EXISTENCE AND ITERATIVE APPROXIMATION OF SOLUTIONS OF SOME SYSTEMS OF VARIATIONAL INEQUALITIES AND INCLUSIONS

SYED HUZOORUL H. KHAN
Department of Mathematics
Aligarh Muslim University
Aligarh-202002
INDIA
E-mail: huzoorkhan@yahoo.com

Abstract: In this paper, we consider a system of general variational inclusions (SGVI) in $q$-uniformly smooth Banach spaces. Using proximal-point mapping technique, we prove the existence and uniqueness of solution and suggest a Mann type perturbed iterative algorithm for SVLI. We also discuss the convergence criteria and stability of Mann type perturbed iterative algorithm. Further, we consider a system of parametric general variational inclusions (SPGVI) corresponding to SGVI and discuss the continuity of the solution. Finally, we consider a system of generalized variational inequality problems (SGVIP) in Hilbert spaces. We prove an existence theorem for auxiliary problems of SGVIP. By exploiting this theorem, an algorithm for the SGVIP is constructed. Further, we prove the existence of a unique solution of SGVIP and discuss the convergence analysis of the algorithm.

The techniques and results presented here improve the corresponding techniques and results for the variational inequalities and inclusions in the literature.

Keywords: System of general variational inclusions; system of parametric general variational inclusions; system of generalized variational inequality problems; proximal-point mapping technique; auxiliary problems; Mann type perturbed iterative algorithm; convergence criteria; stability; sensitivity analysis.

2000 AMS Subject Classifications: 49J40, 47N10, 65K10, 90C47
1. Introduction

Variational inequality theory introduced by Stampacchia [90] and Fichera [34] independently, in early sixtees in potential theory and mechanics, respectively, constitutes a significant extension of variational principles. It has been shown that the variational inequality theory provides the natural, descent, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problem arising in inelasticity, economics, transportations, optimization, control theory and engineering sciences [6-10,19,35-37,65,72]. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative behavior of solutions to important classes of problems. On the other hands, it enables us to develop highly efficient and powerful numerical methods to solve, for example, obstacle, unilateral, free and moving boundary value problems. In last four decades, considerable interest has been shown in developing various classes of variational inequalities, both for its own sake and for its applications.

In 1968, Brézis [11] initiated the study of the existence theory of a class of variational inequalities later known as variational inclusions, using proximal-point mappings due to Moreau [70]. Variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. For application of variational inclusions, see for example [7,22].

One of the most important and interesting problem in the theory of variational inequalities is the development of numerical methods which provide an efficient and implementable algorithm for solving variational inequalities and its generalizations. Some of such methods are projection methods and its various forms, linear approximation, descent and Newton’s methods, and the methods based on auxiliary principle technique. The method based on proximal-point mapping is a generalization of projection method and has been widely used to study the existence of solution and to develop iterative algorithms for variational inclusions, see for example [1,14,23,24,28,40-42,46-50,53,62,78,82,88,89,97,98]. In recent past, the methods based on different classes of proximal-point mappings have been developed to study the existence of solutions and to discuss convergence and stability analysis of iterative algorithms, for various classes of variational (like) inclusions, see for example [2,13,24,28-30,42,47,49,52,55,56,97].

It is worth mentioning that the projection method and its variant forms cannot be extended for constructing iterative algorithms for variational-like inequalities, since it is not possible to find the projection, and variational inequalities with non differential terms. To overcome this drawback, one uses usually the auxiliary principle technique which does not depend on the projection mapping. This technique deals with finding a suitable auxiliary problem for the original problem. Further, this auxiliary problem is used to construct an algorithm for solving the original problem. Glowinski, Lions and Tremolieres [37] introduced this technique and used it to study the existence of a solution of mixed variational inequality. Later, Noor [73,74,76,77], Huang and Deng [44], Chidume, Kazmi and Zeg-
eye [12], Kazmi and Khan [57,63] and Zeng et al. [99-101] extended this technique to suggest and analyze a number of algorithms for solving various classes of variational inequalities.

In recent years, much attention has given to develop general techniques for the sensitivity analysis of solution set of various classes of variational inequalities (inclusions). From the mathematical and engineering point of view, sensitivity properties of various classes of variational inequalities can provide new insight concerning the problem being studied and can stimulate ideas for solving problems. The sensitivity analysis of solution set for variational inequalities have been studied extensively by many authors using quite different techniques. By using the projection technique; Defermos [21], Mukherjee and Verma [71], Yen [94], Liu et al. [67], Park and Jeong [87], Ding [25,26] and Ding and Luo [27] studied the sensitivity analysis of solution of some classes of variational inequalities. By using the implicit function approach that makes use of so called normal mappings, Robinson [91] studied the sensitivity analysis of solution for variational inequalities in finite-dimensional spaces. By using resolvent operator technique, Adly [1], Noor [79,81], Agarwal et al. [3], Kazmi and Khan [54,58,59] and Kazmi et al. [61] studied the sensitivity analysis of solution of some classes of variational inequalities.

In 1985, Pang [86] showed that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a variational inequality defined on the product sets. He decomposed the original variational inequality into a system of variational inequalities and discuss the convergence of method of decomposition for system of variational inequalities. Later, it was noticed that variational inequality over product sets and the system of variational inequalities both are equivalent, see for applications, [6,33,72,86]. Since then many authors, see for example [4,18,33,45,64,72] studied the existence theory of various classes of system of variational inequalities by exploiting fixed-point theorems and minimax theorems. On the other hand, only a few iterative algorithms have been constructed for approximating the solution of system of variational inequalities. Recently, Verma [92] studied the approximate solvability of a system of nonlinear strongly monotone variational inequalities based on system of projection methods. Very recently, Kazmi and Bhat [48], Kazmi and Khan [55], and Cho et al. [16] extended the work of Verma [92] to systems of variational (-like) inclusions using the systems of proximal-point mapping methods. Further, Kazmi and Khan [57] studied the system of variational inequalities using auxiliary principle.

We remark that most of the work on approximate solvability of variational (-like) inclusions and systems of variational (-like) inequalities has been done in the setting of Hilbert spaces.

Motivated by recent work going in this direction, we consider a system of general variational inclusions (in short, SGVI) involving m-accretive mappings in real q-uniformly smooth Banach spaces. Using proximal-point mapping technique, we prove the existence and uniqueness of solution and suggest a Mann type perturbed iterative al-
gorithm for SGVI. Furthermore, we discuss the convergence criteria and stability of Mann type perturbed iterative algorithm.

Next we consider a system of parametric general variational inclusions (in short SPGVI) associated with SGVI and study the behaviour and sensitivity analysis of the solution set for SPGVI. Further, the Lipschitz continuity of solution set of SPGVI is proved under suitable conditions.

Finally, we consider a system of generalized variational inequality problems (in short, SGVIP) involving nondifferentiable terms and relaxed cocoercive mapping, and its related auxiliary problems in real Hilbert spaces. An existence theorem for auxiliary problems is established. Furthermore, we prove the existence of a unique solution of SGVIP and discuss the convergence analysis of the algorithm. For related work of variational inequalities involving relaxed cocoercive mappings, see for example [15,83,84]. The technique and results presented in this paper generalize and unify the corresponding techniques and results given in [1,2,15,16,25,27,44,71,74,76,79,81,87,92,93].

2. System of Variational Inclusions

Let $E$ be a real Banach space equipped with norm $\| \cdot \|$, let $E^*$ be the topological dual space of $E$; let $\langle \cdot , \cdot \rangle$ be the dual pair between $E$ and $E^*$ and let $2^E$ be the power set of $E$.

Definition 1[95]. For $q > 1$, a mapping $J_q : E \to 2^{E^*}$ is said to be generalized duality mapping, if it is defined by

$$J_q(x) = \{ f \in E^* : \langle x, f \rangle = \| x \|^q \},$$

for all $x \in E$. \hspace{1cm} (1)

In particular, $J_2$ is the usual normalized duality mapping on $E$. It is well known (see, e.g., [95]) that

$$J_q(x) = \| x \|^{q-2} J_2(x), \quad \forall \ x(\neq 0) \in E.$$

Note that if $E \equiv H$, a real Hilbert space, then $J_2$ becomes the identity mapping on $H$.

Definition 2[17]. A Banach space $E$ is called smooth if, for every $x \in E$ with $\| x \| = 1$, there exists a unique $f \in E^*$ such that $\| f \| = f(x) = 1$.

The modulus of smoothness of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$, defined by

$$\rho_E(\tau) = \sup \left\{ \frac{(\| x + y \| + \| x - y \|)}{2} - 1 \right\}$$

for all $x, y \in E$, $\| x \| = 1$, $\| y \| = \tau$.

Definition 3[36]. The Banach space $E$ is said to be

(i) uniformly smooth, if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0;$$

(ii) $q$-uniformly smooth, for $q > 1$, if there exists a constant $c > 0$ such that

$$\rho_E(\tau) \leq c \tau^q, \quad \tau \in [0, \infty).$$

It is well known (see, e.g., [95]) that $L_q$ or $l_q$ is $q$-uniformly smooth, if $1 < q \leq 2$ and 2-uniformly smooth, if $q \geq 2$.

Note that if $E$ is uniformly smooth, $J_q$ becomes single-valued. In the study of characteristic inequalities in $q$-uniformly smooth Banach spaces, Xu [95] established the following lemma.
Lemma 1. Let $q > 1$ be a real number and let $E$ be a smooth Banach space. Then $E$ is $q$-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for every $x, y \in E$,

$$\|x + y\|^q \leq \|x\|^q + q \langle y, J_q(x) \rangle + c_q \|y\|^q. \quad (3)$$

Now, we define the following concept.

Definition 4. A multi-valued mapping $M : E \to 2^E$ is said to be

(i) accretive, if

$$\langle u - v, J_q(x - y) \rangle \geq 0,$$

for all $x, y \in E$ and for all $u \in Mx, v \in My$;

(ii) strictly accretive, if

$$\langle u - v, J_q(x - y) \rangle > 0,$$

for all $x, y \in E$ and for all $u \in Mx, v \in My$ and equality holds if and only if $x = y$;

(iii) $\gamma$-strongly accretive, if there exists $\gamma > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq \gamma \|x - y\|^2,$$

for all $x, y \in E$ and for all $u \in Mx, v \in My$;

(iv) $m$-accretive, if $M$ is accretive and $(I + \rho M)(E) = E$ for any $\rho > 0$, where $I$ stands for identity mapping.

Definition 5. Let multi-valued mapping $M : E \to 2^E$ be $m$-accretive, then proximal point mapping, denoted by $J^M_\rho$, for the $m$-accretive mapping $M$ is defined as follows:

$$J^M_\rho(z) = (I + \rho M)^{-1}(z), \quad \forall z \in E,$$

where $\rho > 0$ is a constant.

It is well known that $J^M_\rho$ is single-valued and nonexpansive mapping.

Throughout the rest of Section 2 unless otherwise stated, we assume that, for $i = 1, 2$, $E_i$ is $q_i$-uniformly smooth Banach space with norm $\| \cdot \|$.

Let for each $i = 1, 2$, $g_i, A_i, B_i : E_i \to E_i$ be nonlinear mappings; let $F : E_1 \times E_2 \to E_1, G : E_1 \times E_2 \to E_2$ be two nonlinear mappings, and let $M : E_1 \to 2^{E_1}, N : E_2 \to 2^{E_2}$ be $m_i$-accretive mapping, respectively, such that $g_1(x) \in \text{domain } M(\cdot)$ for all $x \in E_1$, $g_2(x) \in \text{domain } N(\cdot)$ for all $x \in E_2$. We consider the following system of general variational inclusions (SGVI):

Find $(x, y) \in E_1 \times E_2$ such that

$$F(A_1(x), A_2(y)) + M(g_1(x)) \ni \theta_1; \quad G(B_1(x), B_2(y)) + N(g_2(y)) \ni \theta_2, \quad (4)$$

where $\theta_1$ and $\theta_2$ are zero vectors of $E_1$ and $E_2$, respectively.

If $E_1 = E_2 \equiv H$, Hilbert space, $A_i \equiv B_i$, $F \equiv G, M \equiv N, g_1 = g_2$, then SGVI (4) reduces to a problem studied by Adly [1].

2.1. Existence of Solution of SGVI

We define the following concepts which shall be used in the sequel.

Definition 6. Let $A_i,g_i : E_i \to E_i$ and let $F : E_1 \times E_2 \to E_1$ be nonlinear mappings. We say that

(a) the mapping $x \to F_1(A_1(x), A_2(y))$ is $\alpha_1$-strongly accretive with respect to $g_1$ if, for all $x_1, x_2 \in E_1, y \in E_2$,

$$\langle F(A_1(x_1), A_2(y)) - F(A_1(x_2), A_2(y)), J_{g_1}(g_1(x_1) - g_1(x_2)) \rangle \geq \alpha \|g_1(x_1) - g_1(x_2)\|^q_1$$
Lemma 2. For any \((x, y) \in E_1 \times E_2\), \((x, y)\) is a solution of SGVI (4) if and only if \((x, y)\) is a fixed point of the map \(Q: E_1 \times E_2 \to E_1 \times E_2\) defined by
\[
Q(x, y) = (T(x, y), S(x, y)),
\]
for all \((x, y) \in E_1 \times E_2\), where \(T: E_1 \times E_2 \to E_1\) and \(S: E_1 \times E_2 \to E_2\) and defined by
\[
T(x, y) = x - g_1(x) + J^M_{\rho_1}[g_1(x) - \rho_1 F(A_1(x), A_2(y))] \tag{5}
\]
\[
S(x, y) = y - g_2(x) + J^N_{\rho_2}[g_2(y) - \rho_2 G(B_1(x), B_2(y))],
\]
\[
\rho_1, \rho_2 > 0 \text{ are constants; } J^M_{\rho_1} = (I + \rho_1 M(\cdot))^{-1}; J^N_{\rho_2} = (I + \rho_2 N(\cdot))^{-1}.
\]

**Theorem 1.** For \(i = 1, 2\), let \(E_i\) be \(q_i\)-uniformly smooth Banach space, let \((x, y) \to F(A_1(x), A_2(y))\) be \(\alpha_1\)-strongly accretive in the first argument and \((\beta_1, \sigma_1)\)-Lipschitz continuous with respect to \(g_1\); let \((x, y) \to G(B_1(x), B_2(y))\) be \(\alpha_2\)-strongly accretive in the second argument and \((\beta_2, \sigma_2)\)-Lipschitz continuous with respect to \(g_2\); let \(M : E_1 \to 2^{E_1}\) and \(N : E_2 \to 2^{E_2}\) be such that \(M(\cdot)\) and \(N(\cdot)\) are \(m_1\)-accretive and \(m_2\)-accretive mappings, respectively. Suppose that \(\rho_1, \rho_2 > 0\) satisfy the following condition:
\[
(1 - 2\rho_1 q_1 \alpha_1 + c_{q_1} \rho_1^{\beta_1} |\beta_1|^{1/q_1}) + \rho_2 \beta_2 < 1
\]
\[
(1 - 2\rho_2 q_2 \alpha_2 + c_{q_2} \rho_2^{\beta_2} |\beta_2|^{1/q_2}) + \rho_1 \sigma_1 < 1
\]
where \(c_{q_1}\) and \(c_{q_2}\) are constants. Then SGVI (4) has a solution. In addition, if for each \(i = 1, 2\), \(g_i\) is one to one, the solution is unique.

**Proof.** Consider the mappings for each \(i = 1, 2, f_i, h_i : E_i \to E_i\) defined by
\[
\begin{align*}
      f_1(x) &= A_1(\bar{x}), \\
   \bar{x} &= \text{an arbitrary element of } g_1^{-1}(x) \\
      f_2(y) &= A_2(\bar{y}), \\
   \bar{y} &= \text{an arbitrary element of } g_2^{-1}(y) \\
      h_1(x) &= B_1(\bar{x}), \\
   \bar{x} &= \text{an arbitrary element of } g_1^{-1}(x) \\
      h_2(y) &= B_2(\bar{y}), \\
   \bar{y} &= \text{an arbitrary element of } g_2^{-1}(y)
\end{align*}
\]
then we easily observe that \(f_i\) and \(h_i\) are well defined. Hence we see that SGVI (4) is equivalent to system of variational inclusions (SVI): Find \((x, y) \in E_1 \times E_2\) such that
\[
F(f_1(x), f_2(y)) + M(x) \ni \theta_1 \\
G(f_1(x), f_2(y)) + N(y) \ni \theta_2
\]
Further, by definition of \(F, G, f_i, h_i\) and given assumptions, we have that the mapping \((x, y) \to F(f_1(x), f_2(y))\) is \(\alpha_1\)-strongly accretive in the first argument and \((\beta_1, \sigma_1)\)-Lipschitz continuous, i.e.,
\[
\langle F(f_1(x_1), f_2(y_1)) - F(f_1(x_2), f_2(y_2)), x_1 - x_2 \rangle \geq \alpha_1 \|x_1 - x_2\|_1^{\alpha_1},
\]
and
\[
\|F(f_1(x_1), f_2(y_1)) - F(f_1(x_2), f_2(y_2))\| \leq \beta_1 \|x_1 - x_2\|_1 + \sigma_1 \|y_1 - y_2\|_2.
\]
Similarly, we have that the mapping
\[(x, y) \to G(f_1(x), f_2(y))\]
is $\alpha_2$-strongly accretive in the second argument and $(\beta_2, \sigma_2)$-Lipschitz continuous.

Furthermore, as in Lemma 2, we can easily see that SVI (7) has a solution if and only if the mapping $Q': E_1 \times E_2 \to E_1 \times E_2$ defined by $Q'(x, y) = (T'(x, y), S'(x, y))$ for all $(x, y) \in E_1 \times E_2$, has a fixed point, where
\[
T'(x, y) = J_{\rho_1}^M [x - \rho_1 F(f_1(x), f_2(y))],
\]
\[
S'(x, y) = J_{\rho_2}^N [y - \rho_G(h_1(x), h_2(y))],
\]
and $\rho_1$, $\rho_2 > 0$ are constants.

Hence, it is sufficient to show that the mapping $Q'$ defined by (8) have unique fixed point. Since $J_{\rho_1}^M$ is nonexpansive, we have the following estimate:
\[
\|T'(x, y_1) - T'(x, y_2)\|_1
\leq \|J_{\rho_1}^M [x - \rho_1 F(f_1(x_1), f_2(y_1))] - J_{\rho_1}^M [x_2 - \rho_1 F(f_1(x_2), f_2(y_2))]\|
\leq \|x - x_2 - \rho_1 [F(f_1(x_1), f_2(y_1))] - F(f_1(x_2), f_2(y_2))\|_1 + \rho_1 \|F(f_1(x_2), f_2(y_2)) - F(f_1(x_2), f_2(y_1))\|_1
\]
\[
\begin{align*}
&-2\rho_1 q_1 \alpha_1 \|x_1 - x_2\|_1^q + c_{q_1} \rho_1^q \beta_1 q_1 \|x_1 - x_2\|_1^q \\
&\leq \sigma_1 \|y_1 - y_2\|_2
\end{align*}
\]
and
\[
\begin{align*}
\|F(f_1(x_2), f_2(y_1)) - F(f_1(x_2), f_2(y_2))\|_1 \\
&\leq \sigma_1 \|y_1 - y_2\|_2
\end{align*}
\]
From (9), (10) and (11), we have that
\[
\|T'(x, y_1) - T'(x, y_2)\|_1 \leq (1 - 2\rho_1 q_1 \alpha_1 + c_{q_1} \rho_1^q \beta_1 q_1) \|x_1 - x_2\|_1 + \rho_1 \sigma_1 \|y_1 - y_2\|_2.
\]
Next, we have the following estimate:
\[
\|S'(x, y_1) - S'(x, y_2)\|_2 = \|J_{\rho_2}^N [y_1 - \rho_G(h_1(x_1), h_2(y_1))] - J_{\rho_2}^N [y_2 - \rho_G(h_1(x_2), h_2(y_2))]\|_2
\leq \|y_1 - y_2 - \rho_G(h_1(x_1), h_2(y_1)) - \rho_G(h_1(x_2), h_2(y_2))\|_2 + \rho_2 \|G(h_1(x_1), h_2(y_1)) - G(h_1(x_2), h_2(y_2))\|_2 \leq (1 - 2\rho_2 q_2 \alpha_2 + c_{q_2} \rho_2^q \beta_2 q_2)^{1/q_2} \|y_1 - y_2\|_2 + \rho_2 \beta_2 \|x_1 - x_2\|_1.
\]
From (12) and (13), we have
\[
\|T'(x, y_1) - T'(x, y_2)\|_1 + S'(x, y_1) - S'(x, y_2)\|_2 \\
\leq \max\{k_1, k_2\} \|x_1 - x_2\|_1 + \|y_1 - y_2\|_2
\]
where $k_1 = l_1 + \rho_2 \beta_2$ and $k_2 = l_2 + \rho_1 \sigma_1$.
\[
l_1 := [1 - 2\rho_1 q_1 \alpha_1 + c_{q_1} \rho_1^q \beta_1 q_1]^{1/q_1};
\]
\[
l_2 := [1 - 2\rho_2 q_2 \alpha_2 + c_{q_2} \rho_2^q \beta_2 q_2]^{1/q_2}.
\]
Now, define the norm $\|\cdot\|_*$ on $E_1 \times E_2$ by
\[
\|(x, y)\|_* = \|x\|_1 + \|y\|_2, \quad (x, y) \in E_1 \times E_2.
\]
Since $(E_1 \times E_2, \|\cdot\|_*)$ is a Banach space. Hence it follows from (14)-(15) that
\[
\|Q'(x, y_1) - Q'(x, y_2)\|_* \\
\leq \max\{k_1, k_2\} \|(x_1, y_1) - (x_2, y_2)\|_*
\]
Since \( \max\{k_1, k_2\} < 1 \) by condition (4), it follows from (17) that \( Q' \) is a contraction mapping. Hence, by Banach contraction principle, there exists a unique \((x, y) \in E_1 \times E_2\) such that

\[
Q'(x, y) = (x, y),
\]

which implies that SVI (7) has unique solution and hence SGVI (4) has a solution. Further, it is clear that if \( g_1 \) and \( g_2 \) are one-to-one mapping, SGVI (4) has a unique solution. This completes the proof.

### 2.2. Convergence and Stability of Mann Type Perturbed Iterative Algorithm 1

First we recall the following definitions and result

**Definition 7**[5]. Let \( \{M^n\}, n \in \mathbb{N} \) be a sequence of \( m \)-accretive mappings on a Banach space \( E \). The sequence \( \{M^n\} \) is said to be graph convergent to \( M \), an \( m \)-accretive mapping defined on \( E \), denoted by \( M^n \xrightarrow{G} M \), if for every \((u, v) \in G(M)\), there exists a sequence \((u_n, v_n) \in G(M^n)\) such that \( u_n \to u \) and \( v_n \to v \) strongly in \( E \).

**Lemma 3**[5]. Let \( E \) be any Banach space; let \( \{M^n\} \) be a sequence of \( m \)-accretive mappings and let \( M \) be \( m \)-accretive mapping on \( E \), then the following are equivalent:

(a) \( \forall u, v \in E, \exists (u_n, v_n) \in G(M^n) \) such that \( u_n \to u \) and \( v_n \to v \);

(b) \( \forall \rho > 0, \forall u \in E, (I + \rho M^n)^{-1}(u) \to (I + \rho M)^{-1}(u) \);

(c) \( \exists \rho > 0, \forall u \in E, (I + \rho M^n)^{-1}(u) \to (I + \rho M)^{-1}(u) \),

where \( I \) is the identity mapping on \( E \).

**Definition 8** (stability). Let \( E_1, E_2 \) be real Banach spaces; let \( T : E_1 \times E_2 \to E_1 \) and \( S : E_1 \times E_2 \to E_2 \). Let \( Q : E_1 \times E_2 \to E_1 \times E_2 \) be defined as \( Q(x, y) = (T(x, y), S(x, y)) \) for any \((x, y) \in E_1 \times E_2\), and let \((x_0, y_0) \in E_1 \times E_2\). Assume that \((x_{n+1}, y_{n+1}) = f(Q, x_n, y_n) = (g(T, x_n, y_n), g(S, x_n, y_n))\) define an iteration procedure which yields a sequence of points \((x_n, y_n)\) in \( E_1 \times E_2 \). Suppose that \( F(Q) = \{(x, y) \in E_1 \times E_2 : (x, y) = Q(x, y)\} \neq \emptyset \) and \((x_n, y_n)\) converges to some \((p, q) \in F(Q)\). Let \((u_n, v_n)\) be an arbitrary sequence in \( E_1 \times E_2 \) and \( \epsilon_n = \|(u_{n+1}, v_{n+1}) - f(Q, x_n, y_n)\| \), for all \( n \geq 0 \). If \( \lim_{n \to \infty} \epsilon_n = 0 \) implies that \( \lim_{n \to \infty} (u_n, v_n) = (p, q) \), then the iteration procedure defined by \((x_{n+1}, y_{n+1}) = f(Q, x_n, y_n)\) is said to be \( Q \)-stable or stable with respect to \( Q \). If \( \sum_{n=0}^{\infty} \epsilon_n < +\infty \) implies that \( \lim_{n \to \infty} (u_n, v_n) = (p, q) \), then the iterative procedure \((x_n, y_n)\) is said to be almost \( Q \)-stable.

**Remark 1.**

(i) Definition 7 can be viewed as an extension of concept of stability of iteration procedure given by Harder and Hicks [39], see also [68,85]. Harder and Hicks [39] demonstrated the importance of investigating the stability of various iteration procedures.

(ii) Recently, some stability results of iteration procedures for variational inequalities (inclusions) have been established by various authors, see for example [2,43,49,51, 53,60,68,69], but most of them are in the setting of Hilbert
space.

(iii) A stable iteration procedure is almost stable but conversed is not true in general, see [85].

Next, we recall the following lemma, due to Liu [66].

**Lemma 4.** Let \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) be three sequences of non-negative real numbers satisfying:

\[
a_{n+1} \leq (1 - \lambda_n) a_n + b_n \lambda_n + c_n,
\]

for all \( n \geq 0 \), where \( \sum_{n=0}^{\infty} \lambda_n = \infty, \lambda_n \in [0, 1] \),

\[
\lim_{n \to \infty} b_n = 0, \quad \sum_{n=0}^{\infty} c_n < \infty. \quad \text{Then} \quad \lim_{n \to \infty} a_n = 0.
\]

In view of Lemma 2, the fixed-point formulation of SVI (7) allow us to suggest the following iterative algorithm:

**Mann Type Perturbed Iterative Algorithm (MTPIA) (1).** For given \( (x_0, y_0) \in E_1 \times E_2 \), compute approximate solution \( (x_n, y_n) \) given by iterative schemes:

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n J_{\rho_1}^{M_n}[x_n - \rho_1 F(f_1(x_n), f_2(y_n))]
\]

\[
y_{n+1} = (1 - \alpha_n)y_n + \alpha_n J_{\rho_2}^{N_n}[y_n - \rho_2 G(h_1(x_n), h_2(y_n))]
\]

where \( n = 0, 1, 2, \ldots; \rho_1, \rho_2 > 0 \) are constants; \( \{\alpha_n\} \) is a sequence of real numbers such that \( \alpha_n \in [0, 1] \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty; \{M_n\} \) is a sequence of \( m_1 \)-accretive mappings approximating \( M \); \( \{N_n\} \) is a sequence of \( m_2 \)-accretive mappings approximating \( N \).

**Theorem 2 (Convergence).** Let, for \( i = 1, 2; E_i, q_i, F, G, M, N \) be the same as in Theorem 1 and let condition (2) of Theorem 1 hold. If the sequence \( \{M_n\} \) and \( \{N_n\} \) are graph-convergent to \( M \) and \( N \), respectively, then the approximate solution \( (x_n, y_n) \) generated by MTPIA 1, converges strongly to the solution of SGVI (4).

**Proof.** It follows from Theorem 1 that there exists a unique solution \( (x, y) \in E_1 \times E_2 \) of SVI (7) which is the solution of SGVI (4) and unique when \( g_i \) is one to one for each \( i = 1, 2 \).

Setting \( P(x, y) := x - P_1(f_1(x), f_2(y)) \), from MTPIA 1 and nonexpansivity of \( J_{\rho_1}^{M_n} \), we have

\[
\|x_{n+1} - x\|_1 = (1 - \alpha_n)\|x_n - x\|_1 + \alpha_n J_{\rho_1}^{M_n}(P(x, y)) - J_{\rho_1}^{M_n}(P(x, y)) + \alpha_n \|P(x, y) - P(x, y)\|_1
\]

\[
\|x_{n+1} - x\|_1 = (1 - \alpha_n)\|x_n - x\|_1 + \alpha_n J_{\rho_1}^{M_n}(P(x, y)) - J_{\rho_1}^{M_n}(P(x, y)) + \alpha_n \|P(x, y) - P(x, y)\|_1
\]

\[
\leq (1 - \alpha_n)\|x_n - x\|_1 + \alpha_n \|J_{\rho_1}^{M_n}(P(x, y)) - J_{\rho_1}^{M_n}(P(x, y))\|_1 + \alpha_n \|P(x, y) - P(x, y)\|_1
\]

\[
\leq (1 - \alpha_n)\|x_n - x\|_1 + \alpha_n \|P(x, y) - P(x, y)\|_1.
\]

Next, we have the following estimate:

\[
\|P(x, y) - P(x, y)\|_1 \leq l_1\|x_n - x\|_1 + \rho_1 \|y_n - y\|_2.
\]

From (21) and (22), we have

\[
\|x_{n+1} - x\|_1 \leq (1 - \alpha_n)\|x_n - x\|_1
\]

\[
+ \alpha_n \|l_1\|x_n - x\|_1 + \rho_1 \|y_n - y\|_2 + \alpha_n \|l_2\|y_n - y\|_2 + \alpha_n \|y_n - y\|_2
\]

\[
\leq (1 - \alpha_n)\|x_n - x\|_1
\]

where \( l_1 \) is given by (15). Similarly we have

\[
\|y_{n+1} - y\|_2 \leq (1 - \alpha_n)\|y_n - y\|_2 + \alpha_n \|l_2\|y_n - y\|_2
\]
where \( \xi_n = \|J_{\rho_2}^N[y - \rho_2G(h_1(x), h_2(y))] - J_{\rho_2}^N[y - \rho_2G(h_1(x), h_2(y))]\|_2 \), and \( l_2 \) is given by (16).

From (23) and (24), we have
\[
\|(x_{n+1}, y_{n+1}) - (x, y)\|_* \leq (1 - \alpha_n)
\]
\[
\|(x_n, y_n) - (x, y)\|_* + \alpha_n \max \{k_1, k_2\}
\]
\[
\|(x_n, y_n) - (x, y)\|_* + \alpha_n (\epsilon_n + \xi_n)
\]
\[
= [1 - \alpha_n (1 - \max \{k_1, k_2\})] \|(x_n, y_n) - (x, y)\|_* + \alpha_n (\epsilon_n + \xi_n)
\]
where \( k_1 = l_1 + \rho_2 \beta_2, k_2 = l_2 + \rho_1 \sigma_1 \).

Setting:
\[
a_n = \|(x_n, y_n) - (x, y)\|_*;
\]
\[
\lambda_n = (1 - \max \{k_1, k_2\}) \alpha_n;
\]
\[
b_n = (1 - \max \{k_1, k_2\})^{-1} (\epsilon_n + \xi_n);
\]
\[
c_n = 0, \quad \forall \ n.
\]

From condition (5), it follows that \( \max \{k_1, k_2\} < 1 \). Clearly \( \{\lambda_n\} \subset [0, 1] \) such that \( \sum_{n=0}^{\infty} \lambda_n = \infty \), since \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Further, it follows from the graph convergence of \( \{M_n\} \) and \( \{N_n\} \), and hence \( \lim_{n \to \infty} b_n = 0 \).

Thus, it follows from Lemma 2.4 that \( a_n \to 0 \) as \( n \to \infty \), i.e.,
\[
\|(x_n, y_n) - (x, y)\|_* \to 0 \quad \text{as} \quad n \to \infty
\]

Thus, \( (x_n, y_n) \) converges strongly to a solution \( (x, y) \) of SGVI (4). This completes the proof.

**Theorem 3(Stability).** Let, for each \( i = 1, 2, \ E_i, g_i, F, G, M, N \) be the same as in Theorem 2.1 and let condition (5) of Theorem 1 hold. Let the sequence \( \{M_n\} \) and \( \{N_n\} \) be graph convergent to \( M \) and \( N \), respectively. Let \( \{(u_n, v_n)\} \) be any sequence in \( E_1 \times E_2 \) and define \( \{\delta_n\} \subseteq [0, \infty) \) by
\[
\delta_n = \|(u_{n+1}, v_{n+1}) - (A, B)\|_*,
\]
where
\[
A := (1 - \alpha_n) u_n + \alpha_n J_{\rho_1}^{M_n} \left[ u_n - \rho_1 F(f_1(u_n), f_2(v_n)) \right];
\]
\[
B := (1 - \alpha_n) v_n + \alpha_n J_{\rho_2}^{N_n} \left[ v_n - \rho_2 G(h_1(u_n), h_2(v_n)) \right];
\]
where \( n = 0, 1, 2, \cdots, \rho_1, \rho_2 > 0 \) are constants; \( \alpha_n \in [0, 1] \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Then
\[(a) \text{ If } \sum_{n=0}^{\infty} \delta_n < \infty, \text{ then } \lim_{n \to \infty} (u_n, v_n) = (x, y);
\]
\[(b) \text{ If } \lim_{n \to \infty} (u_n, v_n) = (x, y), \text{ then } \lim_{n \to \infty} \delta_n = 0,
\]
where \( (x, y) \) is the unique solution of SVLI (4).

**Proof.** It follows from Theorem 1 that there exists a unique solution \( (x, y) \) of SVI (7), which is a solution of SGVI (4), i.e.,
\[
x = J_{\rho_1}^{M} \left[ x - \rho_1 F(f_1(x), f_2(y)) \right]
\]
\[
y = J_{\rho_2}^{N} \left[ y - \rho_2 G(h_1(x), h_2(y)) \right].
\]
Suppose that \( \sum_{n=0}^{\infty} \delta_n < \infty \). From (25)-(27), we have
\[
\|(u_{n+1}, v_{n+1}) - (x, y)\|_*
\]
\[
\|A-x\| \leq \|A-x\|_1 + \|B-y\|_2. \tag{28}
\]

Using the same arguments for obtaining (23) and (24), we have

\[
\|A-x\| \leq (1-\alpha_n)\|u_n-x\| + \alpha_n\left[ l_1\|u_n-x\| + \rho_1\sigma_1\|v_n-y\| + \alpha_n\epsilon_n, \right] \tag{29}
\]

\[
\|B-y\| \leq (1-\alpha_n)\|v_n-y\| + \alpha_n\left[ l_2\|v_n-y\| + \rho_2\beta_2\|u_n-x\| + \alpha_n\xi_n. \right] \tag{30}
\]

From (28) to (30), we have

\[
\|(u_{n+1}, v_{n+1})-(x, y)\|_* \leq \delta_n + [1-(1-\max\{k_1, k_2\})\alpha_n]\|(u_n, v_n)-(x, y)\|_*, \tag{31}
\]

Setting:

\[
an = \|(u_n, v_n)-(x, y)\|_*; \\
\lambda_n = (1-\max\{k_1, k_2\})\alpha_n; \\
b_n = (1-\max\{k_1, k_2\})^{-1}(\epsilon_n+\xi_n); \\
c_n = \delta_n, \ \forall n.
\]

It follows from (5), \(\alpha_n \in [0,1]\) and \(\sum_{n=0}^{\infty} \alpha_n = \infty\) that \(\lambda_n \in [0,1]\) and \(\sum_{n=0}^{\infty} \lambda_n = \infty\). Further it follows from the graph convergence \(\{M_n\}\) and \(\{N_n\}\) and Lemma 2.3 that \(\epsilon_n \rightarrow 0; \ \xi_n \rightarrow 0\) as \(n \rightarrow \infty\), and hence \(\lim_{n \rightarrow \infty} b_n = 0\). Thus, it follows from Lemma 2.4 that \(a_n \rightarrow \infty\) as \(n \rightarrow \infty\), i.e., \(\lim_{n \rightarrow \infty} (u_n, v_n) = (x, y)\). This completes the proof of (a).

Next, assume that \(\lim_{n \rightarrow \infty} (u_n, v_n) = (x, y)\). Then

\[
\delta_n \leq \|(u_{n+1}, v_{n+1})-(x, y)\|_* + \|(A, B)-(x, y)\|_* \\
\leq \|(u_{n+1}, v_{n+1})-(x, y)\|_*
\]

i.e., \(\lim_{n \rightarrow \infty} \delta_n = 0\). This completes the proof of (b).

3. System of Parametric General Variational Inclusions

Let \(\Lambda_1\) and \(\Lambda_2\) be open subsets of \(E_1\) and \(E_2\), respectively, such that \((\Lambda_1, d_1)\) and \((\Lambda_2, d_2)\) are metric spaces, in which the parametric \(\lambda_1\) and \(\lambda_2\) takes values, respectively. For each \(i = 1, 2\), let \(g_i, A_i, B_i : E_i \times \Lambda_i \rightarrow E_i\) be nonlinear mappings; let \(F : E_1 \times E_2 \times \Lambda_1 \rightarrow E_1\), \(G : E_1 \times E_2 \times \Lambda_2 \rightarrow E_2\) be two nonlinear mappings, and let \(M : E_1 \times \Lambda_1 \rightarrow 2^{E_1}\), \(N : E_2 \times \Lambda_2 \rightarrow 2^{E_2}\) be \(m_i\)-accretive mappings, respectively, such that \(g_1(x_1, \lambda_1) \in \text{domain}M(\cdot, \lambda_1)\) for all \((x_1, \lambda_1) \in E_1 \times E_2\) and \(g_2(x_1, \lambda_2) \in \text{domain}M(\cdot, \lambda_2)\) for all \((x_2, \lambda_2) \in E_1 \times E_2\). The system of parametric general variational inclusions (SPGVI) is to find \((x, y) \in E_1 \times E_2\) such that

\[
F(A_1(x_1, \lambda_1), A_2(y_2, \lambda_2), \lambda_1, \lambda_2) + M(g_1(x_1, \lambda_1), \lambda_1, \lambda_2) \\
+ G(B_1(x_1, \lambda_1), B_2(y_2, \lambda_2), \lambda_1, \lambda_2) \tag{32}
\]

\[+ N(g_2(y_2, \lambda_2), \lambda_2, \lambda_1, \lambda_2) \ni \theta_1 \ni \theta_2\]

where \(\theta_1\) and \(\theta_2\) are zero vectors of \(E_1\) and \(E_2\), respectively.
3.1. Existence of Solution and Sensitivity Analysis

First we prove the following technical lemma.

**Lemma 5.** For any given \((x, y) \in E_1 \times E_2\), \((x, y)\) is a solution of SPGVI (3.1) if and only if \((x, y)\) satisfies the relations

\[
g_1(x, \lambda_1) = J_{\rho_1}^{M(\cdot, \lambda_1)}[g_1(x, \lambda_1)]
- \rho_1 F(A_1(x, \lambda_1), A_2(y, \lambda_2), \lambda_1, \lambda_2)]
\]
\[
g_2(y, \lambda_2) = J_{\rho_2}^{N(\cdot, \lambda_2)}[g_2(y, \lambda_2)]
- \rho_2 G(B_1(x, \lambda_1), B_2(y, \lambda_2), \lambda_1, \lambda_2)]
\]

where \(J_{\rho_1}^{M(\cdot, \lambda_1)} = (I + P_1 M(\cdot, \lambda_1))^{-1}\),
\(J_{\rho_2}^{N(\cdot, \lambda_2)} = (I + P_2 N(\cdot, \lambda_2))^{-1}\), and \(\rho_1, \rho_2 > 0\) are constants.

Now, assume that for some \((\bar{x}, \bar{y}) \in \Lambda_1 \times \Lambda_2\), SPGVI (32) has a solution \((\bar{x}, \bar{y})\) and \(K_1 \times K_2\) is a closed sphere in \(E_1 \times E_2\) centered at \((\bar{x}, \bar{y})\), where \(K_1\) is a closed set in \(E_1\) and \(K_2\) is a closed set in \(E_2\).

We are interested in investigating those conditions under which for each \((\lambda_1, \lambda_2)\) is a neighborhood of \((\bar{x}, \bar{y})\), SPGVI (32) has a unique solution \((x(\lambda_1), y(\lambda_2))\) is Lipschitz continuous.

Next, we give the following concepts:

**Definition 8.** The mapping \(g_1 : E_1 \times \Lambda_1 \rightarrow E_1\) is said to be

(i) locally \(k\)-strongly accretive if there exists a constant \(k > 0\) such that

\[
\langle g_1(x_1, \lambda_1) - g_1(x_2, \lambda_1), J_{q_1}(x_1 - x_2) \rangle 
\geq k \|x_1 - x_2\|^q_1
\]

(ii) locally \((\gamma_1, \gamma_2)\)-Lipschitz continuous, if there exist constants \(\gamma_1, \gamma_2 > 0\) such that

\[
\|g_1(x_1, \lambda_1) - g_1(x_2, \lambda_1)\|_1 
\leq \gamma_1 \|x_1 - x_2\|_1 + \gamma_2 \|\lambda_1 - \lambda_2\|_1
\]

for all \(x_1, x_2 \in E_1, \lambda_1, \lambda_2 \in \Lambda_1\).

**Definition 9.** For each \(i = 1, 2\), let \(A_i : E_i \times \Lambda_i \rightarrow E_i\) be nonlinear mappings. The mapping \(F : E_1 \times E_2 \times \Lambda_1 \times \Lambda_2 \rightarrow E_1\) is said to be

(i) locally \(\alpha\)-strongly accretive with respect to \(A_1\) and \(A_2\) in the first argument, if there exists a constant \(\alpha > 0\) such that

\[
\langle F(A_1(x_1, \lambda_1), A_2(y_1, \lambda_2), \lambda_1, \lambda_2) 
- F(A_1(x_2, \lambda_1), A_2(y_2, \lambda_2), \lambda_1, \lambda_2), J_{q_1}(x_1 - x_2) \rangle 
\geq \alpha \|x_1 - x_2\|^{q_1}_1
\]

(ii) locally \((\beta_1, \beta_2, \beta_3, \beta_4)\)-Lipschitz continuous, if there exist constants \(\beta_1, \beta_2, \beta_3, \beta_4 > 0\) such that

\[
\|F(x_1, y_1, \lambda_1, \lambda_2) - F(x_2, y_2, \lambda_1, \lambda_2)\|_1 
\leq \beta_1 \|x_1 - x_2\|_1 + \beta_2 \|y_1 - y_2\|_2 
+ \beta_3 \|\lambda_1 - \lambda_2\|_1 + \beta_4 \|\lambda_1 - \lambda_2\|_2
\]

for all \(x_1, x_2 \in E_1, y_1, y_2 \in E_2, \lambda_1, \lambda_2, \lambda_1, \lambda_2 \in \Lambda_1, \Lambda_2\).

Next, we have following lemma.

**Lemma 6.** For any given \((x, y) \in E_1 \times E_2\), \((x, y)\) is a solution of SPGVI (32) if and only if \((x, y)\) is a fixed point of the map \(Q : E_1 \times E_2 \times \Lambda_1 \times \Lambda_2 \rightarrow E_1 \times E_2\) defined by
\[ Q(x, y, \lambda_1, \lambda_2) = (T(x, y, \lambda_1, \lambda_2), S(x_1, y_1, \lambda_1, \lambda_2)) \]  
(33)

where

\[ T(x, y, \lambda_1, \lambda_2) = x - g_1(x, \lambda_1) + J^{M_1, \lambda_1} \]

\[ \times [g_1(x, \lambda_1) - \rho_1 F(A_1(x, \lambda_1), A_2(y, \lambda_2), \lambda_1, \lambda_2)] \]  
(34)

\[ S(x, y, \lambda_1, \lambda_2) = y - g_2(y, \lambda_2) + J^{N_2, \lambda_2} \]

\[ \times [g_2(y, \lambda_2) - \rho_2 G(B_1(x, \lambda_1), B_2(y, \lambda_2), \lambda_1, \lambda_2)] \]  
(35)

where \( \rho_1, \rho_2 > 0 \) are constants.

Now, we show that the mapping \( Q(x, y, \lambda_1, \lambda_2) \) defined by (3.2) is a contraction mapping with respect to \((x, y)\) uniformly in \((\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2\).

**Theorem 4.** For each \( i = 1, 2 \), let \( E_i \) be \( q_i \)-uniformly smooth Banach spaces; let \( A_i, B_i : E_i \times \Lambda_i \rightarrow E_i \) be \( \alpha_i \)-Lipschitz continuous and \( b_i \)-Lipschitz continuous in the first argument, respectively; let the mappings \( g_i : E_i \times \Lambda_i \rightarrow E_i \) be \( \gamma_i \)-strongly accretive and \( \mu_i \)-Lipschitz continuous in the first argument; let \( F : E_1 \times E_2 \times \Lambda_1 \times \Lambda_2 \rightarrow E_1 \) be \( \alpha \)-strongly accretive and \( (\beta_1, \sigma_1) \)-Lipschitz continuous in first two arguments; let \( G : E_1 \times E_2 \times \Lambda_1 \times \Lambda_2 \rightarrow E_2 \) be \( \beta_2 \)-strongly accretive and \( (\beta_2, \sigma_2) \)-Lipschitz continuous in first two arguments and let \( M : E_1 \times \Lambda_1 \rightarrow 2^{E_1} \) and \( N : E_2 \times \Lambda_2 \rightarrow 2^{E_2} \) be \( m_1 \)-accretive and \( m_2 \)-accretive mapping, respectively such that \( g_1(\cdot, \lambda_1) \in \text{dom} M(\cdot, \lambda_1) \) and \( g_2(\cdot, \lambda_2) \in \text{dom} N(\cdot, \lambda_2) \) for all \( \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2 \). Suppose that \( \rho_1, \rho_2 > 0 \) satisfy

the following condition:

\[ 2(1 - 2q_1 \gamma_1 + c_{q_1}q_1^{\beta_1^1})^{1/q_1} + (1 - 2\rho_1 q_1 \alpha_1 + c_{q_1}q_1^{\beta_1^1} \beta_1^1)^{1/\alpha_1} + \rho_2 \beta_2 b_1 < 1; \]

\[ 2(1 - 2q_2 \gamma_2 + c_{q_2}q_2^{\beta_2^2})^{1/q_2} + (1 - 2\rho_2 q_2 \alpha_2 + c_{q_2}q_2^{\beta_2^2} \sigma_2^2)^{1/\alpha_2} + \rho_1 \sigma_1 a_2 < 1. \]

Then for each fixed \((\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2\), SPGV (32) has the unique solution.

**Proof.** In view of Lemma 5, it is enough to prove that, for a fixed \((\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2\), defined by (33) has a unique fixed point \((x, y)\). For any \((x_1, y_1, \lambda_1, \lambda_2), (x_2, y_2, \lambda_1, \lambda_2) \in E_1 \times E_2 \times \Lambda_1 \times \Lambda_2\), we estimate

\[ \|T(x_1, y_1, \lambda_1, \lambda_2) - T(x_2, y_2, \lambda_1, \lambda_2)\|_1 \]

\[ \leq 2\|x_1 - x_2 -(g_1(x_1, \lambda_1) - g_1(x_2, \lambda_1))\|_1 \]

\[ +\|x_1 - x_2 - \rho_1 [F(A_1(x_1, \lambda_1), A_2(y_1, \lambda_2), \lambda_1, \lambda_2) - F(A_1(x_2, \lambda_1), A_2(y_1, \lambda_2), \lambda_1, \lambda_2)]\| \]

\[ + \rho_1 \|F(A_1(x_2, \lambda_1), A_2(y_1, \lambda_2), \lambda_1, \lambda_2)\| + \rho_1 \|g_1 - g_1\| \]

\[ \leq [2(1 - 2q_1 \gamma_1 + c_{q_1}q_1^{\beta_1^1})^{1/q_1} + (1 - 2\rho_1 q_1 \alpha_1 + c_{q_1}q_1^{\beta_1^1} \beta_1^1)^{1/\alpha_1} + \rho_2 \beta_2 b_1] \|x_1 - x_2\|_1 + \rho_1 \sigma_1 a_2 \|y_1 - y_2\|_2. \]

(37)

Similarly, we have

\[ \|S(x_1, y_1, \lambda_1, \lambda_2) - S(x_2, y_2, \lambda_1, \lambda_2)\|_2 \]

\[ \leq [2(1 - 2q_2 \gamma_2 + c_{q_2}q_2^{\beta_2^2})^{1/q_2} + (1 - 2\rho_2 q_2 \alpha_2 + c_{q_2}q_2^{\beta_2^2} \sigma_2^2)^{1/\alpha_2} + \rho_1 \sigma_1 a_2] \|y_1 - y_2\|_2 + \rho_2 \beta_2 b_1 \|x_1 - x_2\|_1. \]

(38)

From (37)-(38), we have

\[ \|Q(x_1, y_1, \lambda_1, \lambda_2) - Q(x_2, y_2, \lambda_1, \lambda_2)\|_* \]

\[ \leq \max \{k_1, k_2\} \|(x_1, x_2) - (y_1, y_2)\|_* \]

(39)

where \( k_1 = p_1 + \rho_2 \beta_2 b_1, \)

\( k_2 = p_2 + \rho_1 \sigma_1 a_2; \)

\( p_1 := 2(1 - 2q_1 \gamma_1 + c_{q_1}q_1^{\beta_1^1})^{1/q_1} + 2(1 - 2\rho_1 q_1 \alpha_1 + c_{q_1}q_1^{\beta_1^1} \beta_1^1)^{1/\alpha_1} + \rho_2 \beta_2 b_1; \)

\( p_2 := 2(1 - 2q_2 \gamma_2 + c_{q_2}q_2^{\beta_2^2})^{1/q_2} + 2(1 - 2\rho_2 q_2 \alpha_2 + c_{q_2}q_2^{\beta_2^2} \sigma_2^2)^{1/\alpha_2} + \rho_1 \sigma_1 a_2. \)
\[ p_2 := 2(1 - 2q_2 \gamma_2 + c_{q_2} \mu_2^{q_2})^{1/q_2} + 2(1 - 2p_2 q_2 \alpha_2 + c_{q_2} \mu_2^{q_2})^{1/q_2} \]  

\[ (40) \]

\[ + c_{q_2} \rho_1^{q_2} a_1^{q_2} \beta_1^{q_2} \]  

Since \( \max \{k_1, k_2\} < 1 \) by condition (36), it follows from (39) that \( Q \) is a uniform contraction mapping with respect to \( (\lambda, \lambda_2) \in \Lambda_1 \times \Lambda_2 \). Hence by Banach contraction principle, \( Q \) has the unique fixed point \( (x, y) = (x(\lambda_1), y(\lambda_1)) \), which is unique in the unique solution of SPGVI (32). This completes the proof.

Now, we show that the solution \( (x(\lambda_1), y(\lambda_2)) \) of SPGVI (32) is Lipschitz continuous in \( (\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2 \).

**Theorem 5.** For each \( i = 1, 2 \), let the mappings \( A_i, B_i : E_i \times \Lambda_i \to E_i \) be \( (a_i, c_i) \)-Lipschitz continuous \( (b_i, d_i) \)-continuous, respectively; let the mappings \( g_i : E_i \times \Lambda_i \to E_i \) be \( \gamma_i \)-strongly accretive and \( (\mu, e_i) \)-Lipschitz continuous, and let the mappings \( F : E_1 \times E_2 \times \Lambda_1 \times \Lambda_2 \to E_1 \) be \( \alpha_1 \)-strongly accretive and \( (\beta_1, \sigma_1, \eta_1, \nu_1) \)-Lipschitz continuous; let the mappings \( G : E_1 \times E_2 \times \Lambda_1 \times \Lambda_2 \to E_2 \) be \( \alpha_2 \)-strongly accretive and \( (\beta_2, \sigma_2, \eta_2, \nu_2) \)-Lipschitz continuous, and let \( M : E_1 \times \Lambda_1 \to 2^{E_1} \) and \( N : E_2 \times \Lambda_2 \to 2^{E_2} \) be \( m_1 \)-accretive and \( m_2 \)-accretive, respectively, such that \( g_1(\cdot, \lambda_i) \in \text{dom} M(\cdot, \lambda_i) \) and \( g_2(\cdot, \lambda_i) \in \text{dom} N(\cdot, \lambda_i) \) for all \( \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2 \). If the mappings \( \lambda_1 \to J^{M}(\cdot, \lambda_1) \) and \( \lambda_2 \to J^{M}(\cdot, \lambda_2) \) are \( r_1 \)-Lipschitz and \( r_2 \)-Lipschitz continuous, respectively, and if \( \rho_1, \rho_2 > 0 \) satisfy the condition (36) of Theorem 4. Then for each \( (\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2 \), the solution \( (x(\lambda_1), y(\lambda_2)) \) of SPGVI (32) is Lipschitz continuous.

**Proof.** For each \( (\lambda_1, \lambda_2), (\bar{\lambda}_1, \bar{\lambda}_2) \in \Lambda_1 \times \Lambda_2 \), it follows from Theorem 4 that \((x(\lambda_1), y(\lambda_2))\) and \((x(\bar{\lambda}_1), y(\bar{\lambda}_2))\) are solution of SPGVI (32). Now we estimate

\[ ||(x(\lambda_1), y(\lambda_2)) - (x(\bar{\lambda}_1), y(\bar{\lambda}_2))|| \]

\[ \begin{align*}
&= ||Q(x(\lambda_1), y(\lambda_2), \lambda_1, \lambda_2) - Q(x(\bar{\lambda}_1), y(\bar{\lambda}_2), \bar{\lambda}_1, \bar{\lambda}_2)|| \\
&= ||T(x(\lambda_1), y(\lambda_2), \lambda_1, \lambda_2) - T(x(\bar{\lambda}_1), y(\bar{\lambda}_2), \bar{\lambda}_1, \bar{\lambda}_2)||_1 \\
&+ ||S(x(\lambda_1), y(\lambda_2), \lambda_1, \lambda_2) - S(x(\bar{\lambda}_1), y(\bar{\lambda}_2), \bar{\lambda}_1, \bar{\lambda}_2)||_2 \\
&= p_1 ||x(\lambda_1) - x(\bar{\lambda}_1)||_1 + \rho_1 \sigma_1 a_2 ||y(\lambda_2) - y(\bar{\lambda}_2)||_2 \\
&+ ||T(x(\lambda_1), y(\lambda_2), \lambda_1, \lambda_2) - T(x(\bar{\lambda}_1), y(\bar{\lambda}_2), \bar{\lambda}_1, \bar{\lambda}_2)||_1 \\
&+ ||S(x(\lambda_1), y(\lambda_2), \lambda_1, \lambda_2) - S(x(\bar{\lambda}_1), y(\bar{\lambda}_2), \bar{\lambda}_1, \bar{\lambda}_2)||_2 \\& \leq p_1 ||x(\lambda_1) - x(\bar{\lambda}_1)||_1 + \rho_1 \sigma_1 a_2 ||y(\lambda_2) - y(\bar{\lambda}_2)||_2 \\
&+ ||T(x(\lambda_1), y(\lambda_2), \lambda_1, \lambda_2) - T(x(\bar{\lambda}_1), y(\bar{\lambda}_2), \bar{\lambda}_1, \bar{\lambda}_2)||_1 \\
&+ ||S(x(\lambda_1), y(\lambda_2), \lambda_1, \lambda_2) - S(x(\bar{\lambda}_1), y(\bar{\lambda}_2), \bar{\lambda}_1, \bar{\lambda}_2)||_2 \\
&\leq (e_1(1 + r_1) + \rho_1 r_1 \eta_1) ||\lambda_1 - \bar{\lambda}_1||_1 \\
&+ \rho_1 r_1 \nu_1 ||\lambda_1 - \bar{\lambda}_1||_2 \\
&\leq (e_1(1 + r_1) + \rho_1 r_1 \eta_1) ||\lambda_1 - \bar{\lambda}_1||_1 \\
&+ \rho_1 r_1 \nu_1 ||\lambda_1 - \bar{\lambda}_1||_2 \\
&\leq (e_1(1 + r_1) + \rho_1 r_1 \eta_1) ||\lambda_1 - \bar{\lambda}_1||_1 \\
&+ \rho_1 r_1 \nu_1 ||\lambda_1 - \bar{\lambda}_1||_2 \\
&\leq (e_1(1 + r_1) + \rho_1 r_1 \eta_1) ||\lambda_1 - \bar{\lambda}_1||_1 \\
&+ \rho_1 r_1 \nu_1 ||\lambda_1 - \bar{\lambda}_1||_2 \\
\end{align*} \]  

\[ (42) \]
From (43) and (44), it follows that
\[
\|T(x(\lambda), (y(\lambda)), \lambda, \lambda)\|_1
-\|T(x(\lambda), (y(\lambda)), \lambda, \lambda)\|_1
\leq p_1\|x(\lambda)-x(\lambda)\|_1 + \rho_1\sigma_2\|y(\lambda)\|
-\|y(\lambda)\|_2 + (e_1(1+r_1)+\rho_1r_1\eta_1)\|\lambda_1-\lambda_2\|_1
+ \rho_1r_1\nu_1\|\lambda_2-\lambda_2\|_2
\]
(45)

Similarly, we have
\[
\|S(x(\lambda), (y(\lambda)), \lambda, \lambda)\|_2
-\|S(x(\lambda), (y(\lambda)), \lambda, \lambda)\|_2
\leq p_2\|y(\lambda)-y(\lambda)\|_2 + \rho_2\beta_2b_1\|x(\lambda)\|
-\|x(\lambda)\|_1 + (e_2(1+r_2)+\rho_2r_2\eta_2)\|\lambda_2-\lambda_2\|_2
+ \rho_2r_2\nu_2\|\lambda_1-\lambda_1\|_1
\]
(46)

From (42), (43) and (46), we have
\[
\|(x(\lambda), y(\lambda))-(x(\lambda), y(\lambda))\|_*
\leq \max\{k_1, k_2\}\|(x(\lambda), y(\lambda))\|
-(x(\lambda), y(\lambda))\|_* + \max\{l_1, l_2\}\|(\lambda, \lambda)\|
-(\lambda, \lambda)\|_*
\]
(47)

where \(k_1, k_2\) are given by (40) and
\[
l_1 := e_1(1+r_1) + \rho_1r_1\eta_1 + \rho_2r_2\nu_2;
l_2 := e_2(1+r_2) + \rho_2r_2\eta_2 + \rho_1r_1\nu_1.
\]

Thus
\[
\|(x(\lambda), y(\lambda))-(x(\lambda), y(\lambda))\|_*
\leq \left(\frac{\max\{l_1, l_2\}}{1-\max\{k_1, k_2\}}\right)\|(\lambda, \lambda)-(\lambda, \lambda)\|_*
\]
(48)

Since \(l_1, l_2 > 0\) and by condition (36), \(\max\{k_1, k_2\} < 1\), then (48) implies that \((x(\lambda), y(\lambda))\) is \(\theta\)-Lipschitz continuous in \((\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2\), where
\[
\theta = \frac{\max\{l_1, l_2\}}{1-\max\{k_1, k_2\}} > 0.
\]

4. System of Generalized Variational Inequalities

Let \(I = \{1, 2\}\) be an index set and for each \(i \in I\), let \(H_i\) be a real Hilbert space whose inner product and norm are denoted by \(\langle.,.\rangle_i\) and \(\|\|_i\), respectively. For each \(i \in I\), let \(F_i : H_1 \times H_2 \to H_i\), \(g_i : H_i \to H_i\), \(A_i : H_i \to H_i\), \(B_i : H_i \to H_i\) be non-linear mappings, then we consider the following system of generalized variational inequality problems (SGVIP):

Find \((x, y) \in H_1 \times H_2\) such that
\[
\langle F_1(A_1(x), A_2(y)), g_1(v_1)-g_1(x)\rangle_1
+ b_1(x, g_1(v_1)) - b_1(x, g_1(x)) \geq 0,
\]
(48)

for all \(v_1 \in H_1\);
\[
\langle F_2(B_1(x), B_2(y)), g_2(v_2)-g_2(y)\rangle_2
+ b_2(y, g_2(v_2)) - b_2(y, g_2(y)) \geq 0,
\]
(49)

for all \(v_2 \in H_2\),

where for each \(i \in I\), the bifunction \(b_i : H_i \times H_i \to R\), which is not necessarily differentiable, satisfies the following properties:

(i) \(b_i\) is linear in the first argument;
(ii) \(b_i\) is bounded, that is, there exists a constant \(\nu_i > 0\) such that
\[
b_i(u_i, v_i) \leq \nu_i\|u_i\| \|v_i\|, \forall u_i, v_i \in H_i;
\]
(iii) \(b_i(u_i, v_i) - b_i(u_i, w_i) \leq b_i(u_i, v_i - w_i)\), for all \(u_i, v_i, w_i \in H_i\);
(iv) for each fixed, \(u_i \in H_i\), \(b_i(u_i, g_1(\cdot)) : H_i \to R\) is convex.

We remark that for suitable choices of the mappings \(F_i, g_i, A_i, B_i, b_i\) and the spaces \(H_i\) for \(i = 1, 2\), SGGV (48) and (49) reduces to various classes of variational inequalities.
We need the following definition and assumption in the sequel:

**Definition 10.** A mapping $F_1 : H_1 \times H_2 \to H_1$ is said to be

(i) **relaxed ($d, r$)-cocoercive with first argument** if there exist constants $d, r > 0$ such that

$$\langle F_1(x_1, x_2) - F_1(y_1, x_2), x_1 - y_1 \rangle_1 \geq (-d)\|F(x_1, x_2) - F(y_1, x_2)\|_1^2 + r\|x_1 - y_1\|_1^2,$$

for all $x_1, y_1 \in H_1, x_2 \in H_2$;

(ii) **($\beta_1, \xi_1$)-Lipschitz continuous** if there exist constants $\beta_1, \xi_1 > 0$ such that

$$\|F_1(x_1, x_2) - F_1(y_1, y_2)\|_1 \leq \beta_1\|x_1 - y_1\|_1 + \xi_1\|x_2 - y_2\|_2,$$

for all $x_1, y_1 \in H_1, x_2, y_2 \in H_2$.

**Assumption 1.** For each $i \in I$, the mappings $F_i : H_1 \times H_2 \to H_i$, $A_i, B_i, g_i : H_i \times H_i \to H_i$ satisfying the following conditions:

(i) For all $(x, y) \in H_1 \times H_2$, there exists a constant $\tau_i > 0$ such that

$$\|F_i(x, y)\|_i \leq \tau_i(\|x\|_1 + \|y\|_2);$$

(ii) for any given $x_1 \in H_1, x_2 \in H_2$, the mapping

$$y_i \to \langle F_i(A_i(x_1), B_i(x_2)), g_i(y_i) - g_i(x_i) \rangle$$

is concave and continuous.

### 4.1. Auxiliary problems and algorithm

First, related to SGVIP (48) and (49), we consider the auxiliary problems and then establish an existence theorem for these auxiliary problems.

**Auxiliary problems:** Given $(x_1, x_2) \in H_1 \times H_2$, find $z_1 \in H_1, z_2 \in H_2$ such that

$$\langle z_1 - x_1, y_1 - z_1 \rangle_1 + \rho_1 \langle F_1(A_1(x_1), A_2(x_2)), g_1(z_1) - g_1(x_1) \rangle \geq 0,$$

for all $y_1 \in H_1$;

$$\langle z_2 - x_2, y_2 - z_2 \rangle_2 + \rho_2 \langle F_2(B_1(x_1), B_2(x_2)), g_2(z_2) - g_2(x_2) \rangle \geq 0,$$

for all $y_2 \in H_2$, where $\rho > 0$ is a constant.

**Theorem 6.** For each $i \in I$, let the mapping $g_i : H_i \to H_i$ be $\delta$-Lipschitz continuous, and let the bifunction $b_i$ satisfy the properties (i)-(iv). If Assumption 1 holds, then the Auxiliary problems (50) and (51) are solvable.

The proof of Theorem 6 follows on the similar lines given in the proof of Theorem 4 [57] and hence omitted.

Now, based on Theorem 6, we construct an algorithm for SGVIP (48) and (49).

For given $(x_0, y_0) \in H_1 \times H_2$, we know from Theorem 4.1 that the Auxiliary problems (50) and (51) have solutions, say, $(x_1, y_1) \in H_1 \times H_2$, i.e.,

$$\langle x_1 - x_0, v_1 - x_1 \rangle_1 + \rho F_1(A_1(x_0), A_2(y_0)), g_1(v_1) - g_1(x_1) \rangle_1 + \rho [b_1(x_0, g_1(v_1)) - b_1(x_0, g_1(x_1))] \geq 0, \forall v_1 \in H_1;$$

$$\langle y_1 - y_0, v_2 - y_1 \rangle_2 + \rho F_2(B_1(x_0), B_2(y_0)), g_2(v_2) - g_2(y_1) \rangle \geq 0, \forall v_2 \in H_2.$$

By Theorem 6 again, for $(x_1, y_1) \in H_1 \times H_2$, the Auxiliary problems (50) and (51) have solutions $(x_2, y_2)$, i.e.,

$$\langle x_2 - x_1, v_1 - x_2 \rangle_1 + \rho F_1(A_1(x_1), A_2(y_1)), g_1(v_1) - g_1(x_2) \rangle_1 + \rho [b_1(x_2, g_1(v_1)) - b_1(x_2, g_1(x_2))] \geq 0, \forall v_1 \in H_1;$$

$$\langle y_2 - y_1, v_2 - y_2 \rangle_2 \geq 0, \forall v_2 \in H_2.$$
\[-g_i(x_2) + \rho [b_1(x_1, g_1(v_1)) - b_1(x_1, g_1(x_2))],\]
\[\geq 0, \ \forall v_1 \in H_1;\]
\[\langle y_2 - y_1, v_2 - y_2 \rangle + r F_2(B_1(x_1), B_2(y_1)), g_2(v_2) - g_2(y_2) \rangle + \rho [b_2(y_1, g_2(v_2)) - b_2(y_1, g_2(y_2))],\]
\[\geq 0, \ \forall v_2 \in H_2.\]

By induction, we have the following algorithm for SGVIP (48) and (49) as follows:

**Algorithm 1.** For given \((x_0, y_0) \in H_1 \times H_2,\) compute an approximate solution \((x_n, y_n)\) in \(H_1 \times H_2\) satisfying the following conditions:
\[
\langle x_{n+1} - x_n, v_1 - x_{n+1} \rangle + \rho F_1(A_1(x_n), A_2(y_n)),
\]
\[v_1 - x_{n+1} + \rho [b_1(x_n, g_1(v_1)) - b_1(x_n, g_1(x_{n+1}))],\]
\[\geq 0, \ \forall v_1 \in H_1;\]
\[
\langle y_{n+1} - y_n, v_2 - y_{n+1} \rangle + \rho F_2(B_1(x_n), B_2(y_n)),
\]
\[v_2 - y_{n+1} + \rho [b_2(y_n, g_2(v_2)) - b_2(y_n, g_2(y_{n+1}))],\]
\[\geq 0, \ \forall v_2 \in H_2,\] where \(\rho > 0\) is a constant.

### 4.2. Existence of unique solution and convergence analysis

**Theorem 7.** Let the mapping \(F_1 : H_1 \times H_2 \rightarrow H_1\) be relaxed \((d, r)\)-cocoercive in the first argument and \((\beta_1, \xi_1)\)-Lipschitz continuous; let the mapping \(F_2 : H_1 \times H_2 \rightarrow H_2\) be relaxed \((d_2, r_2)\)-cocoercive in second argument and \((\beta_2, \xi_2)\)-Lipschitz continuous; let for each \(i = 1, 2,\) the mapping \(g_i : H_i \rightarrow H_i\) be relaxed \((e_i, \delta_i)\)-cocoercive and \(\sigma_i\)-Lipschitz continuous; let the mappings \(A_i, B_i \in a_i, c_i\)-Lipschitz continuous respectively; let the bifunction \(b_i\) satisfy the properties \((i)-(iv)\), and let Assumption 1 hold. If the following conditions hold for \(\rho > 0:\)
\[
\rho \geq \frac{A_2(y_n) + \rho (F_1(A_1(x_n), A_2(y_n)), b_1(x_n, g_1(v_1))), -b_1(x_n, g_1(x_{n+1}))]}{\rho},
\]
\[\geq 0, \ \forall v_1 \in H_1;\]
\[
\langle y_{n+1} - y_n, v_2 - y_{n+1} \rangle + \rho F_2(B_1(x_n), B_2(y_n)),
\]
\[v_2 - y_{n+1} + \rho [b_2(y_n, g_2(v_2)) - b_2(y_n, g_2(y_{n+1}))],\]
\[\geq 0, \ \forall v_2 \in H_2,\]
where \(A\) is obtained from Algorithm 1 strongly converges to \((x^*, y^*)\), where \((x^*, y^*)\) is a solution of SGVIP (48) and (49).

**Proof.** For any \((v_1, v_2) \in H_2 \times H_2,\) it follows from (52) and (53) that
\[
\langle x_n - x_{n-1}, v_1 - x_{n-1} \rangle + \rho F_1(A_1(x_{n-1}), A_2(y_{n-1})),
\]
\[A_2(y_{n-1}) = 1, b_1(x_{n-1}, g_1(v_1))), -b_1(x_{n-1}, g_1(x_{n-1})),\]
\[\geq 0, \ \forall v_1 \in H_1;\]
\[
\langle y_{n+1} - y_n, v_2 - y_{n+1} \rangle + \rho F_2(B_1(x_n), B_2(y_n)),
\]
\[v_2 - y_{n+1} + \rho [b_2(y_n, g_2(v_2)) - b_2(y_n, g_2(y_{n+1}))],\]
\[\geq 0, \ \forall v_2 \in H_2,\]
and
\[
\langle x_{n+1} - x_n, v_1 - x_{n+1} \rangle + \rho F_1(A_1(x_n), A_2(y_n)),
\]
\[g_1(v_1) - g_1(x_{n+1})), -b_1(x_n, g_1(x_{n+1})),\]
\[\geq 0, \ \forall v_1 \in H_1;\]
\[
\langle y_{n+1} - y_n, v_2 - y_{n+1} \rangle + \rho F_2(B_1(x_n), B_2(y_n)),
\]
\[g_2(v_2) - g_2(y_{n+1})), -b_2(y_n, g_2(v_2)),\]
\[\geq 0, \ \forall v_2 \in H_2.\]
Taking $v_1 = x_{n+1}$ in (55) and $v_1 = x_n$ in (57), respectively, we get

$$
\langle x_n - x_{n-1}, x_{n+1} - x_n \rangle_1 + \rho \langle F_1(A_1(x_{n-1}), A_2(y_{n-1})), g_1(x_{n+1}) \rangle_1 + \rho [b_1(x_{n-1}), g_1(x_{n+1})] \rangle_1 \geq 0, \quad (59)
$$

$$
\langle x_n - x_{n-1}, x_{n+1} - x_n \rangle_1 + \rho \langle F_1(A_1(x_n), A_2(y_n)), g_1(x_{n+1}) \rangle_1 + \rho [b_1(x_n), g_1(x_{n+1})] \rangle_1 \geq 0, \quad (60)
$$

Adding (59) and (60), we get

$$
\langle x_n - x_{n+1}, x_{n-1} - x_n \rangle_1 \leq \langle x_{n-1} - x_n, x_n - x_{n+1} \rangle_1 \leq \rho \left[ F_1(A_1(x_{n-1}), A_2(y_{n-1})) - F_2(A_1(x_n), A_2(y_n)) \right] \leq \rho \left[ F_1(A_1(x_{n-1}), A_2(y_{n-1})) - F_1(A_1(x_n), A_2(y_n)) \right] \leq (1 - 2\rho r_1) \left\| x_{n-1} - x_n \right\|^2 + (\rho^2 + 2\rho d_1) \left\| F_1(A_1(x_{n-1}), A_2(y_{n-1})) - F_1(A_1(x_n), A_2(y_n)) \right\|^2
$$

$$
\leq (1 - 2\rho r_1) \left\| x_{n-1} - x_n \right\|^2 + (\rho^2 + 2\rho d_1) \beta_1^2 a_1^2 \left\| x_{n-1} - x_n \right\|^2
$$

Again since $g_1$ is relaxed $(\epsilon_1, \delta_1)$-cocoercive and $\sigma_1$-Lipschitz continuous, we estimate that

$$
\left\| x_{n-1} - x_n \right\|^2 \leq (1 - 2\delta_1 + (1 + 2\epsilon_1)\sigma_1^2) \left\| x_n - x_{n+1} \right\|^2
$$

(62)
From (61), (62) and (63), we have
\[
\|x_n - x_{n+1}\|_1 \leq \left\{ \sigma_1(1 - 2\rho r_1 + (\rho^2 + 2\rho d_1)\beta_1^2 a_1^2)^{1/2} \\
+ (1 - 2\delta_1 + (1 + 2\epsilon_1)\sigma_1^2)^{1/2} + \rho \sigma_1 v_1 \right\}
\]
where
\[
\theta_1 := \sigma_1(1 - 2\rho r_1 + (\rho^2 + 2\rho d_1)\beta_1^2 a_1^2)^{1/2} \\
+ \rho \sigma_2 \beta_1 c_1 + (1 - 2\delta_1 + (1 + 2\epsilon_1)\sigma_1^2)^{1/2} + \rho \sigma_1 v_1,
\]
\[
\theta_2 := \sigma_2(1 - 2\rho r_2 + (\rho^2 + 2\rho d_2)\beta_2^2 c_2)^{1/2} + \rho \sigma_1 \alpha_2 + (1 - 2\delta_2 + (1 + 2\epsilon_2)\sigma_2^2)^{1/2} + \rho \sigma_2 v_2.
\]

Taking \(v_2 = y_{n+1}\) in (4.9) and \(v_2 = y_n\) in (4.11), respectively, and then adding the results, we have
\[
\|y_n - y_{n+1}\|_2^2 \leq \|y_{n+1} - y_n\| - \rho \|F_2(B_1(x_{n-1}), B_2(y_{n-1})) - F_2(B_1(x_n, B_2(y_n)))\|_2
\]
\[
\|g_2(y_n) - g_2(y_{n+1})\|_2 + \|y_{n-1} - y_n\|_2 \|y_n - y_{n+1}\|_2
\]
\[
\leq \max\{\theta_1, \theta_2\}\{\|x_{n-1} - x_n\|_1 + \|y_{n-1} - y_n\|_2\}.
\]

Now, defined the norm \(\|\cdot\|_*\) on \(H_1 \times H_2\) by
\[
\|(x, y)\|_* = \|x\|_1 + \|y\|_2, \quad \forall (x, y) \in H_1 \times H_2.
\]

It observe that \((H_1 \times H_2, \|\cdot\|_*)\) is a Banach space. Hence, (67) implies that
\[
\{(x_n, y_n) - (x_{n+1}, y_{n+1})\|_*
\]
\[
\leq \max\{\theta_1, \theta_2\}\{\|x_{n-1} - x_n\|_1 - \|x_n - y_n\|_1\}.
\]

By condition (54), it follows that \(\theta_1, \theta_2 \in (0, 1)\) and hence (68) implies that \(\{(x_n, y_n)\}\) is a Cauchy sequence in \(H_1 \times H_2\) as \(n \to \infty\). From (52) and (53) and from the fact that \(F_1, A_i, B_i, g_i, b_i\) for \(i = 1, 2\) are continuous, we have, \(\forall v_1 \in H_1, v_2 \in H_2,\)
\[
\langle F_1(A_1(x^*), A_2(y^*)), g_1(v_1) - g_1(x^*) \rangle_1 + b_1(x^*, g_1(v_1)) - b_1(x^*, g_1(x^*)) \geq 0, \quad \forall v_1 \in H_1;
\]
\[
+ \langle F_2(B_1(x^*), B_2(y^*)), g_2(v_2) - g_2(y^*) \rangle_2 + b_2(y^*, g_2(v_2)) - b_2(y^*, g_2(y^*)) \geq 0, \quad \forall v_2 \in H_2;
\]

Therefore \((x^*, y^*)\) is a solution of SGVIP (4.1) and (4.2). This completes the proof.

5. Conclusion

We have considered a system of general variational inclusions (SGVI) in \(q\)-uniformly smooth Banach spaces. Using
proximal-point mapping technique, we have proved the existence and uniqueness of solution and suggested a Mann type perturbed iterative algorithm for SVLI. We also discussed the convergence criteria and stability of Mann type perturbed iterative algorithm. Further, a system of parametric general variational inclusions (SPGVI) corresponding to SGVI has been considered and discussed the continuity of the solution. Finally we have considered a system of generalized variational inequality problems (SGVIP) in Hilbert spaces and established an existence theorem for auxiliary problems related to SGVIP. By exploiting this theorem, an algorithm for the SGVIP is constructed. We also discussed the existence of a unique solution of SGVIP and the convergence analysis of the algorithm. The techniques and results presented here improve the corresponding techniques and results for the variational inequalities and inclusions in the literature.

Using the techniques presented in this paper, one can extend the results of this paper for the system of $n$-general variational inclusions, system of $n$-parametric general variational inclusions, and system of $n$-generalized variational inequality problems. Moreover, it is of further research interest to extend the techniques presented here for iterative approximations of solution of considered systems involving multivalued-valued mappings.

References


dualité, Ann. Inst. Fourier (Grenoble), 18(1) (1968), 115-175.


[44] N.-J. Huang and C.-X. Deng, Auxiliary principle and iterative algorithms for


