \frac{\theta}{k}, \frac{\theta}{k - \frac{1}{2}} \right), \quad k \in \mathbb{N}_0.
\]

The sine curves may be damped or undamped. A distinction is made between the major cycle of the first period and the minor cycles The corresponding patterns of components are shown in Fig. C.4.

References:


\(^{14}\)According to this concept, the parameters may be chosen in order to achieve some desired properties for the system. Let the complex number be \( \rho = x + jy \), the \((x,k)\)-root plateau represents the sets of parameter values for which the characteristic equation has \( k \) pseudo roots greater than \( x \). The equations of the \((x,k)\)-root plateau on the \( b \)-line are

\[
\begin{align*}
\text{Re} \; D(\rho) &= x + be^{-x} \cos y, \\
\text{Im} \; D(\rho) &= y - be^{-x} \sin y.
\end{align*}
\]

\(^{15}\)Fig. C.2 shows a state of a dynamic interactive MATHEMATICA plotting, with automatic sliders and controls.

\(^{16}\)A degenerate cycle with infinite period.


