A novel Nonlinear Implicit Sliding Surface controller design for Inertia Wheel Pendulum

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Abstract— Control design for non-affine nonlinear systems is one of the most difficult problems due to the lack of mathematical tools. The Inertia Wheel Pendulum is a benchmark example of non-affine systems, resulting after coordinate transformations. The under actuation property, posing problems in exact feedback linearization makes design of the control law for this a challenging task. Partial feedback linearization reduces a part of the system linear but leaves the core system non-affine in nature. A novel nonlinear controller design fusing recently introduced Sliding Surface Control technique with Implicit Control of the nonlinear core is presented to tackle the issue. The task of the nonlinear controller is not only to stop the wheel but also to stabilize the pendulum at its unstable upright equilibrium. The design procedure is simpler and more intuitive than currently available sliding surfaces, integrator backstepping or energy shaping designs. Stability is analyzed by decomposing the system into a cascade of linear and nonlinear sub-systems. Advantages over existing controller designs are analyzed theoretically and verified using numerical simulations.

Keywords- Inertia Wheel Pendulum, Multiple Sliding Surfaces, Underactuated Mechanical Systems, Backstepping, Implicit control

I. INTRODUCTION

A nonlinear controller design for stabilization of benchmark nonlinear underactuated mechanical system: Inertia Wheel Pendulum (IWP) is presented with stability analysis. The system posses a non-affine nonlinear core complicating the control law designs considerably. In fact, it is impossible to handle the control problem of the non-affine nonlinear system directly because, in general, even if it is known that the inverse exists, it is impossible to construct it analytically. The presented design uses implicit controller design technique after decomposing the system into simpler cascades that reduces the controller design procedure significantly. Stability is analyzed using the theory of cascaded systems. To the best of our knowledge it is the first of its type stabilization controller for IWP type Underactuated Mechanical Systems (UMS)

UMS are mechanical control systems with fewer actuators (i.e. controls) than configuration variables. These systems arise in real life applications, such as space and undersea robots, mobile robots, snake-type and swimming robots, acrobatic robot, flexible robots, walking, brachiating, and gymnastic robots and very recently in Micro Electro mechanical Systems. IWP first introduced by Spong et al., is a Benchmark nonlinear UMS, mainly for Energy Shaping and Damping Injection based approaches. In , a supervisory hybrid/switching control strategy is applied to asymptotic stabilization of the inertia-wheel pendulum around its upright equilibrium point. First, a passivity-based controller swings up the pendulum. Then, a balancing controller, obtained by Jacobian linearization or (local) exact feedback linearization stabilizes the pendulum around its upright position.

Global stabilization of IWP system using Integrator Backstepping procedure (IBS) is already known. IBS, based on results obtained by Sontag and Sussman , is a powerful step-by-step design tool. However it suffers the problem of “explosion of terms” besides putting stringent condition on certain system functions (being C¹ at least). Multiple Sliding Surfaces (MSS) control , a procedure similar to IBS, avoids this issue but falls short of integrator backstepping in terms of theoretical rigor, as the need for analytical differentiation is pushed to a numerical one.

Concept of Dynamic Surface Control (DSC), a dynamic extension to MSS, introduced by Swaroop et al. resolves these issues by using low pass filters. A fusion of DSC and Control Lyapunov Method has been used successfully for stabilization of IWP successfully by authors ; however this system has another challenge present in its non-affine nature of the nonlinear core. The problem is further complicated as the involved function is not invertible. We have used Implicit controllers techniques to stabilize IWP and show the design method resulting in a less complicated control law. The designed controller is simpler than the IBS design and doesn’t require a supervisory controller like the one by Spong et al and presents a straightforward approach to the problem of non-affine control input present in nonlinear core.

The paper opens formally with Section II, containing the dynamical model of IWP along with necessary coordinate transformations, making the note self contained. Controller design and stability discussion that is the major topic of this note appears in section III. Section IV presents simulation results comparing controller performance to existing designs with study of initial condition effects on the system stability followed by concluding remarks in Section V.
II. Dynamical Model

The IWP as depicted in II is a planar inverted pendulum with a rotating wheel on the end. Only the joint on the base is unactuated thus, the pendulum is to be controlled through wheel rotation only.

The controller task is to stabilize the pendulum in its upright equilibrium position while the wheel stops rotating. The specific angle of rotation of the wheel is not important.

Dynamic model of IWP can be obtained easily by Euler Lagrange method. Using configuration variables as shown in I the Lagrangian for IWP is

\[ L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q) \]

the potential energy function is given as

\[ V(q) = w \cos(q) \]

where \( w = (m_1 l_1 + m_2 l_2) g \)

IWP is a Flat Underactuated Mechanical Systems with kinetic symmetry thus all the Christoffel Symbols associated with \( M \) vanish and the inertial matrix \( M \) is constant and

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q(q) u \]

gives equations of motion as

\[
\begin{bmatrix}
    I_1 + I_2 & I_1 & 0 \\
    I_1 & I_2 & 0 \\
    0 & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
    \dot{q}_1 \\
    \dot{q}_2 \\
    \dot{\theta} 
\end{bmatrix}
+ \begin{bmatrix}
    -w \sin(q) \\
    0 \\
    0 
\end{bmatrix} = \begin{bmatrix}
    \tau 
\end{bmatrix}
\]

Where
- \( I_1, I_2 \): Moment of inertia of the pendulum and the wheel (Kg m²)
- \( m_1, m_2 \): Mass of the Pendulum and the wheel (kg)
- \( L_1 \): Length of the Pendulum (m)
- \( l_1 \): Distance to the center of the mass (m)
- \( q_1 \): Angle that the Pendulum makes with the vertical
- \( q_2 \): Angle of the wheel
- \( \tau \): Input Torque applied on the Wheel (Nm)

Underactuated property denies use of feedback linearization so to simplify the model for controller design colocated partial feedback linearization is employed using the following change of control

\[ \tau = \alpha u + \beta \]

where \( \alpha = (m_2 - m_2 m_1 / m_1) \), \( \beta = (m_2 w / m_1) \sin(q) \)

A. Remarks

System after coordinate transformation is a cascade interconnection of a linear double integrator subsystem and a nonlinear core subsystem, in strict feedback form. This form is more amenable to several standard controller design techniques like IBS, MSS and DSC. Carefully note that the nonlinear part has a virtual input appearing from linear system in non-affine manner. The function is not invertible making the control law design very difficult. In next section we demonstrate how this problem can be circumvented using implicit control design technique.

III. Controller Design

We first design the controller for the inner sub system i.e. the non-affine nonlinear core using the technique of implicit
controllers, which in fact is the main emphasis of this paper and then design MSS controller for the lower part.

Core dynamics of inertia wheel pendulum after the coordinate transformation are given as following

\[ \dot{z}_1 = w \sin(z_2) \]

Assuming \( z_2 \) as the virtual control input let \( z_{2d} \) is the required control law that stabilizes. Note it enters in a non-affine fashion with the vector field noninvertible globally. Let us consider the first order non-affine nonlinear systems as

\[ \dot{z}_i = \sin(z_2) = f(z_i, u) \]

where \( z_i \) and \( u \in R \) are the state and input respectively. In general, \( z_i = \theta \phi \left( z_1, u \right) \in R \) is a non-affine function of both \( z_1 \) and \( u \), \( \theta \) and \( \phi \) being dimensionally compatible constant parameters and known regressor, respectively. To find a stabilizing control add and subtract \( bu \) on the right hand side of equation

\[ \dot{z}_1 = \left[ \theta^T \phi_1(z_1, u) - bu \right] + bu \]

Consider the implicit control \( u \) given by

\[ u = -\frac{1}{b} \left[ \theta^T \phi_1(z_1, u) + Kz_1 \right] + u, \quad K > 0 \]

where \( b \) is a design constant. From , we know that \( u \) is actually solved by

\[ \theta^T \phi_1(z_1, u) + Kz_1 = 0 \]

Accordingly, we have \( \dot{z}_1 = -Kz_1 \), which shows that the closed-loop system is stable and \( z_1 \) will exponentially converge to zero. In theory, the existence of the solution for \( u \) is guaranteed as the controllability condition is satisfied, S.S. . The scheme is especially suitable for discrete time controllers where the \( u \) and \( x \) at right hand side of the equation are available from last clock sample, as shown for IWP in following.

Design of control law follows directly from results obtained as and the desired virtual control is given by

\[ z_{2d}(k + 1) = -\frac{1}{b} \left[ w \sin(z_{2d}(k)) + Kz_1 \right] + z_{2d}(k), \quad K > 0 \]

A. Outer subsystem controller design

To stabilize \( z_2 \) is required to follow the trajectory given as. Applying MSS technique, we design a control law for the linear subsystem that generates the desired trajectory. It’s

trivial to verify that necessary assumptions for MMS are satisfied by regarding the system and by regarding the trajectory i.e.

- \( f \) is a \( C^1 \) function in it’s arguments
- The desired trajectory is bounded and sufficiently smooth
- System has no uncertainties

Design procedure:

Take the first sliding surface \( S_1 \) as the error in generation of stabilization function by \( z_1 \) then

\[ S_1 := z_2 - z_{2d} \]

\[ \dot{S}_1 = \dot{z}_2 - \dot{z}_{2d} = \frac{1}{m_1} \left( z_1 - \frac{1}{m_1} \right) - \dot{z}_{2d} \]

Now \( z_3 \) is chosen as next virtual control to drive \( S_1 \) to zero i.e.

\[ z_{3d} = \frac{1}{m_1} \left( K_1 S_1 + \frac{1}{m_1} z_1 - \dot{z}_{2d} \right) \]

Similarly defining the second surface \( S_2 \) as

\[ S_2 := z_3 - z_{3d} \]

\[ \dot{S}_2 = \dot{z}_3 - \dot{z}_{3d} = u - \dot{z}_{3d} = -K_2 S_2 \]

As the control law chosen to derive \( S_2 \) to zero

\[ u = \dot{z}_{3d} - K_2 S_2 \]

Notice that the direct calculation of \( \dot{z}_{3d}(t) \) and \( \dot{z}_{3d}(t) \) required at this step by the conventional backstepping design procedure leads to complexity due to “explosion of terms”. Motivated by MSS technique this problem is dealt by numerical differentiation, i.e. \( \dot{z}_{3d}(t) = (z_{3d}(t) - z_{3d}(t - \Delta T)) / \Delta T \)

With modern high speed digital electronics the processing speed can be set very high as compared to the slowly evolving dynamics of the mechanical system. The upper bound of error for this calculation is \( O(h^2) \leq M_3(h^2 / 6) \) where

\[ M_3 = \max_{x \in \mathbb{R}^n} \| f^{(3)}(x) \| \]

Thus by keeping \( \Delta T := h \) sufficiently small the error can practically be made very close to zero.
Taking \( V = \frac{1}{2}(K_1 S_1^2 + K_2 S_2^2) \) as a Lyapunov function candidate for closed loop system, \( \dot{V} \) is given as

\[
\dot{V} = S_1 \dot{S}_1 + S_2 \dot{S}_2 \\
\dot{V} = -K_1 S_1^2 - K_2 S_2^2 \leq 0
\]

This shows the system Globally Asymptotically stable for all \( K_1, K_2 > 0 \) and theorem 4.11 Hassan K. Khalil, shows that for linear systems asymptotic stability of the origin is equivalent to exponential stability, this can cater easily for errors in numerical differentiations ignored in closed loop model. The map \( f(\varepsilon, S_1) : \mathbb{R}^2 \rightarrow \mathbb{R} \) is \( C^1 \), and we have already shown after that the driven system is Globally exponentially Stable and globally Lipschitz. Thus it is trivial to show that composition satisfies all the conditions to avoid peaking, and the origin of the composition is globally asymptotically stable, Sussman H.J. et al.

A comparison to existing designs and reveals the ease of design and simplicity of obtained control law.

IV. SIMULATION RESULTS

For fair performance comparisons we use same system parameters as R. Olfati and Spong i.e. \( m_1 = 4.83 \times 10^{-3} \), \( m_{12} = m_{21} = m_{22} = 32 \times 10^{-6} \), and \( w = 379.26 \times 10^{-3} \).

As obvious from analysis, \( K_i \) can be set moderately high for faster convergence rates. Contrary to conventional \( K_i \) tuning external layer is not needed to converge faster than the internal necessarily. Filter time constant \( \Delta T \) controls boundary layer error hence it must be set as low as possible. However, DAC/ADC sample time and actuator saturation must be kept in mind as smaller values increase control effort peaks and also make control signal noisy. Following controller parameters were used for simulations \( b_i = 2 \), \( K = 3 \), \( K_1 = 4 \), \( K_2 = 6 \) and \( \Delta T = 0.001 \).

As depicted II the nonlinear controller aggressively stabilizes pendulum from its downward stable equilibrium point to its upright unstable equilibrium point with negligible transients, while wheel stops rotating, IV

Swing up is faster than the design by R. Olfati that requires more coordinate transformations and IBS exhibiting the phenomenon of explosion of terms. The structure is also simpler than spong’s design as it requires no supervisory switching controller.
V. CONCLUSIONS

A novel controller design is presented for stabilization of IWP employing a fusion of Implicit Controller design method with MSS technique. The presented architecture demonstrates the potential of implicit controllers handling the non-affine structures appearing in UMS after collocated partial feedback linearization. Model is brought to strict feedback before application of design techniques. Stability of the system is analyzed theoretically by considering it as a cascaded connection of two exponentially stable systems. Design simplicity is shown and Controller performance is compared to existing designs through both theoretically and simulation studies. It is concluded that the presented controller architecture with a simpler design procedure while leading to a less complicated Control Law, achieves faster stabilization and presents a straightforward method for handling non-affine systems. Further improvements envisioned are Robustification and generalization of scheme to whole subclass.

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