Mutually Independent Hamiltonian Cycles in $k$-ary $n$-cubes when $k$ is odd

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Abstract: The $k$-ary $n$-cubes, $Q^k_n$, is one of the most well-known interconnection networks in parallel computers. Let $n \geq 1$ be an integer and $k \geq 3$ be an odd integer. It has been shown that any $Q^k_n$ is a $2n$-regular, vertex symmetric and edge symmetric graph with a hamiltonian cycle. In this article, we prove that any $k$-ary $n$-cube contains $2n$ mutually independent hamiltonian cycles. More specifically, let $v_i \in V(Q^k_n)$ for $0 \leq i \leq |Q^k_n| - 1$ and let $(v_0, v_1, \ldots, v_{|Q^k_n| - 1})$ be a hamiltonian cycle of $Q^k_n$. We prove that $Q^k_n$ contains $2n$ hamiltonian cycles of the form $(v_0, v_l, \ldots, v_0)$ for $0 \leq l \leq 2n - 1$, where $v_i \neq v_i^l$ whenever $l \neq l'$. The result is optimal since each vertex of $Q^k_n$ has exactly $2n$ neighbors.

Key Words: $k$-ary $n$-cubes, hamiltonian cycle, mutually independent, hypercubes.

1 Introduction

The hypercube family, denoted by $Q_n$, is one of the most well-known and popular interconnection networks due to its excellent properties such as the recursive structure, symmetry, small diameter, low degree, easy routing and so on. See [3, 7, 11] and their references. Therefore, numerous studies have been devoted to the hypercube family. See [2, 4, 6, 13, 14] for example.

The $k$-ary $n$-cube, denoted by $Q^k_n$, is an enlarged family of $Q_n$ that preserves many pleasing properties of the hypercubes. More precisely, $Q^k_n$ is a graph with $k^n$ vertices, and each vertex $u$ can be labeled by an $n$-bit string, $u = (u(n - 1), u(n - 2), \ldots, u(1), u(0))$, where $0 \leq u(i) \leq k - 1$ for all $0 \leq i \leq n - 1$. There is an edge between two vertices $u = (u(n - 1), u(n - 2), \ldots, u(1), u(0))$ and $v = (v(n - 1), v(n - 2), \ldots, v(1), v(0))$ in $Q^k_n$ if and only if there exists some $i$, $0 \leq i \leq n - 1$, such that $u(i) = v(i) \pm 1$ (mod $k$) and $u(j) = v(j)$ for all $j \neq i$. Obviously, the hypercube $Q_n$ is the same as $Q_2^n$ and hence is a subclass of the $k$-ary $n$-cube family. The following recursive interpretation of $Q^k_n$ can be found in [10]. First, we define a cycle of $k$ vertices labeled $0, 1, \ldots, k - 1$ to be a graph with edges between $i$ and $i + 1$ (mod $k$) for $i = 0, 1, \ldots, k - 1$. When $k = 1$, the “cycle” is a vertex. When $k = 2$, it consists of two vertices sharing an edge. When $k \geq 3$, it is a conventional cycle. The recursive structure of $k$-ary $n$-cubes, where $k \geq 3$, is as follows:

- A $k$-ary 1-cube is a cycle of $k$ vertices. Without loss of generality, we place the $k$ vertices on a line and call the leftmost vertex the 0th position vertex and the rightmost vertex the $(k - 1)$th position vertex.

- A $k$-ary $n$-cube contains $k$ composite subcubes, each of which is a $k$-ary $(n - 1)$-cube placed from left to right. For each position $i = \{0, \ldots, k^{n-1} - 1\}$, edges between composite subcubes are defined by connecting all $k$ $i$th position vertices into a cycle. See Figure 1 for an illustration.

It is shown that $Q^k_n$ is a bipartite graph when $k$ is an even integer [5].

The concept of mutually independent hamiltonian cycles arises from the following application [8]. If $k$
pieces of data must be sent from a message center $u$, and the data has to be processed at each intermediate receiver (and the process takes time), then the existence of mutually independent cycles from $u$ guarantees that there will be no waiting time at a processor. Recently, many studies about mutually independent hamiltonian cycles on hypercubes and other interconnection networks are published \cite{8, 9, 12, 13}. In this article, we prove that any $k$-ary $n$-dimensional cube $Q^n_k$ contains $2n$ mutually independent hamiltonian cycles for any integer $n \geq 1$ and any odd integer $k \geq 3$. The result is optimal since each vertex of $Q^n_k$ has exactly $2n$ neighbors. The article is organized as follows. In Section 2, we introduce the graph terminologies and notations. In Section 3, we present the main results. Our conclusion is given in the last section.

## 2 Preliminaries

For the graph definitions and notations we follow \cite{1}. $G = (V, E)$ is a graph if $V$ is a finite set and $E$ is a subset of $\{ (u, v) \mid (u, v) \in V \}$ is an unordered pair of $V$. We say that $V$ is the vertex set and $E$ is the edge set of $G$. Two vertices $u$ and $v$ are adjacent if $(u, v) \in E$. The total number of vertices of $G$ is denoted by $|V(G)|$ or $|G|$. For a vertex $u$ of $G$, we denote the degree of $u$ by $\deg(u) = |\{ v \mid (u, v) \in E \}|$. A graph $G$ is $k$-regular if, for every vertex $u \in G$, $\deg(u) = k$. A path is represented by $(v_0, v_1, v_2, \ldots, v_k)$, where any two consecutive vertices are connected by an edge. We also write the path $(v_0, v_1, v_2, \ldots, v_k)$ as $(v_0, P_1, v_1, v_2+1, \ldots, v_j, P_2, v_i, v_k)$, where $P_1$ is the path $(v_0, v_1, \ldots, v_j)$ and $P_2$ is the path $(v_j, v_{j+1}, \ldots, v_{k-1}, v_k)$. If a path $P = (v_0, v_1, v_2, \ldots, v_{k-1}, v_k)$, then $P^{-1}$ denotes the path $(v_k, v_{k-1}, \ldots, v_2, v_1, v_0)$. A hamiltonian path between $u$ and $v$, where $u$ and $v$ are two distinct vertices of $G$, is a path joining $u$ to $v$ that visits every vertex of $G$ exactly once. Two paths $P_1 = (u_0, u_1, \ldots, u_m)$ and $P_2 = (v_0, v_1, \ldots, v_m)$ from $a$ to $b$ are independent if $u_0 = v_0 = a$, $u_m = v_m = b$, and $u_i \neq v_i$ for $1 \leq i \leq m-1$ \cite{8}. Paths with the same number of vertices from $a$ to $b$ are mutually independent if every two different paths are independent \cite{8}. In \cite{4}, two paths $P'_1 = (u_0, u_1, \ldots, u_m)$ and $P'_2 = (v_0, v_1, \ldots, v_m)$ are full-independent if $u_i \neq v_i$ for all $0 \leq i \leq m$. Similarly, paths with the same number of vertices are mutually full-independent if every two different paths are full-independent.

A cycle is a path of at least three vertices such that the first vertex is the same as the last vertex. A hamiltonian cycle of $G$ is a cycle that traverses every vertex of $G$ exactly once. A hamiltonian graph is a graph with a hamiltonian cycle. The length of a cycle $C$ is the number of edges/vertices in $C$. Two cycles $C_1 = (u_0, u_1, \ldots, u_k, u_0)$ and $C_2 = (v_0, v_1, \ldots, v_k, v_0)$ beginning at $s$ are independent if $u_0 = v_0 = s$ and $u_i \neq v_i$ for $1 \leq i \leq k$ \cite{8}. Cycles beginning at $s$ with the same length are mutually independent if every two different cycles are independent. A graph $G$ is said to contain $n$ mutually independent hamiltonian cycles if there exist $n$ hamiltonian cycles in $G$ beginning at any vertex $s$ such that the $n$ cycles are mutually independent. Recently, there are many studies in mutually independent hamiltonian cycles. Readers can refer to \cite{4, 8, 9, 12, 13}.

## 3 Main Results

**Theorem 1** There exist two mutually independent hamiltonian cycles in $Q^n_k$, where $k \geq 3$ is an odd integer.

**Proof.** By definition, $Q^n_k$ is a cycle with $k$ vertices. Let the graph of $Q^n_k$ be $(V, E)$, where $V = \{ i \mid 0 \leq i \leq k-1 \}$ and $E = \{ (i, i+1 \mod k) \mid 0 \leq i \leq k-1 \}$. Let $C_1 = \{ (0, 1, 2, \ldots, k-1, 0) \}$ and $C_2 = \{ (0, k-1, k-2, \ldots, 2, 1, 0) \}$. Obviously, $C_1$ and $C_2$ are two mutually independent hamiltonian cycles of $Q^n_k$.

**Lemma 2** There exist four mutually independent hamiltonian cycles in $Q^n_{2k}$, denoted by $C^n_{2k}$ for $0 \leq i \leq 3$, such that $(2, 2), (2, 0) \in C^n_{2k}$. $(0, 0), (0, 0) \in C^n_{2k}$, $(0, 0), (2, 2) \in C^n_{2k}$, and $(0, 2), (2, 2) \in C^n_{2k}$.

**Proof.** We construct the four mutually independent hamiltonian cycles below, where the edges required in the lemma are underlined.

\[
C^n_{0} = \langle (1, 1), (2, 1), (2, 2), (2, 0), (1, 0), (0, 0), (0, 1), (2, 1), (1, 1) \rangle;
\]

\[
C^n_{1} = \langle (1, 1), (1, 0), (2, 0), (0, 0), (0, 1), (0, 2), (1, 2), (2, 2), (1, 1) \rangle;
\]

\[
C^n_{2} = \langle (1, 1), (0, 1), (0, 0), (2, 2), (1, 2), (2, 2), (2, 1), (2, 0), (1, 0), (1, 1) \rangle;
\]

\[
C^n_{3} = \langle (1, 1), (1, 2), (0, 2), (2, 2), (2, 1), (2, 0), (0, 0), (0, 1), (1, 1) \rangle.
\]

The lemma is proved.

We shall rewrite the four mutually independent hamiltonian cycles in Lemma 2 to simplify the proof in Lemma 3 as follows:

\[
C^n_{3} = \langle (1, 1), P^n_{3}, (2, 2), (0, 2), \overline{P^n_{3}}, (1, 1) \rangle;
\]

\[
C^n_{1} = \langle (1, 1), P^n_{3}, (2, 0), (0, 0), \overline{P^n_{3}}, (1, 1) \rangle;
\]

\[
C^n_{2} = \langle (1, 1), P^n_{3}, (0, 0), (0, 2), \overline{P^n_{3}}, (1, 1) \rangle;
\]

\[
C^n_{3} = \langle (1, 1), P^n_{3}, (0, 2), (2, 2), \overline{P^n_{3}}, (1, 1) \rangle.
\]
Similarly for

patterns to facilitate our derivation in the following:

Note that

Let

\[ f(0, i, j) = \langle (0, i), (0, i+1), (0, i+2), \ldots, (0, j-1), (0, j) \rangle; \]
\[ s(i, j) = \langle (k-1, i), (k+1, i), (k+1, i+2), \ldots, (k-1, j), (k-1, j) \rangle; \]
\[ w(i, j) = \langle (i, 0), (i+1, 0), (i+2, 0), \ldots, (j-1, 0), (j, 0) \rangle; \]
\[ e(i, j) = \langle (i, k-1), (i+1, k-1), (i+2, k-1), \ldots, (j-1, k-1), (j, k-1) \rangle; \]
\[ N^{-1}(i, j) = \langle (0, 0), (0, j-1), (0, j-2), \ldots, (0, i+1), (0, i) \rangle; \]
\[ s^{-1}(i, j) = \langle (k-1, i), (k-1, j-1), (k-1, j-2), \ldots, (k+1, i), (k+1, i) \rangle; \]
\[ w^{-1}(i, j) = \langle (j, 0), (j+1, 0), (j+2, 0), \ldots, (i+1, 0), (i, 0) \rangle; \]
\[ e^{-1}(i, j) = \langle (j, k-1), (j+1, k-1), (j+2, k-1), \ldots, (i+1, k-1), (i, k-1) \rangle. \]

Please see Figure 2 for an illustration in \( Q_2^5 \). Define a function \( f : Q_2^k \rightarrow Q_2^{k+2} \) as follows. If \( x = (i, j) \in V(Q_2^k) \), then \( f(x) = (i+1, j+1) \) and \( f(x) \in V(Q_2^{k+2}) \); if \( m \geq 2 \) and \( P = \langle x_1, x_2, \ldots, x_m \rangle \) is a path of \( Q_2^k \), then \( f(P) = \langle f(x_1), f(x_2), \ldots, f(x_m) \rangle \) is a path of \( Q_2^{k+2} \).

**Lemma 3** There exist four mutually independent hamiltonian cycles in \( Q_2^5 \), \( \{ C^5_0, C^5_1, C^5_2, C^5_3 \} \), that satisfy the following requirements:

1. For \( 0 \leq i \leq 3 \), \( C^5_i \) is a cycle extended by the hamiltonian cycle \( C^3_0 \) of \( Q_2^5 \) extended by \( C^3_0 \) of \( Q_2^2 \).

   - \( C^5_0 \) contains \( (4,0), (4,4) \), \( C^5_1 \) contains \( (0,0), (4,0) \), \( C^5_2 \) contains \( (0,4), (0,0) \), and \( C^5_3 \) contains \( (4,4), (0,4) \).

   **Proof.** By brute force, we construct the four mutually independent hamiltonian cycles in \( Q_2^5 \) extended by \( Q_2^2 \) as follows.

   \[ C^5_0 = \langle (f_1(1,1), f(0,0)), (f((2,2), (4,3), S^{-1}(3,0), (4,0), (4,4), E^{-1}(4,0), (0,4), N^{-1}(4,0), (0,0), W(0,3), (3,0), f((2,0)), f((0,0)), f((1,1))) \rangle; \]
   \[ C^5_1 = \langle (f_1(1,1), f(0,0)), (f((2,0), (3,0), W^{-1}(3,0), (0,0)), (0,4), S((0,4), (4,4), E^{-1}(4,0), (0,4), N^{-1}(4,1), (0,1), f((0,0)), f((0,0)), f((1,1))) \rangle; \]
   \[ C^5_2 = \langle (f_1(1,1), f(0,0)), (f((0,0), (1,4), N(1,4), (0,4), (0,0), W((0,4), (4,4), S((0,4), (4,4), E^{-1}(4,1), (1,4), f((0,2)), f((0,0)), f((1,1))) \rangle; \]
   \[ C^5_3 = \langle (f_1(1,1), f(P_3^2), f((0,2), (1,4), E(1,4), (4,4), (0,4), N^{-1}(4,0), (0,0), W((0,4), (4,4), (0,0), S((0,3), (4,3), f((2,2)), f((0,0)), f((1,1))) \rangle. \]

   Note that the underlined edges are required by the lemma. Please see Figure 3 for an illustration. Obviously, \( C^5_i \)'s satisfy the two requirements in the lemma, where \( 0 \leq i \leq 3 \).

**Theorem 4** Let \( k \geq 5 \) be an odd integer. There exist four mutually independent hamiltonian cycles in \( Q_2^k \), \( \{ C^k_0, C^k_1, C^k_2, C^k_3 \} \), that satisfy the following requirements:

1. For \( 0 \leq i \leq 3 \), \( C^k_i \)'s are cycles extended by hamiltonian cycles \( C^{k-2}_i \)'s of \( Q_2^{k-2} \). More specifically, \( f(e) \) is an edge on \( C^k_1 \) for
all \( e \in E(Q_k^{k-2}) \cap C_k^{k-2} \cup \{(k-3, k-3), (k-3, 0), (k-3, 0), (0, 0)\} \).

2. \( C_k^0 \) contains \((k-1, 0), (k-1, k-1)\), \( C_k^1 \) contains \((0, 0), (k-1, 0)\), \( C_k^2 \) contains \((0, k-1), (0, 0)\), and \( C_k^3 \) contains \((k-1, k-1), (0, 0)\).

**Proof.** The theorem will be proved using mathematical induction. With Lemma 3, we know that the theorem holds for \( Q_5^k \) for \( k = 5 \). Using the induction hypothesis, we assume the theorem holds when \( k = k^* - 2 \), where \( k^* \) is an odd integer and \( k^* \geq 7 \). We want to prove that the statement is true when \( k = k^* \).

Suppose the four mutually independent hamiltonian cycles of \( Q_{k^*}^{k^*-2} \) that satisfy the requirements are \( C_i^{k^*-2} \), \( 0 \leq i \leq 3 \). Namely, \( C_i^{k^*-2} \)'s satisfy the following:

(i) For \( 0 \leq i \leq 3 \), \( C_i^{k^*-2} \) is a cycle extended by the hamiltonian cycle \( C_{i-1}^{k^*-4} \) of \( Q_{k^*}^{k^*-4} \). More specifically, \( f(e) \) is an edge on \( C_{i-1}^{k^*-4} \) for all \( e \in E(Q_{k^*}^{k^*-4}) \cap C_{k^*-4} \setminus \{((k^*-5, k^*-5), (k^*-5, 0))\}, (k^*-5, 0)), ((0, 0), (0, 0)), (0, k^*-5))\}.

(ii) \( C_0^{k^*-2} \) contains \((k^*-3, 0), (k^*-3, k^*-3)\), \( C_1^{k^*-2} \) contains \((0, 0), (k^*-3, 0)\), \( C_2^{k^*-2} \) contains \((0, k^*-3), (0, 0)\), and \( C_3^{k^*-2} \) contains \((k^*-3, k^*-3), (0, k^*-3)\).

Define \( t = \frac{k^*-3}{2} \). By (ii) above, let

\[
\begin{align*}
C_0^{k^*-2} & = \langle (t, t), P_0^{k^*-2}, (2t, 2t), (2t, 0), P_0^{k^*-2}, (t, t) \rangle; \\
C_1^{k^*-2} & = \langle (t, t), P_1^{k^*-2}, (2t, 0), (0, 0), P_0^{k^*-2}, (t, t) \rangle; \\
C_2^{k^*-2} & = \langle (t, t), P_2^{k^*-2}, (0, 0), (2t, 2t), P_0^{k^*-2}, (t, t) \rangle; \\
C_3^{k^*-2} & = \langle (t, t), P_3^{k^*-2}, (0, 2t), (2t, 2t), P_0^{k^*-2}, (t, t) \rangle.
\end{align*}
\]

When \( k = k^* \), we construct the mutually independent hamiltonian cycles of \( Q_k^{k^*-2} \) as follows:

\[
\begin{align*}
C_0^{k^*} & = \langle f(t, t), f(P_0^{k^*-2}), f((2t, 2t)), (k^*-1, k^*-2), S^{-1}(k^*-2, 0), (k^*-1, 0), (k^*-1, k^*-1) \rangle, \\
E^{-1}(k^*-1, 0), (0, k^*-1), N^{-1}(k^*-1, 0), (0, 0), W(0, k^*-2), (k^*-2, 0), f((2t, 0)), \\
f((t, t)), f(P_0^{k^*-2}) \rangle; \\
C_1^{k^*} & = \langle f(t, t), f(P_1^{k^*-2}), f((2t, 0)), (k^*-2, 0), W^{-1}(k^*-2, 0), (0, 0), (k^*-1, 0), S(0, k^*-1), \\
(k^*-1, k^*-1), E^{-1}(k^*-1, 0), (0, k^*-1) \rangle; \\
C_2^{k^*} & = \langle f((t, t), f(P_2^{k^*-2}), f((2t, 0)), (k^*-2, 0), W^{-1}(k^*-2, 0), (0, 0), (k^*-1, 0), S(0, k^*-1), \\
(k^*-1, k^*-1), E^{-1}(k^*-1, 0), (0, k^*-1) \rangle; \\
C_3^{k^*} & = \langle f((t, t), f(P_3^{k^*-2}), f((2t, 0)), (k^*-2, 0), W^{-1}(k^*-2, 0), (0, 0), (k^*-1, 0), S(0, k^*-1), \\
(k^*-1, k^*-1), E^{-1}(k^*-1, 0), (0, k^*-1) \rangle.
\end{align*}
\]

It is easy to check that \( C_i^{k^*} \), \( 0 \leq i \leq 3 \), satisfy the requirements in the theorem. Therefore, Theorem 4 is proved by the mathematical induction.

**Lemma 5** Let \( k \in \{3, 5, 7\} \). There exist four mutually independent hamiltonian cycles in \( Q_k^{k} \) of the form \( C_{i}^{k} = \langle x_{i}(0), x_{i}(1), x_{i}(2), x_{i}(3), \ldots, x_{i}(k^*-2), x_{i}(0) \rangle \), where \( 0 \leq i \leq 3 \), such that we can find four mutually full-independent hamiltonian paths between \( x_{i}(2) \) and \( x_{i}(3) \).

**Proof.** There are three cases.

**Case 1.** \( k = 3 \). Let \( C_{i}^{k} \), \( 0 \leq i \leq 3 \), be the same as in Lemma 2. With the four edges \( ((2, 2), (2, 0)), ((2, 0), (0, 0)), ((0, 0), (0, 2)) \) and \( ((0, 2), (2, 2)) \), we can construct four mutually full-independent hamiltonian paths as follows:

\[
\begin{align*}
HP_0^{3} & = \langle (2, 2), (0, 2), (1, 2), (1, 1), (2, 1), (0, 1), (0, 0), (1, 0), (2, 0) \rangle; \\
HP_{1}^{3} & = \langle (2, 0), (2, 1), (2, 2), (1, 2), (1, 1), (0, 1), (0, 2), (0, 0) \rangle; \\
HP_{2}^{3} & = \langle (0, 0), (1, 0), (2, 0), (2, 1), (2, 2), (1, 2), (1, 1), (0, 1), (0, 2) \rangle; \\
HP_{3}^{3} & = \langle (0, 2), (1, 2), (1, 1), (0, 1), (0, 0), (1, 0), (2, 0), (2, 1), (2, 2) \rangle.
\end{align*}
\]

**Case 2.** \( k = 5 \). Let \( C_{i}^{k} \), \( 0 \leq i \leq 3 \), be the same as in Lemma 3. With the four edges \( ((3, 3), (4, 3)), ((3, 1), (3, 0)), ((1, 1), (0, 1)) \) and \( ((1, 3), (1, 4)) \), we can construct four mutually full-independent hamiltonian paths as follows:

\[
\begin{align*}
HP_{0}^{5} & = \langle (3, 3), (3, 4), (2, 4), (2, 3), (1, 3), (1, 4), (0, 4), (0, 3), (0, 2), (0, 1), (0, 0), (1, 0), (2, 0), (3, 0), \\
(3, 1), (2, 1), (1, 1), (1, 2), (2, 2), (3, 2), (4, 2), (4, 1), (4, 0), (4, 4), (4, 3) \rangle; \\
HP_{1}^{5} & = \langle (3, 1), (4, 1), (0, 1), (1, 1), (2, 1), (2, 2), (1, 0), (0, 0), (0, 4), (0, 3), (0, 2), (1, 2), (1, 3), (1, 4) \rangle.
\end{align*}
\]
Case 3. $k = 7$. Let $C^*_n$, where $0 \leq i \leq 3$, be the same as in Theorem 4. With the four edges \((2, 3, (4, 3), (3, 3), (3, 4), (4, 4), (4, 4), (3, 0));\) \[(1, 1), (1, 2), (1, 3), (1, 4), (2, 4), (2, 3), (2, 2), (2, 1), (3, 1), (3, 2), (3, 3), (4, 4), (4, 3), (4, 2), (4, 1), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 1));\) \[H P^*_2 = \{(1, 3), (0, 3), (0, 4), (0, 0), (1, 1), (0, 1), (1, 4), (4, 0), (3, 0), (2, 0), (1, 0), (0, 0), (0, 4), (0, 3), (0, 2), (0, 1));\) \[H P^*_3 = \{(1, 3), (0, 3), (0, 4), (0, 0), (1, 1), (0, 1), (1, 4), (4, 0), (3, 0), (2, 0), (1, 0), (0, 0), (0, 4), (0, 3), (0, 2), (0, 1));\) these four paths are much more lengthy and hence are skipped here.

**Lemma 6** Let $k \geq 7$ be an odd integer. For $0 \leq i \leq 3$, there exist four mutually independent hamiltonian cycles in $Q^*_k$ of the form $C^*_n = \langle x_i^0(0), x_i^1(1), x_i^2(2), \ldots, x_i^{4k(n-1)}, x_i^0(0) \rangle$ such that one can construct four mutually full-independent hamiltonian paths $H P^*_2$ between the four edges \[(x_i(2), x_i(3)) \mid 0 \leq i \leq 3\].

**Proof.** The lemma will be proved using mathematical induction. Obviously, the base case for $k = 7$ is proved by Lemma 5. With Theorem 4, we know that the four mutually independent hamiltonian cycles of $Q^*_k$ can be obtained using the four mutually independent hamiltonian cycles of $Q^{k-2}_k$. Therefore, the corresponding mutually full-independent hamiltonian paths between $(x_i(2), x_i(3))$ for $0 \leq i \leq 3$ are constructed in a very similar way. The proof is skipped here.

**Theorem 7** Let $n \geq 2$ be an integer and $k \geq 3$ be an odd integer. Let $i$ be an integer with $0 \leq i \leq 2n - 1$. There exist $2n$ mutually independent hamiltonian cycles in $Q^*_n$ of the form $C^*_n = \langle x(0), x_1(1), x_2(2), \ldots, x_i(k^n - 1), x(0) \rangle$ such that one can find $2n$ mutually full-independent hamiltonian paths between $x_i(j)$ and $x_i(j + 1)$ for some $1 \leq j \leq k^n - 2$.

**Proof.** The theorem will be proved using mathematical induction. With Lemma 5 and 6, we know that the theorem holds for $Q^*_2$ for any odd integer $k \geq 3$ and $j = 2$. Note that $Q^*_2$ consists of $k$ copies of $Q^*_n$, where $n \geq 3$. Let them be $Q^*_{n-1}, Q^*_{n-1}, \ldots, Q^*_{n-1}$.

With the induction hypothesis, we assume that the theorem holds for each $Q^*_{n-1}$, where $0 \leq k' < k - 1$.

More precisely, let $N' = |V(Q^*_{n-1})|$, which equals $k^{n-1}$. Then there exist $2n - 2$ mutually independent hamiltonian cycles in $Q^*_n$. Let them be $C^*_i = \langle u, y_i(1), y_i(2), \ldots, y_i(N' - 2), y_i(N') \rangle$ in $Q^*_{n-1}$ for $0 \leq i \leq 2n - 3$. Moreover, there exist $2n - 2$ mutually full-independent hamiltonian paths of the form \[(y_i(1) = x_i(0), x_i(1), \ldots, x_i(N' - 1) = y_i(2))\] in $Q^*_n$.

Let $R^*_i = \{x_i(0), x_i(1), \ldots, x_i(N' - 1), x_i(1), \ldots, x_i(N' - 2) \mid 0 \leq i \leq 2n - 3\}$. Obviously, $R$ is a set of mutually full-independent hamiltonian paths of $Q^*_{n-1}$. By deleting $x_i(0)$ from $R^*_i$, we define $R^*_i = \{x_i(1), \ldots, x_i(N') \}$ for $0 \leq i \leq 2n - 3$. Then \[R^*_i \mid 0 \leq i \leq 2n - 3\] is a set of mutually full-independent paths of $Q^*_{n-1}$.

Thus $C^*_i = \langle u, y_i(0), \ldots, y_i(1), x_i(0), \ldots, x_i(N' - 1), x_i(1), x_i(1), \ldots, x_i(N' - 1) \rangle, R^*_i \rangle$ for $0 \leq i \leq 2n - 3$. In addition, $\{C^*_i \mid 0 \leq i \leq 2n - 3\}$ is a set of mutually independent hamiltonian cycles of $Q^*_n$.

In every $Q^*_{n-1}$, there is a hamiltonian cycle $C^*_{n-1} = \langle u, y_i(0), \ldots, y_i(N' - 1), y_i(N') \rangle, u \rangle$. Let $L^*_{i} = \langle y_i(0), y_i(1), y_i(2), \ldots, y_i(N' - 3), y_i(N' - 2) \rangle$ and let $L^*_{i} = \langle y_i(0), y_i(1), y_i(2), \ldots, y_i(N' - 3), y_i(N' - 2) \rangle$. Then let $C_{2n-2}$ be $\langle u, u^{-1}, y_i(1), y_i(2), \ldots, y_i(N' - 3), y_i(N' - 2) \rangle$.
2), \[ u^{k-1}, u^{k-2}, y^{k-2}(0), y^{k-3}(0), y^{k-4}(0), L^{k-4}, y^{k-4}(N'-2), y^{k-5}(N'-2), (L^{k-5})^{-1}, y^{k-5}(0), y^{2l+1}(0), L^{2l+1}, y^{2l+1}(N' - 2), y^{2l}(N' - 2), (L^{2l})^{-1}, y^{2l}(0), \ldots, y^{l}(0), L^{1}, y^{l}(N') \]

\[ y^{k-1}(N' - 3), y^{k-2}(N' - 3), (L^{k-2})^{-1}, y^{k-2}(1), y^{k-3}(1), L^{k-3}, y^{k-3}(N' - 3), y^{k-3}(N' - 2), u^{k-3}, u^{k-3}, \ldots, u^{1}, u^{0} \]

where \( 0 \leq l \leq \frac{k-3}{2} \) is an integer. Please see Figure 5 for an illustration.

Finally, let \( C_{2n-1} = (0, 1, y^{1}(0), L^{1}, y^{0}(N') - 2), (1, 0, y^{0}(0), y^{1}(0), (L^{1})^{-1}, y^{1}(0), L^{1}, y^{1}(N') - 2), y^{k-2}(N' - 2), (L^{k-2})^{-1}, y^{k-2}(0), y^{2l}(0), L^{2l}, y^{2l}(N' - 2), y^{2l+1}(N' - 2), (L^{2l+1})^{-1}, \ldots, y^{l}(0), L^{2}, y^{0}(N' - 2), u^{2}, u^{3}, \ldots, u^{k-1}, u^{0} \]

where \( 2 \leq l \leq \frac{k-3}{2} \) is an integer.

With \( \{ C_i \mid 0 \leq i \leq 2n - 1 \} \), we obtain the 2n mutually independent hamiltonian cycles of \( Q_n^k \). The theorem is proved.

4 Conclusion

Let \( n \geq 1 \) be an integer and \( k \geq 3 \) be an odd integer. There exist \( 2n \) mutually independent hamiltonian cycles in \( Q_n^k \). Moreover, let the \( 2n \) mutually independent hamiltonian cycles be \( C_1 = (x_i(0), x_i(1), x_i(2), \ldots, x_i(k^{n} - 1), x_i(0)) \), where \( 0 \leq i \leq 2n - 1 \). We show that there exists \( 2n \) mutually full-independent hamiltonian paths between \( x_i(j) \) and \( x_i(j + 1) \) for some \( j \). The result is optimal since each vertex in \( Q_n^k \) has exactly \( 2n \) neighbors.

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