DIFFERENTIAL EQUATIONS ON METRIC GRAPH

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Differential Equations On Metric Graph

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Preface

This is a development report for the investigation of the partial differential equation networks. In this report, we mainly discuss the stabilization of the wave equations with variable coefficients defined on the metric graphs. These networks might be discontinuous or with circuits.

In the past decades, there have been a lot of literature studying the controllability, observability and stabilization for the elastic system. Some nice results and innovative approaches have obtained. For examples, Rolewicz in [95] investigated the controllability of systems of strings; Cox and Zuazua in [22] studied decay rate of the energy of single string system; Xu and Guo in [111] studied the stabilization of a string with interior pointwise control and obtained the Riesz basis property of the eigenfunctions of the system; more recent papers for single string system, we refer to [128], [120] and the reference therein. While single Timoshenko beam treated as two weakly internal-coupled vibrating strings have been studied under various boundary conditions, for instance, see [110], [112], [113],[114] and [115]. Under different control laws, the exponential stabilization and the Riesz basis property of those systems were obtained. There were some nice results for the serially connected strings system, here we refer to literature [15], [62], [68] and [71], in which the authors used the multiplier approach to obtain stabilization for the wave equations by boundary control. In particular, under certain conditions, Liu et al in [71] obtained the exponential stabilization for a long chain of vibrating strings. Guo et al in [43] gave an abstract sufficient condition to deal with Riesz basis generation and apply it to serially connected strings. For other type of serially connected elastic system such as Euler-Bernoulli beams and Timoshenko beams, many authors had made great effort on the control and stabilization of the system, for instance, see [96], [16], [103] and [119].

The study of the differential equations on graphs (or networks) was derived from distinct science background. The questions arise the high-tech such as chip interconnect problem and electron motion in a molecule. The differential equations on graphs was investigated in [87] and [40] for the scattering problem of the free electrons. Since then, there were a great deal papers studying the properties of the differential equations, we refer to two works [90] and [12], in which the authors gave a brief review of results on this aspect. As for spectral problem of the differential equations on the graphs, there were many nice results, we refer to the works of J. von Below, F. Ali Mehmeti and S. Nicaise, please see [10], [1] ,[30] and the references therein. The elastic networks are important class of the differential equations on graphs. As to the modelling and control problem of the elastic networks, we refer to the early works [75] and [102]. More recent development on controllability, observability and stabilization of the network of strings, we refer to a book [34] and a report [129], in where there is a complete list of references on the study of network of strings.

We observe that most of the literature aforementioned mainly deal with the differential equations on graphs with constant coefficients and system continuity, there are a few works discussing the system with variable coefficients and discontinuity. Therefore, we choose the networks with variable coefficients as our research object. Our project includes the following two aspects:

(1) Stabilization of elastic network with variable coefficients;

(2) Identification of the network structure.

In the first aspect, we mainly discuss design of the feedback controllers involving the location of controllers and availability of controllers and stability analysis of the close loop system. In the second part, our attention focuses on identifying the shape of the network structure by measurement. These questions have important application in the real world.

This report is merely a development report for investigation of one dimensional wave networks. The first four chapters are basic materials on the elastic networks. Chapters 5–9 are on the control and stabilization of networks of strings. These works are finished recently. As to the networks of Euler-Bernoulli beams and Timoshenko beams, we will give an investigation report on them in the future. This research is supported by the Natural Science Foundation of China Grant NSFC-60874034 and partially supported by WSEAS.

Contents

Chapter 1

Complex Dynamic Systems

1.1 Distributed electronic circuits

1.1.1 Distributed parameter model of interconnects

With the rapid increase of the signal frequency and decrease of the feature sizes of high-speed electronic circuits, interconnects have become a dominant factor in determining circuit performance and reliability in deep sub-micron designs. On-chip interconnect problems have been interested many researchers, and become hot topics in the area of advanced CAD (computer aided design) techniques [18]. The interconnects can be modeled as lumped, distributed, or full-wave models directly according to the operating frequency, rise time of signal, interconnect structure, etc. Generally speaking, at lower frequencies, i.e., the length of the interconnect lines is electrically small at the frequency of interest, interconnect lines could be modeled using lumped RC (first-order system for monotonic waveform) or RLC (second-order system needed for ringing phenomena) circuit model [28]. To model the interconnect lines more precisely, a large number of lumped sections are often needed, which leads to circuit equations with very large dimension, high CPU intensive and memory exhaustive simulations.

At relatively high signal speed, electrical length of interconnect lines becomes a significant fraction of the operating wavelength, interconnect line reveals its distributed nature, giving rise to signal distorting effects that do not exist at lower frequencies, which results in the fact that the conventional lumped models are inadequate to capture the dynamic characteristic of interconnect lines and distributed parameter system models are needed. In [11] and [105], On-chip interconnects are modeled as a distributed RLC parameter model, and the approximation model obtained is not complicated due to the fact that the dielectric loss G is ignored. Unfortunately, this can not be neglected in many practical occasions especially in high frequency domain. [98] expanded the admittance matrix of RLCG transmission line analytically in terms of poles and residues. Often, from the system design point of view, the solution to Maxwell's equations may be given by the so-called quasi-transverse electro-magnetic modes (TEM), and it can be characterized by distributed parameters R, L, C, and G [4]. In general, a transmission line is presented by Telegrapher's equations.

Let us recall a single nonuniform RLCG interconnect line under consideration. A real interconnect system consists of single interconnect line of length ℓ and ground. Suppose the line is inhomogeneous, which implies that the Resistant R, inductance L, capacitance C and conductance for unit length (dielectric loss) G are the position-dependent. For a RLCG transmission line system, let $v(x,t)$ and $i(x,t)$ respectively be voltage and current at position $x \in (0,\ell)$ at time t. Using Kirchhoff's laws of the voltage and current, the equivalent circuit equations can be written into

$$\frac{\partial v(x,t)}{\partial x} + L(x)\frac{\partial i(x,t)}{\partial t} = -R(x)i(x,t), \tag{1.1.1}$$

$$\frac{\partial i(x,t)}{\partial x} + C(x)\frac{\partial v(x,t)}{\partial t} = -G(x)v(x,t). \tag{1.1.2}$$

The coefficients R, L, C, and G are distributed parameters for RLCG transmission line. If $R = 0$ and $G = 0$, the transmission line is lossless, see [59] and [60].

The current and voltage at the near end are $i(0,t)$ and $v(0,t)$ respectively, and the current and voltage at the far end are $i(\ell,t)$ and $v(\ell,t)$ respectively.

1.1.2 Complex circuit equations with distributed elements

Let us consider a hybrid system of lump and distributed elements, which have two simple circuits with RLCG transmission lines, see Fig.1.1.1.

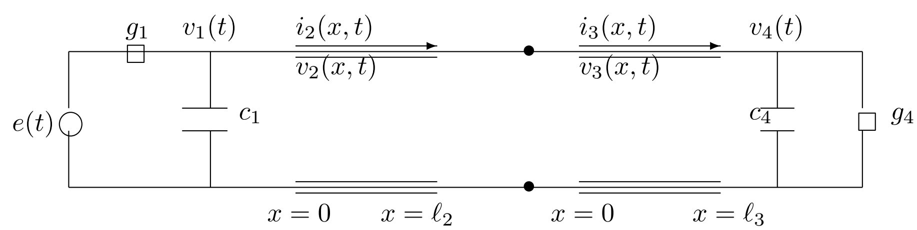

Fig.1.1.1 An electronic circuit network

The behavior of the system is governed by

$$\begin{cases} c_1\frac{dv_1(t)}{dt} = g_1(e_1 - v_1)(t) - \frac{1}{z_0}v_1(t) + \frac{1}{z_0}i(0,t) \\ \frac{\partial v_2(x,t)}{\partial x} + L_2(x)\frac{\partial i_2(x,t)}{\partial t} = -R_2(x)i_2(x,t), \quad x \in (0,\ell_2) \\ \frac{\partial i_2(x,t)}{\partial x} + C_2(x)\frac{\partial v_2(x,t)}{\partial t} = -G_2(x)v_2(x,t), \quad x \in (0,\ell_2) \\ v_2(0,t) = v_1(t), v_2(\ell_2,t) = v_3(0,t) \\ \frac{\partial v_3(x,t)}{\partial x} + L_3(x)\frac{\partial i_3(x,t)}{\partial t} = -R_3(x)i_3(x,t), \quad x \in (0,\ell_3) \\ \frac{\partial i_3(x,t)}{\partial x} + C_3(x)\frac{\partial v_3(x,t)}{\partial t} = -G_3(x)v_3(x,t), \quad x \in (0,\ell_3) \\ v_3(\ell_3,t) = v_4(t) \\ c_4\frac{dv_4(t)}{dt} = -g_4(v_4)(t) + \frac{1}{z_4}v_4(t) + \frac{1}{z_4}i_3(\ell_3,t), \end{cases} \tag{1.1.3}$$

where g_1 and g_4 are nonlinear functions

1.2 Saint Venant network

1.2.1 Saint Venant equations

Dynamics of open-water channels are usually described by Saint-Venant equations which are nonlinear PDEs representing mass and momentum balance along the channel. The Saint-Venant equations constitute a so-called 2×2 system of one-dimensional balance laws.

Let us consider a pool of a prismatic open channel with a rectangular cross section and a constant non-zero slope. Let $H(t, x)$ denote the water depth at position x of channel at time t and $V(t, x)$ denote the horizontal water velocity at the time instant t and the location x along the channel, L be the length of the pool. The dynamics of the system are described by the Saint-Venant equations

$$\frac{\partial}{\partial t}\begin{pmatrix} H \\ V \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} HV \\ \frac{1}{2}V^2 + gH \end{pmatrix} + \begin{pmatrix} 0 \\ g[S_f(H,V) - S_b] \end{pmatrix} = 0, \quad x \in (0, L), \tag{1.2.1}$$

where S_b is the bottom slope and g the gravity constant. $S_f(H, V)$ is the so-called friction slope for which various empirical models are available in the engineering literature. The simplest model is

$$S_f(H,V) = C\frac{V^2}{H} \tag{1.2.2}$$

where C is a constant friction coefficient. In this case, the steady-state (or equilibrium) of (1.2.1) is a constant state $(H^*, V^*)^T$ that satisfies the relation

$$S_f(H^*, V^*) = S_b, \quad \text{or} \quad S_bH^* = C(V^*)^2. \tag{1.2.3}$$

1.2.2 Saint-Venant network

Let us consider a system of navigable rivers or irrigation channels (see e.g. [23] [78]). Under the power of gravity the water is transported along the channel through successive pools separated by automated gates that are used to regulate the water flow, as shown in Fig.1.2.1. Suppose that the channel has n pools, the dynamics are described by Saint-Venant equations

$$\frac{\partial}{\partial t}\begin{pmatrix} H_j \\ V_j \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} H_jV_j \\ \frac{1}{2}V_j^2 + gH_j \end{pmatrix} + \begin{pmatrix} 0 \\ g[C_jV_j^2H_j^{-1} - S_b] \end{pmatrix} = 0, \quad x \in (0, L) \tag{1.2.4}$$

where $j = 1, 2, \cdots, n$.

Further we assume that all the pools have a rectangular section with the same width W. The system (1.2.4) is subject to a set of $2n$ boundary conditions that are distributed into three subsets:

1) The flow continuity condition between the pools: a first subset of $n-1$ conditions expresses the natural physical constraint of flow-rate conservation between the pools (the flow that exits pool j is equal to the flow that enters pool $j+1$)

$$H_j(t, L)V_j(t, L) = H_{j+1}(t, 0)V_{j+1}(t, 0), \quad j = 1, 2, \cdots, n-1. \tag{1.2.5}$$

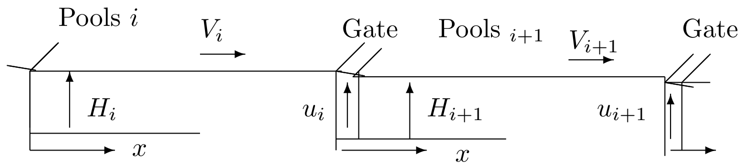

Fig.1.2.1 n-series pools: Lateral view of successive pools of an open-water channel with overflow gates

2) The nodal condition at every gate: a second subset of n boundary condition is made up of the equations that describe the gate operations. A standard gate model is given by the algebraic relation

$$H_j(t,L)V_j(t,L) = kG\sqrt{[H_j(t,L) - u_j(t)]^3}, \quad j = 1,2,\cdots,n. \tag{1.2.6}$$

where kG is a positive constant coefficient and $u_j(t)$ denotes the weir elevation which is a control input (see Fig.1.2.1.).

3) The last boundary condition imposes the value of the canal inflow rate that denotes by $Q_0(t)$

$$H_1(t,0)V_1(t,0) = Q_0(t). \tag{1.2.7}$$

Depending on the application, $Q_0(t)$ may be viewed as a control input (in irrigation channels) or as a disturbance input (in navigable rivers).

A steady-state (or equilibrium) is a constant state $(H_j^*, V_j^*)(j = 1,2,\cdots,n)$ that satisfies the relations

$$S_j H_j^* = C_j (V_j^*)^2, \quad j = 1,2,\cdots,n. \tag{1.2.8}$$

The subcritical flow condition is

$$gH_j^* - (V_j^*)^2 > 0, \quad j = 1,2,\cdots,n. \tag{1.2.9}$$

REMARK 1.2.1 *This model is taken from [14], in which the authors concerned with the exponential stability (in L^2-norm) of the classical solutions of the linearized Saint-Venant equations. The stability of systems of one-dimensional conservation laws has been analyzed for a long time in the literature. The most recent results can be found in [24] where it is shown that the stability is guaranteed if the Jacobian matrix of the boundary conditions satisfies an appropriate sufficient dissipative condition.*

1.3 Ramp-metering modelling in road traffic networks

1.3.1 The LWR Model

In the fluid paradigm for road traffic modelling, the traffic state is usually represented by a macroscopic variable $\rho(t,x)$ which represents the density of the vehicles (# veh/km) at time t and at position x along the road. $q(t,x)$ is the traffic flux representing the flow rate of the vehicles at (t,x). By the definition, one has $q(t,x) = \rho(t,x)v(t,x)$ where $v(t,x)$ is the velocity of the vehicles at (t,x). Then the traffic dynamics are represented by a conservation law

$$\partial_t \rho(t,x) + \partial_x q(t,x) = 0 \tag{1.3.1}$$

this expresses the conservation of the number of vehicles on a road segment without entries or exits. The basic assumption of the so-called LWR model (see e.g. [50, Chapter 3]) is that the drivers instantaneously adapt their speed to the local traffic density, which is expressed by a function $v(t,x) = V(\rho(t,x))$. The LWR traffic model is therefore written as

$$\partial_t \rho(t,x) + \partial_x(\rho(t,x)V(\rho(t,x))) = 0. \tag{1.3.2}$$

According to the physical observations, the velocity-density relation is a monotonic decreasing function ($dV/d\rho < 0$) on the interval $[0, \rho_m]$ with properties that

1) $V(0) = V_m$, the maximal vehicle velocity when the road is empty;

2) $V(\rho_m) = 0$, the velocity is zero when the density is maximal, the vehicles are stopped and the traffic is totally congested.

The flux $q(\rho) = \rho V(\rho)$ is a non-monotonic function with $q(0) = 0$ and $q(\rho_m) = 0$, which is maximal at some critical value ρ_c that separates free-flow and traffic congestion: the traffic is flowing freely when $\rho < \rho_c$ while the traffic is congested when $\rho > \rho_c$.

1.3.2 The LWR network

Ramp-metering Control Problem. Let us now consider the highway network made up of nine road segments with four entries and three exits, whose structure is shown as in Fig. 1.3.1.

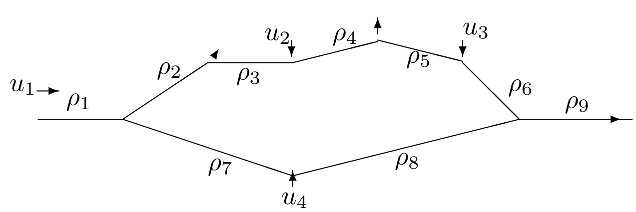

Fig. 1.3.1. A highway network

The densities and flows on the road segments are denoted ρ_j and q_j, $j \in \{1, 2, \cdots, 9\}$. The flow rate u_1 is a disturbance input and the flow rates u_2, u_3, u_4 at the three other entries are inputs.

The traffic dynamics are described by a set of LWR models:

$$\partial_t \rho_j(t,x) + \partial_x(\rho_j(t,x)V(\rho_j(t,x))) = 0, \quad j \in \{1, 2, \cdots, 9\} \tag{1.3.3}$$

Under free-flow conditions, the flows $q_j(\rho_j) = \rho_j V(\rho_j)$ are monotonic increasing functions and hence there are inverse function $\rho_j = S(q_j)$. Thus the model for the network of Fig.1.3.1 can be rewritten into a set of kinematic wave equations

$$\partial_t q_j(t,x) + c(q_j(t,x))\partial_x q_j(t,x) = 0, \quad c(q_j) > 0, \tag{1.3.4}$$

with the boundary conditions

$$\begin{cases} q_1(t,0) = u_1(t), & q_2(t,0) = \alpha q_1(t,L), \\ q_3(t,0) = \beta q_2(t,L), & q_4(t,0) = q_3(t,L) + u_2(t), \\ q_5(t,0) = \gamma q_4(t,L), & q_6(t,0) = q_5(t,L) + u_3(t), \\ q_7(t,0) = (1-\alpha)q_1(t,L), & q_8(t,0) = q_7(t,L) + u_4(t), \\ q_9(t,0) = q_6(t,L) + q_8(t,L) & \end{cases}$$

where α, β and γ are traffic splitting factors at the diverging junction and two exits of the network.

REMARK 1.3.1 *This example is taken from [13], in where the objective is to analyze the stability of this network under a feedback ramp metering strategy which consists in using traffic lights for modulating the entry flows u_i. The motivation behind such control strategy is that a temporary limitation of the flow entering a highway can prevent the appearance of traffic jams and improve the network efficiency (possibly at the price of temporary queue formation at the ramps).*

1.4 Transport system

Herein we consider a transport problem of population in some region. $V = \{a_1, a_2, \cdots, a_m\}$ denotes a set of large towns in the region under consideration. $E = \{e_1, e_2, \cdots, e_n\}$ denotes a set of the transport lines of population, in which each line e_j is of length ℓ_j. Suppose that there is no birth in the transport process, and that the velocity of transmission along the line e_j is c_j that is position independent. Let us consider change of population in the region.

Let $p_j(s,t)$ denote the density (number) of transport population at time t and position $s \in e_j$. Then the change of transport population along line e_j at a small time Δt is

$$p_j(s + c_j\Delta t, t + \Delta t) - p_j(s,t) = -\mu_j(s)p_j(s,t)\Delta t$$

where $\mu_j(s)$ denotes the death rate (mortality) of the transport population. Then the dynamical equation is given by

$$\frac{\partial p_j(s,t)}{\partial t} + c_j\frac{\partial p_j(s,t)}{\partial s} = -\mu_j(s)p_j(s,t). \tag{1.4.1}$$

Suppose that $w(a_i, t)$ is the total number of population at time t in town a_i, then the change of population in this town is

$$\frac{dw(a_i,t)}{dt} = -\mu(a_i)w(a_i,t) + b(a_i)w(a_i,t) + \sum_{j\in J^+(a_i)} p_i(\ell_j,t) - \sum_{k\in J^-(a_i)} p_k(0,t) \tag{1.4.2}$$

where $b(a_i)$ is the birth rate of population in the town a_i, which includes the birth rate and available birth age. $\mu(a_i)$ is the death rate. $J^+(a_i)$ is a index set of the line entering the town a_i, and $J^-(a_i)$ is a index set of the line leaving the town.

Note that each town has its environment capability. One can assume that the environment capability for population is D_i in town a_i. If $w(a_i,t) < D_i$, the transport population may be admissible to settle down, otherwise there are some population joining transport. Therefore, the outgoing population of $p_k(0,t)$,$k \in J^-(a_i)$, is described by

$$p_k(0,t) = \begin{cases} w^{(1)}_{i,k} w(a_i,t), & w(a_i,t) < D_i \\ w^{(2)}_{i,k} w(a_i,t), & w(a_i,t) \geq D_i. \end{cases} \tag{1.4.3}$$

where $w^{(j)}_{i,k}, j = 1,2$ are rate coefficients.

EXAMPLE 1.4.1 *Let G be of the structure shown as in Fig. 1.4.1. The transport process takes place along the edge of G. a_2, a_4 and a_5 are the center cities.*

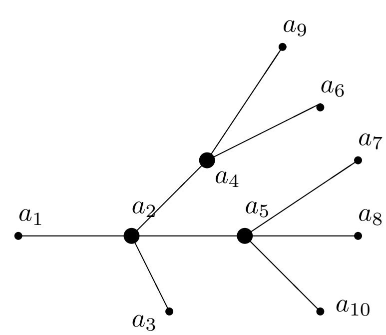

Fig. 1.4.1 transport tree with center cities a_2, a_4 and a_5

1.5 Elastic grid

1.5.1 String equation with tip mass

Let a string of length ℓ be homogeneous with density m and tension T. Suppose that the string is fixed at one end and attached a tip mass M at another end. Let $w(x,t)$ denote the displacement of the string at position x and at time t depart from its equilibrium position and $v(t)$ denote the displacement of the mass. Obviously, it holds that $w(\ell,t) = v(t)$ due to the mass M has same displacement as that of the endpoint of the string. The Newton's law says $Mv''(t) = -Tw_x(\ell, t)$. Thus the motion of the hybrid system is governed by the partial differential equation and ordinary differential equation

$$\begin{cases} mw_{tt}(x,t) = Tw_{xx}(x,t), \quad x \in (0,\ell) \\ w(0,t) = 0, \quad w(\ell,t) = v(t) \\ Mv''(t) = -Tw_x(1,t) \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \\ v'(0) = v_0, \end{cases} \tag{1.5.1}$$

1.5.2 Elastic gird

Let us consider an elastic grid whose structure is shown as Fig.1.5.1, where $\bullet$ denote the elastic vibrator with mass M, different vibrator may have different mass, and the $\rceil$ denote the elastic support.

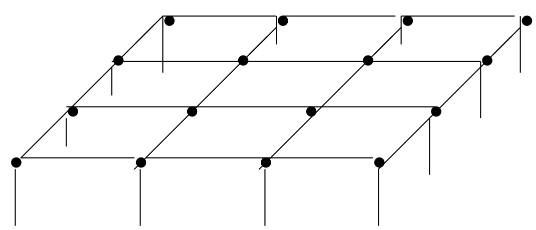

Fig.1.5.1. An elastic grid

Let $V = \{a_1, a_2, \cdots, a_{16}\}$ denote the set of vertices. At each vertex $a_j \in V$, the vibrator has mass $M_j = M(a_j)$. The vibrators M_i and M_j are connected by a nonuniform string e_k. Suppose that $w_k(x,t)$ denote the displacement of the string e_k depart from its equilibrium position, which satisfies the wave equation, i.e.,

$$m_k(s)w_{k,tt}(s,t) = (T_k(s)w_{k,s}(s,t))_s - q_k(s)w_k(s,t), \quad s \in (0,\ell) \tag{1.5.2}$$

where $m_j(s)$ is the mass density and $T_j(s)$ is the tension, they are positive continuous differential functions, and $q_j(s)$ are nonnegative functions (or called potentials).

For node $a_j \in V$, let $v_j(t) = v(a_j,t)$ denote the displacement of elastic vibrator M_j, then the strings jointed with M_j have property

$$v(a_j,t) = w_k(\ell,t) = w_i(0,t), \quad k \in J^+(a_j), \quad i \in J^-(a_j) \tag{1.5.3}$$

where $J^+(a_j)$ denotes the index set of the strings with $x = \ell$ end jointed M_j and $J^-(a_j)$ denotes the index set of the strings with $x = 0$ end jointed M_j.

At the interior node a_j, the dynamic behavior of the vibrator is governed by the differential equation

$$M(a_j)v_{j,tt}(t) + \left[\sum_{k\in J^+(a_j)} T_k(1)w_{k,s}(\ell,t) - \sum_{i\in J^-(a_j)} T_i(0)w_{i,s}(0,t)\right] = 0. \tag{1.5.4}$$

At the exterior node a_j, the network has an elastic support, the motion of vibrator is described by

$$M(a_j)v_{j,tt}(t) + \left[\sum_{k\in J^+(a_j)} T_k(1)w_{j,s}(\ell,t) - \sum_{i\in J^-(a_j)} T_i(0)w_{i,s}(0,t)\right] + k(a_j)v_j(t) = 0 \quad (1.5.5)$$

where $k(a_j)$ are the Hooke's law constants. The energy function of the system is defined by

$$\begin{aligned}\mathcal{E}(t) \quad &= \quad \frac{1}{2}\sum_{j=1}^{n}\int_{e_j}[T_j(s)|w_{j,s}(s,t)|^2 + q_j(s)|w_j(s,t)|^2]ds + \frac{1}{2}\sum_{j=1}^{n}\int_{e_j} m_j(s)|w_{j,t}(s,t)|^2 ds \\ &\quad + \frac{1}{2}\sum_{a\in V} M(a)|v_t(a,t)|^2 + \frac{1}{2}\sum_{a\in\partial G} k(a)|v(a,t)|^2.\end{aligned}$$

This is a conservation system.

1.6 Modeling elastic system

In this section we model an elastic structure, whose motion retains in a plane.

1.6.1 Elastic network

R is an elastic structure made of n members, i.e., $R = \bigcup_{k=1}^{n} R^k$, where $R^k(x,t)$ denotes centric axis of the k-th member in space position x at time t, which is a vector-valued function on $x \in [0, \ell_k]$ where ℓ_k is its nature length. Notations $R^k_x(x,t)$ and $R^k_t(x,t)$ denote the partial differential with respect to x and t, respectively. In particular, when $x = 0$ or ℓ_k, $R^k_x(x,t)$ denotes the single side partial differential.

Let V be a set composed of all nodes of R. For $a \in V$, the index set $J(a)$ is defined as

$$J(a) = \{i; a \text{ is an endpoint of } R^i\}.$$

If $\#J(a) = 1$, the number of members in $J(a)$, then a is called a simple node, otherwise a is said to be a multiple node. Clearly, $\sum_{a\in V}\#J(a) = 2n$. For each $k \in J(a)$, $\varepsilon_k(a) = -1$ or 1 denote the node a to be the initial node ($x = 0$) or terminal node ($x = \ell_k$) of R^k, respectively.

Let $\rho(x)$ be the mass density distribution of the elastic structure. Suppose that the elastic structure always retains its motion in a plane. Then the total mechanical energy (including the kinetic energy, the elastic energy and the potential energy) of the elastic structure is

$$\mathcal{L}(R) = \frac{1}{2}\sum_{i=1}^{n}\int_0^{\ell_i}\left[\rho_i|R^i_t|^2 + h_i(|R^i_x| - 1)^2 + EI_i\kappa_i^2\right]dx \quad (1.6.1)$$

where h_i are the Hooke's law constants, EI_i are physical constants and κ_i is the curvature of the i-th member:

$$\kappa_i = \frac{(|R^i_x|^2|R^i_{xx}|^2 - (R^i_x \cdot R^i_{xx})^2)^{\frac{1}{2}}}{|R^i_x|^3}. \quad (1.6.2)$$

Let R have an equilibrium position which coincides with a planar graph $G = (V, E)$ whose vertices $V = \{a_1, a_2, \cdots, a_N\}$ and edges $E = \{e_1, e_2, \cdots e_n\}$. For each segment e_i, it also is used to represent the unit vector whose tail corresponds to $x = 0$. In such an equilibrium in which each element is straight (a line) and hence curvature is 0, the i-th member is of the form

$$R_0^i + xe_i, \quad x \in (0, \ell_i)$$

where R_0^i is a node position associated with some a_j.

We consider a small vibration of the elastic structure near its equilibrium position. Denote by $R^i(x,t)$ the new position of the i-th member which always retains in the same plane with e_i, then it has the form

$$R^i(x,t) = R_0^i + xe_i + r^i(x,t)$$

where $r^i(x,t)$ is the deformation of the i-th member. Thus we have

$$R_x^i = e_i + r_x^i, \quad R_{xx}^i = r_{xx}^i$$

and

$$\begin{aligned} h_i(|R_x^i| - 1)^2 &= h_i(|R_x^i|^2 - 2|R_x^i| + 1) = h_i(R_x^i \cdot R_x^i + 2\sqrt{R_x^i \cdot R_x^i} + 1) \\ &= h_i\left(1 + 2r_x^i \cdot e_i + r_x^i \cdot r_x^i + 1 - 2\sqrt{1 + 2r_x^i \cdot e_i + r_x^i \cdot r_x^i}\right). \end{aligned}$$

Applying the asymptotic expansion

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{1}{8}x^2 + o(x^2)$$

we get

$$\begin{aligned} & 2\sqrt{1 + 2r_x^i \cdot e_i + r_x^i \cdot r_x^i} = 2\left[1 + \frac{1}{2}\left(2r_x^i \cdot e_i + r_x^i \cdot r_x^i\right) - \frac{1}{8}\left(2r_x^i \cdot e_i + r_x^i \cdot r_x^i\right)^2 + o(|r_x^i|^2)\right] \\ = \; & 2 + 2r_x^i \cdot e_i + r_x^i \cdot r_x^i - (r_x^i \cdot e_i)^2 + o(|r_x^i|^2). \end{aligned}$$

So it holds that

$$\begin{aligned} h_i(|R_x^i| - 1)^2 &= h_i\left(1 + 2r_x^i \cdot e_i + r_x^i \cdot r_x^i + 1 - 2\sqrt{1 + 2r_x^i \cdot e_i + r_x^i \cdot r_x^i}\right) \\ &= h_i(r_x^i \cdot e_i)^2 + o(|r_x^i|^2). \end{aligned}$$

Since

$$\begin{aligned} \kappa_i^2 &= \frac{|R_x^i|^2|R_{xx}^i|^2 - (R_x^i \cdot R_{xx}^i)^2}{|R_x^i|^6} \\ &= \frac{(1 + 2r_x^i \cdot e_i + r_x^i \cdot r_x^i)(r_{xx}^i \cdot r_{xx}^i) - (r_{xx}^i \cdot e_i + r_x^i \cdot r_{xx}^i)^2}{|R_x^i|^6} \\ &= \frac{1}{|R_x^i|^6}\left[[(r_{xx}^i \cdot r_{xx}^i) - (r_{xx}^i \cdot e_i)^2] + 2[(r_x^i \cdot e_i)(r_{xx}^i \cdot r_{xx}^i) - (r_{xx}^i \cdot e_i)(r_x^i \cdot r_{xx}^i)]\right] \\ &\quad + \frac{(r_x^i \cdot r_x^i)(r_{xx}^i \cdot r_{xx}^i) - (r_x^i \cdot r_{xx}^i)^2}{|R_x^i|^6} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{|R_x^i|^6} - 1 &= \frac{1 - |R_x^i|^6}{|R_x^i|^6} \\ &= \frac{1}{|R_x^i|^6}\left[1 - (1 + 2r_x^i \cdot e_i + r_x^i \cdot r_x^i)^3\right] \\ &= -\frac{2r_x^i \cdot e_i + r_x^i \cdot r_x^i}{|R_x^i|^6}\left[1 + (1 + 2r_x^i \cdot e_i + r_x^i \cdot r_x^i) + (1 + 2r_x^i \cdot e_i + r_x^i \cdot r_x^i)^2\right], \end{aligned}$$

so we have

$$\kappa_i^2 = \frac{[(r^i_{xx}\cdot r^i_{xx})-(r^i_{xx}\cdot e_i)^2]+2[(r^i_x\cdot e_i)(r^i_{xx}\cdot r^i_{xx})-(r^i_{xx}\cdot e_i)(r^i_x\cdot r^i_{xx})]+[(r^i_x\cdot r^i_x)(r^i_{xx}\cdot r^i_{xx})-(r^i_x\cdot r^i_{xx})^2)]}{|R^i_x|^6}$$

Substituting above into the energy function (1.6.1) and retaining only quadratic terms yield an asymptotic formula

$$\mathcal{L}_q(\{r^i\}) = \frac{1}{2}\sum_{i=1}^{n}\int_0^{\ell_i}[\rho_i|r^i_t|^2 + H_i r^i_x\cdot r^i_x + K_i r^i_{xx}\cdot r^i_{xx}]dx \tag{1.6.3}$$

where

$$H_i = h_i e_i e_i^T, \qquad K_i = EI_i\left[I - e_i e_i^T\right] \tag{1.6.4}$$

are symmetric matrices, e_i^T denotes the transpose of the column vector e_i. This is a linearization form of nonlinear system R, which expresses the movement of the structure relative to its equilibrium position.

Since $r^i(x,t)$ is the relative displacement of the i-th member, the energy function of the relative movement of the system is

$$E(t) = \frac{1}{2}\sum_{i=1}^{n}\int_0^{\ell_i}[\rho_i r^i_t\cdot r^i_t + H_i r^i_x\cdot r^i_x + K_i r^i_{xx}\cdot r^i_{xx}]dx. \tag{1.6.5}$$

We will deduce the dynamic equations of relative movement of the system by use method of the energy function similar to [96]. With time development the energy change of the system is

$$\begin{aligned}
\frac{dE(t)}{dt} &= \sum_{i=1}^{n}\int_0^{\ell_i}[\rho_i r^i_{tt}\cdot r^i_t + H_i r^i_{tx}\cdot r^i_x + K_i r^i_{txx}\cdot r^i_{xx}]dx\\
&= \sum_{i=1}^{n}\int_0^{\ell_i}[\rho_i r^i_t\cdot r^i_{tt} - H_i r^i_t\cdot r^i_{xx} + K_i r^i_t\cdot r^i_{xxxx}]dx\\
&\quad + \sum_{i=1}^{n}\left\{H_i r^i_t\cdot r^i_x\Big|_0^{\ell_i} + K_i r^i_{tx}\cdot r^i_{xx}\Big|_0^{\ell_i} - K_i r^i_t\cdot r^i_{xxx}\Big|_0^{\ell_i}\right\}\\
&= \sum_{i=1}^{n}\int_0^{\ell_i} r^i_t\cdot[\rho_i r^i_{tt} - H_i r^i_{xx} + K_i r^i_{xxxx}]dx\\
&\quad + \sum_{i=1}^{n} r^i_t\cdot[H_i r^i_x - K_i r^i_{xxx}]\Big|_0^{\ell_i} + \sum_{i=1}^{n} K_i r^i_{tx}\cdot r^i_{xx}\Big|_0^{\ell_i}\\
&= \sum_{i=1}^{n}\int_0^{\ell_i} r^i_t\cdot[\rho_i r^i_{tt} - H_i r^i_{xx} + K_i r^i_{xxxx}]dx\\
&\quad + \sum_{a\in V}\sum_{i\in J(a)} r^i_t(a)\cdot\varepsilon_i(a)[H_i r^i_x(a) - K_i r^i_{xxx}(a)] + \sum_{a\in V}\sum_{i\in J(a)} r^i_{tx}(a)\cdot\varepsilon_i(a)[K_i r^i_{xx}(a)].
\end{aligned}$$

Clearly, there is no external force acting on the system, the energy of the elastic system is conservation, i.e., $\frac{dE(t)}{dt} = 0$. Therefore the motion of each member of the system is governed by the partial differential equation:

$$\rho_i r^i_{tt} = H_i r^i_{xx} - K_i r^i_{xxxx}, \quad x\in(0,\ell_i). \tag{1.6.6}$$

When the geometric structure of the system continues at each multiple node a, the relative displacements satisfy condition

$$r^i(a,t) = r^j(a,t), \quad \forall i,j \in J(a) \tag{1.6.7}$$

which means that the displacement of the structure at node a is continuous. Hence the associated dynamic condition is

$$\sum_{i\in J(a)} \varepsilon_i(a)[H_i r^i_x - K_i r^i_{xxx}](a,t) = 0 \tag{1.6.8}$$

that means the forces balance of the system at a.

Assume that the structure satisfies the prefect geometric condition, i.e.,

$$r^i_x(a,t) = r^j_x(a,t), \quad \forall i,j \in J(a), \tag{1.6.9}$$

which means the each member has same rotation at node a during the whole process, then the corresponding dynamic condition is given by

$$\sum_{i\in J(a)} \varepsilon_i(a) K_i r^i_{xx}(a,t) = 0, \tag{1.6.10}$$

that shows the moment balance of the system at a.

Similarly, when the geometry structure of the system has a gap at node a, but it satisfies condition:

$$\sum_{i\in J(a)} r^i(a,t) = 0, \tag{1.6.11}$$

this implies that the structure is not continuous at node a, then the associated dynamic condition is

$$\varepsilon_i(a)[H_i r^i_x(a,t) - K_i r^i_{xxx}(a,t)] = \varepsilon_j(a)[H_j r^j_x(a,t) - K_j r^j_{xxx}(a,t)], \quad \forall i,j \in J(a) \tag{1.6.12}$$

which means that the forces of the structure at node a remain continuity. In addition, the geometric structure of the system satisfies condition

$$\sum_{i\in J(a)} \varepsilon_i(a) r^i_x(a,t) = 0, \quad (or \sum_{i\in J(a)} r^i_x(a,t) = 0) \tag{1.6.13}$$

then corresponding dynamic condition is given by

$$K_i r^i_{xx}(a,t) = K_j r^j_{xx}(a,t), \quad (or\ \varepsilon_i(a) K_i r^i_{xx}(a,t) = \varepsilon_j(a) K_j r^j_{xx}(a,t)) \quad \forall i,j \in J(a), \tag{1.6.14}$$

that shows the moment continuity.

REMARK 1.6.1 *The equality (1.6.11) can be explained as the flow balance condition. In fact, if $r^i(x,t)$ represents a flow, at the node a we can write it into*

$$\sum_{i\in J(a)} r^i(a,t) = \sum_{i\in J^{in}(a)} r^i(a,t) + \sum_{i\in J^{out}(a)} r^i(a,t) = 0$$

where $J^{in}(a)$ denotes the incoming flows and $J^{out}(a)$ denote the outgoing flows. This is the Kirchhoff's law.

If a is simple node of the system, then the following nodal types are possible:
1) *controlled node*, for instance, it satisfies

$$\begin{cases} r^i(a,t)\cdot e_i = u_i(t), \quad r^i(a,t)\cdot e_i^\perp = 0, \\ r_x^i(a,t)\cdot e_i^\perp = w^i(t) \end{cases} \tag{1.6.15}$$

2) *fixed node*, that means that the structure has neither displacement and nor rotation, i.e.,

$$r^i(a,t) = 0, \quad r_x^i(a,t)\cdot e_i^\perp = 0. \tag{1.6.16}$$

3) *free node*, on which there is no geometric restriction, and hence the dynamic conditions are

$$r_x^i(a,t)\cdot e_i = r_{xx}^i(a,t)\cdot e_i^\perp = r_{xxx}^i(a,t)\cdot e_i^\perp = 0. \tag{1.6.17}$$

Once the geometric structure of the system at nodes are chosen, the associated dynamic condition are then determined uniquely.

REMARK 1.6.2 *As before, we can propose various different joint conditions at a node a, hence it forms different model at the junction.*

1.6.2 Planar motion

Although we have supposed that the elastic structure moves in a fixed plane, we do not appoint which plane the motion retains in. Let graph G be in a plane Π. If the motion of the elastic structure remains in the plane Π, it is called undergoing a planar motion. In this case, the vectors e_i and $e_i^\perp$ are in the same plane Π. Here we consider two types of the structure: inextensible and extensible.

Inextensible structure

Assume that the structure is inextensible and the parameter x is the arc length of member e_i. Then the position function of the member e_i associated arc x at time t is

$$R^i(x,t) = R_0^i + xe_i + u^i(x,t)e_i + w^i(x,t)e_i^\perp, \quad r^i(x,t) = u^i(x,t)e_i + w^i(x,t)e_i^\perp.$$

In this case, the continuity condition of the structure R at the node a is

$$r^i(a,t) = r^j(a,t), \quad \forall i,j \in J(a).$$

For instance, $r^i(\ell_i,t) = r^j(0,t)$, i.e.,

$$u^i(\ell_i,t)e_i + w^i(\ell_i,t)e_i^\perp = u^j(0,t)e_j + w^j(0,t)e_j^\perp.$$

Both sides of the above equality are the different representation of the same point in distinct local coordinate. This relationship also shows that $u^j(0,t)$ (corresponding $w^j(0,t)$) depends upon both $u^i(\ell_i,t)$ and $w^i(\ell_i,t)$. In this case, the function values $u^i(\ell_i,t)$ and $u^j(0,t)$ are not equal. Similarly, so are functions w^i and w^j. Whatever $r(x,t)$ always is continuous, so the function (u^i,w^i) must appear as a coupled pair.

Now we deduce the differential equations satisfied by u^i and w^i. Since $r^i(x,t) = u^i(x,t)e_i + w^i(x,t)e_i^\perp$, so

$$H_i r^i_{xx} = h_i e_i e_i^\tau r^i_{xx} = h_i(r^i_{xx}, e_i)e_i = h_i u^i_{xx}(x,t)e_i$$

and

$$K_i r^i_{xxxx} = EI_i(I - e_i e_i^\tau)r^i_{xxxx} = EI_i r^i_{xxxx} - EI_i(r^i_{xxxx}, e_i)e_i = EI_i w^i_{xxxx} e_i^\perp.$$

Therefore, the vector-valued equation

$$\rho_i r^i_{tt} = H_i r^i_{xx} - K_i r^i_{xxxx}$$

is equivalent to the following scale equations

$$\begin{cases} \rho_i u^i_{tt}(x,t) = h_i u^i_{xx}(x,t), \quad x \in (0,\ell_i) \\ \rho_i w^i_{tt}(x,t) = -EI_i w^i_{xxxx}(x,t), \quad x \in (0,\ell_i) \end{cases} \tag{1.6.18}$$

Define the mappings by

$$\pi_j : [0,\ell_j] \to e_j, \quad j = 1,2,\cdots,N$$

it is called the parametrization realization of edge e_j. For $a \in V$, denote $\pi_i^{-1}(a) = 0$, or ℓ_i if $i \in J(a)$. Set $e_i = (\cos\alpha_i, \sin\alpha_i)$, then $e_i^\perp = (-\sin\alpha_i, \cos\alpha_i)$. Thus the geometric continuity condition (1.6.7) and the dynamic conditions (1.6.8) are respectively

$$\begin{cases} u^i(\pi_i^{-1}(a),t)\cos\alpha_i - w^i(\pi_i^{-1}(a),t)\sin\alpha_i = u^j(\pi_j^{-1}(a),t)\cos\alpha_j - w^j(\pi_j^{-1}(a),t)\sin\alpha_j, \\ u^i(\pi_i^{-1}(a),t)\sin\alpha_i + w^i(\pi_i^{-1}(a),t)\cos\alpha_i = u^j(\pi_j^{-1}(a),t)\sin\alpha_j + w^j(\pi_j^{-1}(a),t)\cos\alpha_j, \\ \forall i,j \in J(a), \end{cases} \tag{1.6.19}$$

and

$$\begin{cases} \sum\limits_{i\in J(a)} \varepsilon_i(a)[h_i u^i_x(\pi_i^{-1}(a),t)\cos\alpha_i + EI_i w^i_{xxx}(\pi_i^{-1}(a),t)\sin\alpha_i] = 0, \\ \sum\limits_{i\in J(a)} \varepsilon_i(a)[h_i u^i_x(\pi_i^{-1}(a),t)\sin\alpha_i - EI_i w^i_{xxx}(\pi_i^{-1}(a),t)\cos\alpha_i] = 0. \end{cases} \tag{1.6.20}$$

The geometric condition (1.6.9) and dynamic condition (1.6.10) become

$$\begin{cases} u^i_x(\pi_i^{-1}(a),t)\cos\alpha_i - w^i_x(\pi_i^{-1}(a),t)\sin\alpha_i = u^j_x(\pi_j^{-1}(a),t)\cos\alpha_j - w^j_x(\pi_j^{-1}(a),t)\sin\alpha_j, \\ u^i_x(\pi_i^{-1}(a),t)\sin\alpha_i + w^i_x(\pi_i^{-1}(a),t)\cos\alpha_i = u^j_x(\pi_j^{-1}(a),t)\sin\alpha_j + w^j_x(\pi_j^{-1}(a),t)\cos\alpha_j \\ \forall i,j \in J(a). \end{cases} \tag{1.6.21}$$

and

$$\begin{cases} -\sum\limits_{i\in J(a)} \varepsilon_i(a) EI_i w^i_{xx}(\pi_i^{-1}(a),t)\sin\alpha_i = 0, \\ \sum\limits_{i\in J(a)} \varepsilon_i(a) EI_i w^i_{xx}(\pi_i^{-1}(a),t)\cos\alpha_i = 0. \end{cases} \tag{1.6.22}$$

Similarly, we also can write the equations (1.6.11)–(1.6.14) into the scale function form. Although the equations obtained are separated about u and w, the geometric conditions and dynamic conditions are coupled at the multiple node. It is a common property of the multi-link structure. Observe that the coordinate functions u and w may be discontinuous at multiple node. But (u, w) together with the direction are continuous, i.e.,

$$u^i(x,t)\cos\alpha_i - w^i(x,t)\sin\alpha_i, \quad u^i(x,t)\sin\alpha_i + w^i(x,t)\cos\alpha_i$$

are continuous functions. So we can not desire that u and w are independent.

Extensible structure

If the structure is extensible, then x is merely a parameter which is independent of arc length. In this case, the position of component R^i associated x is

$$R^i(x,t) = R_0^i + xe_i + w^i(x,t)e_i^\perp, \quad r^i(x,t) = w^i(x,t)e_i^\perp.$$

It only has a displacement along $e_i^\perp$ direction. At the join-point a, one has

$$w^i(\pi_i^{-1}(a),t) = w^j(\pi_j^{-1}(a),t), \quad i,j \in J(a).$$

So the structure function is continuous with respect to x. Note that the tangent vector at x is

$$R_x^i(x,t) = e_i + w_x^i(x,t)e_i^\perp.$$

Usually, at a multiple node a, they do not satisfy the conditions

$$w_x^i(\pi_i^{-1}(a),t) = w_x^j(\pi_j^{-1}(a),t), \quad \forall i,j \in J(a).$$

More practice condition is that there exists a constant group $\{\gamma_i(a), i \in J(a)\}$ which depend upon both the geometric structure and connecting type of the structure such that

$$\sum_{i\in J(a)} \gamma_i(a) R_x^i(a,t) = 0.$$

For example, there is a group numbers $\{\gamma_i(a)\}$ such that $\sum_{i\in J(a)}\gamma_i(a)e_i = 0$. Then the corresponding dynamic condition will become

$$\gamma_i^{-1} w_{xx}^i(\pi_i^{-1}(a),t) = \gamma_j^{-1} w_{xx}^j(\pi_j^{-1}(a),t), \quad \forall i,j \in J(a).$$

In this case, the equations of the system are given by

$$\begin{cases} \rho_i w_{tt}^i(x,t) = -EI_i w_{xxxx}^i(x,t), \quad x \in (0,\ell_i) \\ w^i(\pi_i^{-1}(a),t) = w^j(\pi_j^{-1}(a),t), \quad \forall i,j \in J(a), \\ \sum_{i\in J(a)} \gamma_i(a) w_x^i(\pi_i^{-1}(a),t) = 0, \\ \gamma_i^{-1}(a) EI_i w_{xx}^i(\pi_i^{-1}(a),t) = \gamma_j^{-1}(a) EI_j w_{xx}^j(\pi_j^{-1}(a),t), \quad \forall i,j \in J(a) \\ \sum_{i\in J(a)} EI_i w_{xxx}^i(\pi_i^{-1}(a),t) = 0. \end{cases} \tag{1.6.23}$$

Also we can attach certain dynamic conditions and geometric conditions at the simple nodes.

REMARK 1.6.3 *In the extensible structure one does not see much more structure condition. One can understand the structure to be* $\Gamma = \{(x,w) \mid x \in G\}$.

1.6.3 Vertical motion

Here one considers another type of planar motion of the structure. Let graph G be in a plane Π and $e^{\perp}$ be the unit normal vector of Π. If the motion of each element i in the elastic structure remains in the plane determined by e_i and $e^{\perp}$, one calls it undergoing a vertical movement. For the vertical movement one still divides the structure into two cases: inextensible and extensible. For the inextensible case, its movement includes vertical motion and planar motion in essential, it belongs to the general motion. So we only need to consider the extensible case.

Assume that G is a planar graph, and x is the parameter of the edges, every member e_i of the structure is extensible, the position of the element e_i associated arc x at time t is

$$R^i(x,t) = R_0^i + xe_i + w^i(x,t)e^{\perp}.$$

In this case, the continuity condition of the structure R at the node a is

$$R^i(a,t) = R^j(a,t), \quad \forall i,j \in J(a).$$

It has only a displacement along $e^{\perp}$ direction. So, at an interior node a, one has

$$w^i(\pi_i^{-1}(a),t) = w^j(\pi_j^{-1}(a),t), \quad i,j \in J(a).$$

The corresponding dynamic condition is given by

$$\sum_{i\in J(a)} EI_i w^i_{xxx}(a,t) = 0.$$

Since the tangent vector at x is $R^i_x(x,t) = e_i + w^i_x(x,t)e^{\perp}$, one can require the structure condition at a multiple node a, for instance, ball joint condition, that is, no restriction on rotation of structure. In this case, $w^i_x(\pi_i^{-1}(a),t)$ may be arbitrary, the corresponding dynamic condition is given by

$$EI_i w^i_{xx}(\pi_i^{-1}(a),t) = 0, \quad \forall i \in J(a).$$

Moreover, we impose the fixed node conditions at the simple nodes. Therefore, the motion of the elastic structure is governed by the partial differential equations

$$\begin{cases} \rho_i w^i_{tt}(x,t) = -EI_i w^i_{xxxx}(x,t), \quad x \in (0,\ell_i) \\ w^i(\pi_i^{-1}(a),t) = w^j(\pi_j^{-1}(a),t), \quad \forall i,j \in J(a), \\ EI_i w^i_{xx}(\pi_i^{-1}(a),t) = 0, \quad \forall i \in J(a) \\ \sum\limits_{i\in J(a)} EI_i w^i_{xxx}(\pi_i^{-1}(a),t) = 0 \\ w^i(a,t) = w^i_x(a,t) = 0, \quad i \in J(a), \#J(a) = 1. \end{cases} \tag{1.6.24}$$

with certain initial conditions of the structure.

REMARK 1.6.4 *From discussion above we see that if the structure is extensible, the planar motion and the vertical motion have no difference in the equations involving their node conditions. Such a form is closely related to the differential equation on graphs.*

Now we consider such a case that, at the node a, we divide the rotation angle into two parts: positive and negative. We regard the positive part as incoming flow and negative the outgoing flow. At vertex a_i, the incoming satisfies addition rule, i.e.,

$$w_x(a_i,t) = \sum_{j\in J^{in}(a)} a_{ij}^{+} w_x^j(a_i,t)$$

and the outgoing flow satisfies the transmission rule

$$w_x^j(a_i,t) = a_{ij}^{-} w_x(a_i,t), \quad j \in J^{out}(a_i), \qquad \sum_{j\in J^{out}(a)} a_{ij}^{-} = 1.$$

In this case, the rotation angles at the vertex a_i satisfy

$$\sum_{j\in J(a)} a_{ij} w_x^j(a_i,t) = 0.$$

the corresponding dynamic conditions are

$$w_{xx}(a_i,t) = \sum_{j\in J^{out}(a)} (a_{ij}^{-})^{-1} w_{xx}^j(a_i,t),$$

and

$$w_{xx}^j(a_i,t) = (a_{ij}^{+})^{-1} w_{xx}(a_i,t), \quad j \in J^{in}(a_i).$$

In the above, $w_x(a,t)$ and $w_{xx}(a,t)$ are merely notions, they have no actual meaning.

Suppose that the system has fixed boundary conditions. Then the motion of the system is governed by

$$\begin{cases} \rho_i w_{tt}^i(x,t) = -EI_i w_{xxxx}^i(x,t), \quad x \in (0,\ell_i) \\ w^i(\pi_i^{-1}(a),t) = w^j(\pi_j^{-1}(a),t), \quad \forall i,j \in J(a), \\ w_x(a_i,t) = \sum_{j\in J^{in}(a)} a_{ij}^{+} w_x^j(a_i,t) \\ w_x^j(a_i,t) = a_{ij}^{-} w_x(a_i,t), \quad j \in J^{out}(a_i), \\ w_{xx}(a_i,t) = \sum_{j\in J^{out}(a)} a_{ij}^{-} EI_j w_{xx}^j(a_i,t) \\ w_{xx}^j(a_i,t) = a_{ij}^{+} w_{xx}(a), \quad j \in J^{in}(a) \\ \sum\limits_{i\in J(a)} EI_i w_{xxx}^i(\pi_i^{-1}(a),t) = 0, \\ w^i(a,t) = w_x^i(a,t) = 0, \quad i \in J(a), \#J(a) = 1. \end{cases} \tag{1.6.25}$$

REMARK 1.6.5 *The content of this section comes from research report of the first author, which was completed during visiting the Hong Kong University in 2006. As further works, Xu, Mastorakis and Yung in [122] [123] and [124] discussed the properties of the star-shaped networks of Euler-Bernoulli beams.*

Chapter 2

Graph and Function Defined on Graphs

2.1 Graph theory

2.1.1 Basic notions in graph theory

A graph consists of a set V, a set E and a mapping Φ from E to $V \times V$, denote it by $G = (V, E)$. The elements of V and E are said to be the vertices and edges of the graph respectively, the mapping Φ is called the incidence mapping associated with the graph. If V and E are both finite sets , G is called a finite graph. Otherwise, it is said to be infinite.

Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic to each other if there exists a 1-1 correspondences between V_1 and V_2 and E_1 and E_2 which preserve incidences. If G is isomorphic to a geometric graph $G' \subset \mathbb{R}^n$, then G' is said to be a geometric realization of G. A graph is said to be planar if and only if it has a geometric realization in $\mathbb{R}^2$.

If $e \in E$, $\Phi : e \sim v\&w$, v and w are called the endpoints of e. If $e \sim v\&v$, i.e., v is a sole endpoint of e, then e is said to be a loop. If $e_1 \sim v\&w$ and $e_2 \sim v\&w$, then e_1 and e_2 are called parallel edges. A graph G is said to be simple if it has neither loop and nor parallel edges. A graph G is said to be connected if for every pair of distinct vertices there exits a sequence of edges that join these vertices.

A graph G is said to be a directed graph if $\Phi(e) = (v, w)$ is a ordered pair, the edge e is said to be a directed edge, and v is called the starting vertex (or tail) and w the final vertex (or head). If $v \in V$, there is no edge connected it, then v is said to be an isolated vertex.

Let G be a geometric graph and each $e \in E$ have finite arc length ℓ_e, denote $|e| = \ell_e$. For $e \sim v\&w$, one can parameterize it by its arc length, i.e., $x_e(s) \in e, s \in (0, \ell_e)$ with $x_e(0) = v$, $x_e(\ell_e) = w$, or, $x_e(0) = w, x_e(\ell_e) = v$. With this parameterization, the graph is called a metric graph. If e is a directed edge, then the direction of edge coincides with the parameter increasing. In this manner, the directed edge $\Phi(e) = (v, w)$ always denotes $x_e(s) \in e$, $x_e(0) = v$, $x_e(\ell_e) = w$

Let G be a metric graph with the vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and the edge set $E = \{e_1, e_2, \cdots, e_n\}$. One defines the outgoing incidence matrix (outgoing mapping) Φ^- by

$$\phi_{ij}^- = \begin{cases} 1, & \text{if } x_j(0) = a_i \\ 0, & \text{otherwise} \end{cases} \tag{2.1.1}$$

and the incoming incidence matrix (incoming mapping) Φ^+ by

$$\phi_{ij}^+ = \begin{cases} 1, & \text{if } x_j(\ell_j) = a_i \\ 0, & \text{otherwise.} \end{cases} \tag{2.1.2}$$

Obviously, $\Phi^+ = (\phi_{ij}^+)$ and $\Phi^- = (\phi_{ij}^-)$ are $m \times n$ matrices, they have exactly one nonzero entry in each column if G has no isolated vertex. In addition, they have the following properties.

PROPOSITION 2.1.1 *Let G be a directed graph, then the incoming and outgoing incidence matrices have the following properties:*

1) $\sum_{k=1}^{n} \phi_{ik}^-$ is the number of outgoing edges at a_i;

2) $\sum_{k=1}^{n} \phi_{ik}^+$ is the number of incoming edges at a_i;

3) for each $k \neq j$, $\sum_{i=1}^{n} \phi_{ki}^{\pm}\phi_{ji}^{\pm} = 0$;

4) for each k, $\sum_{i=1}^{n} \phi_{ki}^{+}\phi_{ki}^{-}$ is the number of loop at vertex a_k;

5) for any $k \neq j$, $\sum_{i=1}^{n} \phi_{ki}^{-}\phi_{ji}^{+}$ is the number of parallel edges starting at a_k and ending a_j .

Let G be a directed graph without loop. Then the relation between the vertices and the edges has a matrix representation

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_n \end{array} \\ \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \phi_{23} & \cdots & \phi_{2n} \\ \phi_{31} & \phi_{32} & \phi_{33} & \cdots & \phi_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{m1} & \phi_{m2} & \phi_{m3} & \cdots & \phi_{mn} \end{pmatrix} \end{array}, \tag{2.1.3}$$

denote it by $\Phi = (\phi_{ij})_{m\times n} = \Phi^+ - \Phi^-$, and call it the incidence matrix of G.

REMARK 2.1.1 *When G has a loop, it can not be represented by the incidence matrix.*

Let G be a graph with vertex set V and edge set E. For each $a \in V$, denote by $J(a)$ the index set of edges having incident at vertex a. $\#J(a)$ denotes the number of elements in $J(a)$, it is called the degree of a. If $a \in V$ is an isolated vertex, then $\#J(a) = 0$.

Suppose that G has no isolated vertex, then for each $a \in V$, it holds that $\#J(a) > 0$. One can classify the vertex of V in following manner:

1) a subset of V consists of all vertices satisfying $\#J(a) = 1$, denote by ∂G. ∂G is called the boundary (exterior vertices) of G;

2) a subset of V consists of all vertices satisfying $\#J(a) > 1$, denoted by V_{int}. V_{int} is called the node (interior vertex) set of G.

In the sequel, we consider only the graph G satisfying $\#J(a) > 0$ for all $a \in V$, which means G has no isolated vertex. Hence, it holds that $V = V_{int} \cup \partial G$.

Let G be a directed graph with $V = \{a_1, a_2, \cdots, a_m\}$ and $E = \{e_1, e_2, \cdots, e_n\}$. For each vertex $a \in V$, denote by $J^+(a)$ the index set of the incoming edges, i.e., $j \in J^+(a)$ if there exists an edge e_j such that a is its final vertex (or head). Similarly, denote by $J^-(a)$ the index set of the outgoing edges, $j \in J^-(a)$ if there exists an edge e_j such that a is its starting point(or tail). Then one has $J(a) = J^+(a) \cup J^-(a)$.

EXAMPLE 2.1.1 *Let G be a planar directed graph, whose structure be shown in Fig. 2.1.1*

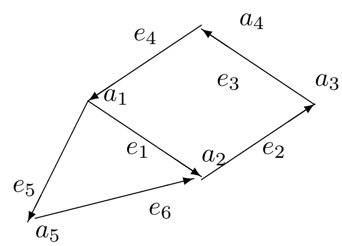

Fig. 2.1.1 A directed graph without boundary

The directed edges are defined by

$$\Phi(e_1) = (a_1, a_2) \quad \Phi(e_2) = (a_2, a_3) \quad \Phi(e_3) = (a_3, a_4)$$
$$\Phi(e_4) = (a_4, a_1) \quad \Phi(e_5) = (a_1, a_5) \quad \Phi(e_6) = (a_5, a_2)$$

The incidence matrix Φ is

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

The index sets of the edges at vertices are

$$J(a_1) = \{4, 1, 5\} \quad J(a_2) = \{1, 6, 2\} \quad J(a_3) = \{2, 3\}$$
$$J(a_4) = \{3, 4\} \qquad J(a_5) = \{5, 6\}$$

and the index sets of incoming and outgoing edges are

$$J^+(a_1) = \{4\} \quad J^-(a_1) = \{1, 5\} \quad J^+(a_2) = \{1, 6\} \quad J^-(a_2) = \{2\}$$
$$J^+(a_3) = \{2\} \quad J^-(a_3) = \{3\} \quad J^+(a_4) = \{3\} \quad J^-(a_4) = \{4\}$$
$$J^+(a_5) = \{5\} \quad J^-(a_5) = \{6\}$$

2.1.2 Algebraic structure of a graph

In preceding subsection one sees that for any one directed graph, it always is represented by the incidence matrices Φ^+ and Φ^-. Conversely, from the incidence matrices Φ^+ and Φ^- one can reconstruct a graph. In this sense, a graph is equivalent to the incidence matrices Φ^+ and Φ^-, which are said to be the structure matrices of G. In this subsection, one will investigate the structure of a graph.

Firstly we calculate the matrices $\Phi^+(\Phi^+)^T$, $\Phi^-(\Phi^-)^T$ and $\Phi^-(\Phi^+)^T$, they are $m \times m$ matrices. Using Proposition 2.1.1, one has

$$\begin{aligned}
\Phi^+(\Phi^+)^T &= \begin{pmatrix} \phi^+_{11} & \phi^+_{12} & \cdots & \phi^+_{1n} \\ \phi^+_{21} & \phi^+_{22} & \cdots & \phi^+_{2n} \\ \cdots & \ddots & \ddots & \vdots \\ \phi^+_{m1} & \phi^+_{m2} & \cdots & \phi^+_{mn} \end{pmatrix} \begin{pmatrix} \phi^+_{11} & \phi^+_{21} & \cdots & \phi^+_{m1} \\ \phi^+_{12} & \phi^+_{22} & \cdots & \phi^+_{m2} \\ \cdots & \ddots & \ddots & \vdots \\ \phi^+_{1n} & \phi^+_{2n} & \cdots & \phi^+_{mn} \end{pmatrix} \\
&= \begin{pmatrix} \sum\limits_{j=1}^n \phi^+_{1j}\phi^+_{1j} & \sum\limits_{j=1}^n \phi^+_{1j}\phi^+_{2j} & \cdots & \sum\limits_{j=1}^n \phi^+_{1j}\phi^+_{mj} \\ \sum\limits_{j=1}^n \phi^+_{2j}\phi^+_{1j} & \sum\limits_{j=1}^n \phi^+_{2j}\phi^+_{2j} & \cdots & \sum\limits_{j=1}^n \phi^+_{2j}\phi^+_{mj} \\ \cdots & \ddots & \ddots & \vdots \\ \sum\limits_{j=1}^n \phi^+_{mj}\phi^+_{1j} & \sum\limits_{j=1}^n \phi^+_{mj}\phi^+_{2j} & \cdots & \sum\limits_{j=1}^n \phi^+_{mj}\phi^+_{mj} \end{pmatrix} \\
&= \begin{pmatrix} \#J^+(a_1) & 0 & \cdots & 0 \\ 0 & \#J^+(a_2) & \cdots & 0 \\ \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \#J^+(a_m) \end{pmatrix} = D_+,
\end{aligned}$$

similarly,

$$\Phi^-(\Phi^-)^T = \begin{pmatrix} \#J^-(a_1) & 0 & \cdots & 0 \\ 0 & \#J^-(a_2) & \cdots & 0 \\ \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \#J^-(a_m) \end{pmatrix} = D_-.$$

Therefore, one has

$$\Phi^+(\Phi^+)^T + \Phi^-(\Phi^-)^T = \begin{pmatrix} \#J(a_1) & 0 & \cdots & 0 \\ 0 & \#J(a_2) & \cdots & 0 \\ \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \#J(a_m) \end{pmatrix} \tag{2.1.4}$$

Denote $\Phi^+(\Phi^+)^T + \Phi^-(\Phi^-)^T = D$ and call it the vertex degree matrix of G. Obviously, if G has no isolated vertex, then D is an invertible matrix.

For $\Phi^-(\Phi^+)^T$, a direct calculation gives

$$\begin{aligned}
\Phi^-(\Phi^+)^T &= \begin{pmatrix} \phi^-_{11} & \phi^-_{12} & \cdots & \phi^-_{1n} \\ \phi^-_{21} & \phi^-_{22} & \cdots & \phi^-_{2n} \\ \cdots & \ddots & \ddots & \vdots \\ \phi^-_{m1} & \phi^-_{m2} & \cdots & \phi^-_{mn} \end{pmatrix} \begin{pmatrix} \phi^+_{11} & \phi^+_{21} & \cdots & \phi^+_{m1} \\ \phi^+_{12} & \phi^+_{22} & \cdots & \phi^+_{m2} \\ \cdots & \ddots & \ddots & \vdots \\ \phi^+_{1n} & \phi^+_{2n} & \cdots & \phi^+_{mn} \end{pmatrix} \\
&= \begin{pmatrix} \sum\limits_{j=1}^n \phi^-_{1j}\phi^+_{1j} & \sum\limits_{j=1}^n \phi^-_{1j}\phi^+_{2j} & \cdots & \sum\limits_{j=1}^n \phi^-_{1j}\phi^+_{mj} \\ \sum\limits_{j=1}^n \phi^-_{2j}\phi^+_{1j} & \sum\limits_{j=1}^n \phi^-_{2j}\phi^+_{2j} & \cdots & \sum\limits_{j=1}^n \phi^-_{2j}\phi^+_{mj} \\ \cdots & \ddots & \ddots & \vdots \\ \sum\limits_{j=1}^n \phi^-_{mj}\phi^+_{1j} & \sum\limits_{j=1}^n \phi^-_{mj}\phi^+_{2j} & \cdots & \sum\limits_{j=1}^n \phi^-_{mj}\phi^+_{mj} \end{pmatrix}.
\end{aligned}$$

Similarly, one has

$$\Phi^+(\Phi^-)^T = \begin{pmatrix} \sum\limits_{j=1}^n \phi^+_{1j}\phi^-_{1j} & \sum\limits_{j=1}^n \phi^+_{1j}\phi^-_{2j} & \cdots & \sum\limits_{j=1}^n \phi^+_{1j}\phi^-_{mj} \\ \sum\limits_{j=1}^n \phi^+_{2j}\phi^-_{1j} & \sum\limits_{j=1}^n \phi^+_{2j}\phi^-_{2j} & \cdots & \sum\limits_{j=1}^n \phi^+_{2j}\phi^-_{mj} \\ \cdots & \ddots & \ddots & \vdots \\ \sum\limits_{j=1}^n \phi^+_{mj}\phi^-_{1j} & \sum\limits_{j=1}^n \phi^+_{mj}\phi^-_{2j} & \cdots & \sum\limits_{j=1}^n \phi^+_{mj}\phi^-_{mj} \end{pmatrix} = (\Phi^-(\Phi^+)^T)^T.$$

Let $\ell(a_k)$ be the subset of E that each element $e \in \ell(a_k)$ is a loop at a_k. Define the loop diagonal matrix D_l by

$$D_l = \begin{pmatrix} \#\ell(a_1) & 0 & \cdots & 0 \\ 0 & \#\ell(a_2) & \cdots & 0 \\ \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \#\ell(a_m) \end{pmatrix}. \tag{2.1.5}$$

Obviously, $\#\ell(a_k) = \sum\limits_{j=1}^n \phi^+_{kj}\phi^-_{kj}$ for each k according to Proposition 2.1.1.

In order to analyze the structure of a graph, one defines the vertex adjacency matrix of G by

$$\begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_m \end{array}
\begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & \cdots & \cdots & a_m \end{array} \\
\begin{pmatrix} 0 & a_{12} & \cdots & \cdots & a_{1m} \\ a_{21} & 0 & \cdots & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & 0 \end{pmatrix} \end{array} \tag{2.1.6}$$

where a_{ij} is defined by

$$a_{ij} = \begin{cases} 1 & \text{there exists an edge connecting } a_i \text{ and } a_j \\ 0, & \text{otherwise} \end{cases} \tag{2.1.7}$$

Denote it by $A = (a_{ij})$, which is a symmetric matrix.

Since $\Phi^+(\Phi^-)^T + \Phi^-(\Phi^+)^T$ is a symmetric matrix, one defines the matrix P by

$$P = \Phi^+(\Phi^-)^T + \Phi^-(\Phi^+)^T - 2D_l - A. \tag{2.1.8}$$

P also is a symmetric matrix and is called the parallel edge pattern of G. Thus one has

$$\Phi^+(\Phi^-)^T + \Phi^-(\Phi^+)^T = 2D_l + P + A.$$

Therefore, for any graph G one has

$$(\Phi^+ - \Phi^-)(\Phi^+ - \Phi^-)^T = D - (2D_\ell + P + A)$$

and

$$(\Phi^+ + \Phi^-)(\Phi^+ + \Phi^-)^T = D + 2D_\ell + P + A.$$

If G is a simple graph (without loop and parallel edges), then

$$\Phi^+(\Phi^-)^T + \Phi^-(\Phi^+)^T = A$$

and

$$\Phi\Phi^T = (\Phi^+ - \Phi^-)(\Phi^+ - \Phi^-)^T = D - A.$$

Summarizing above discussion, one has the following definition.

DEFINITION 2.1.1 *Let G be a directed graph with the vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and the edge set $E = \{e_1, e_2, \cdots, e_n\}$, Φ^+ and Φ^- be the structure matrices of G. The matrix given by*

$$D = \Phi^+(\Phi^+)^T + \Phi^-(\Phi^-)^T = \mathrm{diag}(\#J(a_1), \#J(a_2), \cdots, \#J(a_m)) \tag{2.1.9}$$

is called the vertex degree matrix of G.

The matrix defined by

$$D_l = \mathrm{diag}(\#\ell(a_1), \#\ell(a_2), \cdots, \#\ell(a_m)) \tag{2.1.10}$$

is called the loop degree matrix of G.

The matrix $A = (a_{ij})$ defined by

$$a_{ii} = 0, \quad a_{ij} = \begin{cases} 1 & \textit{there exists an edge connecting } a_i \textit{ and } a_j \\ 0, & \textit{otherwise} \end{cases} \tag{2.1.11}$$

is called the vertex adjacency matrix of G;

The matrix defined by

$$P = \Phi^+(\Phi^+)^T + \Phi^-(\Phi^-)^T - 2D_l - A \tag{2.1.12}$$

is called the parallel edge pattern matrix of G.

From Definition 2.1.1 one sees that the matrices D, D_l, A and P are independent of the direction of G. So they also are character of the undirected graph. Note that the matrix A gives the vertex adjacency pattern. Based on A, using P and D_l one can determine location of loops and parallel edges.

EXAMPLE 2.1.2 *Let G be a planar directed graph, whose structure be shown in Fig. 2.1.2*

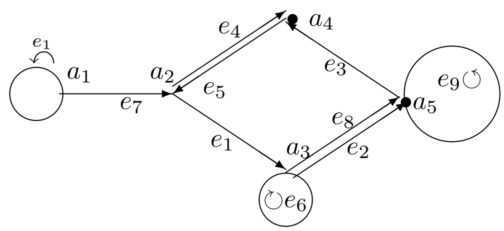

Fig. 2.1.2 A complex directed graph without boundary

The structure matrices are

$$\Phi^- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Phi^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and hence

$$\Phi^+(\Phi^-)^T + \Phi^-(\Phi^+)^T = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 2 \end{bmatrix}$$

The vertex degree diagonal matrix and the loop diagonal matrix of G are

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}, \quad D_l = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

the parallel edge pattern and the vertex adjacency matrix of G

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Using these information one can redraw the picture of G. □

2.1.3 Spectral graph

Spectral graph theory is an important content of algebraic structure of graph G. One of main tasks in graph theory is to deduce essential properties and structure of a graph from its graph spectrum. In this subsection one only introduce several notions in spectral graph theory.

DEFINITION 2.1.2 *Let G be a simple directed graph with the vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and the edge set $E = \{e_1, e_2, \cdots, e_n\}$, Φ^+ and Φ^- be the structure matrices of G. Let D and A be defined as before, and $\Phi = \Phi^+ - \Phi^-$ be the incidence matrix of G. The matrix defined by*

$$\widehat{A} = (\sqrt{D})^{-1} A (\sqrt{D})^{-1} \tag{2.1.13}$$

is said to be the normalized adjacency matrix of G;

The matrix defined by

$$\mathcal{L} = D - A \tag{2.1.14}$$

is said to be the Laplace matrix of G; The matrix

$$\widehat{\mathcal{L}} = (\sqrt{D})^{-1} \mathcal{L} (\sqrt{D})^{-1} \tag{2.1.15}$$

is called the normalized Laplace matrix of G.

The matrix defined by

$$|\mathcal{L}| = D + A \tag{2.1.16}$$

is said to be the signless Laplace matrix of G; The matrix

$$|\widehat{\mathcal{L}}| = (\sqrt{D})^{-1} |\mathcal{L}| (\sqrt{D})^{-1} \tag{2.1.17}$$

is called the normalized signless Laplace matrix of G.

From Definition 2.1.2 we can see that the following result is true.

PROPOSITION 2.1.2 *Let G be a simple direct graph with the vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and the edge set $E = \{e_1, e_2, \cdots, e_n\}$, and let Φ be the incidence matrix of G. Then one has*

1) $\mathcal{L} = \Phi\Phi^T = (\Phi^+ - \Phi^-)(\Phi^+ - \Phi^-)^T = D - A$;

2) $|\mathcal{L}| = (\Phi^+ + \Phi^-)(\Phi^+ + \Phi^-)^T$;

3) $\widehat{\mathcal{L}} = I - \widehat{A}$ and $|\widehat{\mathcal{L}}| = I + \widehat{A}$

hence $\mathcal{L}$, $|\mathcal{L}|$, $\widehat{\mathcal{L}}$ and $|\widehat{\mathcal{L}}|$ are non-negative matrices.

The following proposition gives the spectrum of the matrices $\widehat{A}$, $\widehat{\mathcal{L}}$ and $|\widehat{\mathcal{L}}|$, and the relations among them.

PROPOSITION 2.1.3 *Let G be a simple graph with the vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and the edge set $E = \{e_1, e_2, \cdots, e_n\}$, Φ be the incidence matrix of G. Then the following statements hold:*

1) $\sigma(\widehat{A}) \subset [-1, 1]$, $\sigma(\widehat{A}) = \{\widehat{\lambda}_1, \widehat{\lambda}_2, \cdots, \widehat{\lambda}_m\}$, and $\lambda_m = 1$ is an eigenvalue with eigenvector $\sqrt{D}e$ where $e = [1, 1, 1, \cdots, 1]^T$;

2) $\sigma(\widehat{\mathcal{L}}) \subset [0, 2]$, $\sigma(\widehat{\mathcal{L}}) = \{\widehat{\mu}_1, \widehat{\mu}_2, \cdots, \widehat{\mu}_m\}$, and $\mu_1 = 0$ is an eigenvalue with eigenvector $\sqrt{D}e$;

3) $\sigma(|\widehat{\mathcal{L}}|) \subset [0, 2]$, $\sigma(|\widehat{\mathcal{L}}|) = \{\widehat{\nu}_1, \widehat{\nu}_2, \cdots, \widehat{\nu}_m\}$, and $\nu_m = 2$ is an eigenvalue with eigenvector $\sqrt{D}e$;

4) If anyone of spectra of A, $\widehat{A}$, $\mathcal{L}$, $\widehat{\mathcal{L}}$, $|\mathcal{L}|$, and $|\widehat{\mathcal{L}}|$ is known, so are all the spectra.

EXAMPLE 2.1.3 *Let G be a planar simple graph, whose structure be shown in Fig. 2.1.3*

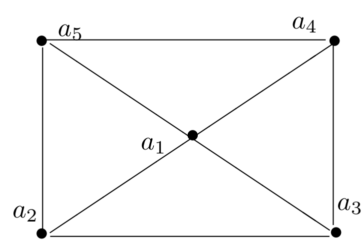

Fig. 2.1.3. A graph with multi-circle

Clearly, the vertex degree matrix D and vertex adjacency matrix A are

$$D = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

For graph G one has the following matrices and their spectra:

$$A=\begin{pmatrix}0&1&1&1&1\\1&0&1&0&1\\1&1&0&1&0\\1&0&1&0&1\\1&1&0&1&0\end{pmatrix},\qquad \widehat{A}=\begin{pmatrix}0&\frac{1}{2\sqrt{3}}&\frac{1}{2\sqrt{3}}&\frac{1}{2\sqrt{3}}&\frac{1}{2\sqrt{3}}\\\frac{1}{2\sqrt{3}}&0&\frac{1}{3}&0&\frac{1}{3}\\\frac{1}{2\sqrt{3}}&\frac{1}{3}&0&\frac{1}{3}&0\\\frac{1}{2\sqrt{3}}&0&\frac{1}{3}&0&\frac{1}{3}\\\frac{1}{2\sqrt{3}}&\frac{1}{3}&0&\frac{1}{3}&0\end{pmatrix}$$

$$\sigma(A)=\{-2,1-\sqrt{5},0,0,1+\sqrt{5}\}\qquad \sigma(\widehat{A})=\{-\tfrac{2}{3},-\tfrac{1}{3},0,0,1\}$$

$$\mathcal{L}=\begin{pmatrix}4&-1&-1&-1&-1\\-1&3&-1&0&-1\\-1&-1&3&-1&0\\-1&0&-1&3&-1\\-1&-1&0&-1&3\end{pmatrix}\qquad \widehat{\mathcal{L}}=\begin{pmatrix}1&\frac{-1}{2\sqrt{3}}&\frac{-1}{2\sqrt{3}}&\frac{-1}{2\sqrt{3}}&\frac{-1}{2\sqrt{3}}\\\frac{-1}{2\sqrt{3}}&1&\frac{-1}{3}&0&\frac{-1}{3}\\\frac{-1}{2\sqrt{3}}&\frac{-1}{3}&1&\frac{-1}{3}&0\\\frac{-1}{2\sqrt{3}}&0&\frac{-1}{3}&1&\frac{-1}{3}\\\frac{-1}{2\sqrt{3}}&\frac{-1}{3}&0&\frac{-1}{3}&1\end{pmatrix}$$

$$\sigma(\mathcal{L})=\{0,3,3,5,5\}\qquad \sigma(\widehat{\mathcal{L}})=\{0,1,1,\tfrac{4}{3},\tfrac{5}{3}\}$$

$$|\mathcal{L}|=\begin{pmatrix}0&1&1&1&1\\1&0&1&0&1\\1&1&0&1&0\\1&0&1&0&1\\1&1&0&1&0\end{pmatrix},\qquad |\widehat{\mathcal{L}}|=\begin{pmatrix}0&1&1&1&1\\1&0&1&0&1\\1&1&0&1&0\\1&0&1&0&1\\1&1&0&1&0\end{pmatrix}$$

$$\sigma(\mathcal{L})=\{1,\tfrac{9-\sqrt{17}}{2},3,3,\tfrac{9+\sqrt{17}}{2}\}\qquad \sigma(\widehat{\mathcal{L}})=\{\tfrac{1}{3},\tfrac{2}{3},1,1,2\}$$

REMARK 2.1.2 *The spectral graph theory has important applications in natural science, for examples, in Astronomy, the astronomers used the stellar spectra to determine the make-up of distant stars; in Chemistry, the eigenvalues were associated with the stability of molecules. More information about the spectral graph theory and its applications we refer to [27],[52] and [76] and the references therein.*

2.1.4 Geometric structure of a graph

Let G be a directed graph with the vertex set $V=\{a_1,a_2,\cdots,a_m\}$ and the edge set $E=\{e_1,e_2,\cdots,e_n\}$, and $\Phi^{\pm}$ be its structure matrices. Assume that G has no isolated vertex. Here one concerns with the connection relation of edge-edge of a graph G. For each $e_j\in E$, e_j^+ denotes its head of e_j(final point), and e_j^- indicates its tail (starting point). One considers a

$2n \times 2n$ matrix Ψ, the edge-edge intersection matrix,

$$\begin{array}{c} \\ e_1^+ \\ e_2^+ \\ \vdots \\ e_n^+ \\ e_1^- \\ e_2^- \\ \vdots \\ e_n^- \end{array}
\left(\begin{array}{cccccccc}
e_1^+ & e_2^+ & \cdots & e_n^+ & e_1^- & e_2^- & \cdots & e_n^- \\
\psi_{11} & \psi_{12} & \cdots & \psi_{1n} & \psi_{1,n+1} & \psi_{1,n+2} & \cdots & \psi_{1,2n} \\
\psi_{21} & \psi_{22} & \cdots & \psi_{2n} & \psi_{2,n+1} & \psi_{2,n+2} & \cdots & \psi_{2,2n} \\
\vdots & \cdots & \cdots & \vdots & \vdots & \cdots & \cdots & \vdots \\
\psi_{n1} & \psi_{n2} & \cdots & \psi_{nn} & \psi_{n,n+1} & \psi_{n,n+2} & \cdots & \psi_{n,2n} \\
\psi_{(n+1),1} & \psi_{(n+1),2} & \cdots & \psi_{(n+1),n} & \psi_{n+1,n+1} & \psi_{n+1,n+2} & \cdots & \psi_{n+1,2n} \\
\psi_{n+2,1} & \psi_{n+2,2} & \cdots & \psi_{n+2,n} & \psi_{n+2,n+1} & \psi_{n+2,n+2} & \cdots & \psi_{n+2,2n} \\
\vdots & \cdots & \cdots & \vdots & \vdots & \cdots & \cdots & \vdots \\
\psi_{2n,1} & \psi_{2n,2} & \cdots & \psi_{2n,n} & \psi_{2n,n+1} & \psi_{2n,n+2} & \cdots & \psi_{2n,2n}
\end{array}\right)$$

where the entries ψ_{ij} are defined by

$$\psi_{ii} = 1, \quad \forall i = 1, 2, \cdots, 2n,$$

and

$$\psi_{rk} = \begin{cases} 1, & \text{if one of } e_i^{\pm} \cap e_j^+ \neq \emptyset \text{ and } e_i^{\pm} \cap e_j^- \neq \emptyset \text{ holds} \\ 0 & \text{if one of } \ e_i^{\pm} \cap e_j^{\pm} = \emptyset \text{ holds,} \end{cases}$$

The matrix Ψ presents the intersection relation of endpoints of edge-edge.

For a directed graph, one can see that if e_k^+ and e_i^+ have a common vertex, then $\sum\limits_{j=1}^{m} \phi_{jk}^+ \phi_{ji}^+ = 1$; if e_k^+ and e_i^+ have no common vertex, it holds that $\sum\limits_{j=1}^{m} \phi_{jk}^+ \phi_{ji}^+ = 0$, i.e.,

$$\sum_{j=1}^{m} \phi_{jk}^+ \phi_{ji}^+ = \begin{cases} 1 & e_k^+ \cap e_i^+ \neq \emptyset, \\ 0 & e_k^+ \cap e_i^+ = \emptyset. \end{cases}$$

Similarly, one has the relations

$$\sum_{j=1}^{m} \phi_{jk}^- \phi_{ji}^- = \begin{cases} 1 & e_k^- \cap e_i^- \neq \emptyset, \\ 0 & e_k^- \cap e_i^- = \emptyset \end{cases}$$

and

$$\sum_{j=1}^{m} \phi_{jk}^+ \phi_{ji}^- = \begin{cases} 1 & e_k^+ \cap e_i^- \neq \emptyset, \\ 0 & e_k^+ \cap e_i^- = \emptyset. \end{cases}$$

In order to represent matrix Ψ using the structure matrices, one now calculates products of the structure matrices of G

$$(\Phi^+)^T \Phi^+ = \begin{pmatrix} \phi_{11}^+ & \phi_{21}^+ & \cdots & \cdots & \phi_{m1}^+ \\ \phi_{12}^+ & \phi_{22}^+ & \cdots & \cdots & \phi_{m2}^+ \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \phi_{1n}^+ & \phi_{2n}^+ & \cdots & \cdots & \phi_{mn}^+ \end{pmatrix} \begin{pmatrix} \phi_{11}^+ & \phi_{12}^+ & \cdots & \cdots & \phi_{1n}^+ \\ \phi_{21}^+ & \phi_{22}^+ & \cdots & \cdots & \phi_{2n}^+ \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \phi_{m1}^+ & \phi_{m2}^+ & \cdots & \cdots & \phi_{mn}^+ \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix}
\sum_{j=1}^{m} \phi_{j1}^{+}\phi_{j1}^{+} & \sum_{j=1}^{m} \phi_{j1}^{+}\phi_{j2}^{+} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j1}^{+}\phi_{jn}^{+} \\
\sum_{j=1}^{m} \phi_{j2}^{+}\phi_{j1}^{+} & \sum_{j=1}^{m} \phi_{j2}^{+}\phi_{j2}^{+} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j2}^{+}\phi_{jn}^{+} \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
\sum_{j=1}^{m} \phi_{jn}^{+}\phi_{j1}^{+} & \sum_{j=1}^{m} \phi_{jn}^{+}\phi_{j2}^{+} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{jn}^{+}\phi_{jn}^{+}
\end{pmatrix} \\
&= \begin{pmatrix}
1 & \sum_{j=1}^{m} \phi_{j1}^{+}\phi_{j2}^{+} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j1}^{+}\phi_{jn}^{+} \\
\sum_{j=1}^{m} \phi_{j2}^{+}\phi_{j1}^{+} & 1 & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j2}^{+}\phi_{jn}^{+} \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
\sum_{j=1}^{m} \phi_{jn}^{+}\phi_{j1}^{+} & \sum_{j=1}^{m} \phi_{jn}^{+}\phi_{j2}^{+} & \cdots & \cdots & 1
\end{pmatrix};
\end{aligned}
$$

$$
\begin{aligned}
(\Phi^{-})^{T}\Phi^{-} &= \begin{pmatrix}
\phi_{11}^{-} & \phi_{21}^{-} & \cdots & \cdots & \phi_{m1}^{-} \\
\phi_{12}^{-} & \phi_{22}^{-} & \cdots & \cdots & \phi_{m2}^{-} \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
\phi_{1n}^{-} & \phi_{2n}^{-} & \cdots & \cdots & \phi_{mn}^{-}
\end{pmatrix}
\begin{pmatrix}
\phi_{11}^{-} & \phi_{12}^{-} & \cdots & \cdots & \phi_{1n}^{-} \\
\phi_{21}^{-} & \phi_{22}^{-} & \cdots & \cdots & \phi_{2n}^{-} \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
\phi_{m1}^{-} & \phi_{m2}^{-} & \cdots & \cdots & \phi_{mn}^{-}
\end{pmatrix} \\
&= \begin{pmatrix}
\sum_{j=1}^{m} \phi_{j1}^{-}\phi_{j1}^{-} & \sum_{j=1}^{m} \phi_{j1}^{-}\phi_{j2}^{-} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j1}^{-}\phi_{jn}^{-} \\
\sum_{j=1}^{m} \phi_{j2}^{-}\phi_{j1}^{-} & \sum_{j=1}^{m} \phi_{j2}^{-}\phi_{j2}^{-} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j2}^{-}\phi_{jn}^{-} \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
\sum_{j=1}^{m} \phi_{jn}^{-}\phi_{j1}^{-} & \sum_{j=1}^{m} \phi_{jn}^{-}\phi_{j2}^{-} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{jn}^{-}\phi_{jn}^{-}
\end{pmatrix} \\
&= \begin{pmatrix}
1 & \sum_{j=1}^{m} \phi_{j1}^{-}\phi_{j2}^{-} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j1}^{-}\phi_{jn}^{-} \\
\sum_{j=1}^{m} \phi_{j2}^{-}\phi_{j1}^{-} & 1 & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j2}^{-}\phi_{jn}^{-} \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
\sum_{j=1}^{m} \phi_{jn}^{-}\phi_{j1}^{-} & \sum_{j=1}^{m} \phi_{jn}^{-}\phi_{j2}^{-} & \cdots & \cdots & 1
\end{pmatrix};
\end{aligned}
$$

$$(\Phi^+)^T\Phi^- = \begin{pmatrix} \phi^+_{11} & \phi^+_{21} & \cdots & \cdots & \phi^+_{m1} \\ \phi^+_{12} & \phi^+_{22} & \cdots & \cdots & \phi^+_{m2} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \phi^+_{1n} & \phi^+_{2n} & \cdots & \cdots & \phi^+_{mn} \end{pmatrix} \begin{pmatrix} \phi^-_{11} & \phi^-_{12} & \cdots & \cdots & \phi^-_{1n} \\ \phi^-_{21} & \phi^-_{22} & \cdots & \cdots & \phi^-_{2n} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \phi^-_{m1} & \phi^-_{m2} & \cdots & \cdots & \phi^-_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_{j=1}^m \phi^+_{j1}\phi^-_{j1} & \sum\limits_{j=1}^m \phi^+_{j1}\phi^-_{j2} & \cdots & \cdots & \sum\limits_{j=1}^m \phi^+_{j1}\phi^-_{jn} \\ \sum\limits_{j=1}^m \phi^+_{j2}\phi^-_{j1} & \sum\limits_{j=1}^m \phi^+_{j2}\phi^-_{j2} & \cdots & \cdots & \sum\limits_{j=1}^m \phi^+_{j2}\phi^-_{jn} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \sum\limits_{j=1}^m \phi^+_{jn}\phi^-_{j1} & \sum\limits_{j=1}^m \phi^+_{jn}\phi^-_{j2} & \cdots & \cdots & \sum\limits_{j=1}^m \phi^+_{jn}\phi^-_{jn} \end{pmatrix}$$

and

$$(\Phi^-)^T\Phi^+ = \begin{pmatrix} \sum\limits_{j=1}^m \phi^-_{j1}\phi^+_{j1} & \sum\limits_{j=1}^m \phi^-_{j1}\phi^+_{j2} & \cdots & \cdots & \sum\limits_{j=1}^m \phi^-_{j1}\phi^+_{jn} \\ \sum\limits_{j=1}^m \phi^-_{j2}\phi^+_{j1} & \sum\limits_{j=1}^m \phi^-_{j2}\phi^+_{j2} & \cdots & \cdots & \sum\limits_{j=1}^m \phi^-_{j2}\phi^+_{jn} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \sum\limits_{j=1}^m \phi^-_{jn}\phi^+_{j1} & \sum\limits_{j=1}^m \phi^-_{jn}\phi^+_{j2} & \cdots & \cdots & \sum\limits_{j=1}^m \phi^-_{jn}\phi^+_{jn} \end{pmatrix} = ((\Phi^+)^T\Phi^-)^T.$$

Therefore, one has

$$\Psi = \begin{pmatrix} (\Phi^+)^T\Phi^+ & (\Phi^+)^T\Phi^- \\ (\Phi^-)^T\Phi^+ & (\Phi^-)^T\Phi^- \end{pmatrix}. \tag{2.1.18}$$

The matrix Ψ has the following property.

PROPOSITION 2.1.4 *Let G be a directed graph with the vertex set V and the edge set $E = \{e_1, e_2, \cdots, e_n\}$. Suppose that G has no isolated vertex. Let the matrix Ψ be given by (2.1.18), then*

1) The rank of matrix Ψ is the number of vertices of G, i.e., $rank(\Psi) = \#V$;

2) $\sum\limits_{i=1}^{2n} \psi_{ij}$ is the number of edges at some vertex. In particular, $\sum\limits_{i=1}^{n} \psi_{ij}$ is the number of incoming edges at the vertex; $\sum\limits_{i=1}^{n} \psi_{n+i,j}$ is the number of outgoing edges at the vertex.

3) Let Ψ_j denote the j^{th} column vector of Ψ, i.e., $\Psi_j = (\psi_{1j}, \psi_{2j}, \cdots, \psi_{2n,j})^T$. Then for any $k, j \in \{1, 2, \cdots, 2n\}$, the inner product $(\Psi_k, \Psi_j)_{\mathbb{R}^{2n}}$ satisfies

$$(\Psi_k, \Psi_j)_{\mathbb{R}^{2n}} = \begin{cases} \psi_{kj} \sum\limits_{i=1}^{2n} \psi_{ik}, & \Psi_k \textit{ and } \Psi_j \textit{ are linearly dependent} \\ 0, & \Psi_k \textit{ and } \Psi_j \textit{ are linearly independent} \end{cases} = \psi_{kj} \sum_{i=1}^{2n} \psi_{ik} \tag{2.1.19}$$

In the case of linearly dependence, it holds that $\Psi_k = \Psi_j$.

The matrix Ψ implicates the relation of vertices of G. Form it one can not see explicitly the adjacency relation of V.

EXAMPLE 2.1.4 *Let* G *be the directed graph given by Example 2.1.1,*

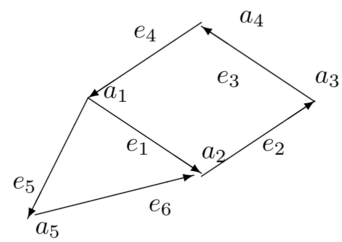

Fig. 2.1.4. A directed graph without boundary

The intersection matrix Ψ is

$$\begin{array}{cc}
 & \begin{array}{cccccc|cccccc} e_1^+ & e_2^+ & e_3^+ & e_4^+ & e_5^+ & e_6^+ & e_1^- & e_2^- & e_3^- & e_4^- & e_5^- & e_6^- \end{array} \\
\begin{array}{c} e_1^+ \\ e_2^+ \\ e_3^+ \\ e_4^+ \\ e_5^+ \\ e_6^+ \\ e_1^- \\ e_2^- \\ e_3^- \\ e_4^- \\ e_5^- \\ e_6^- \end{array} &
\left(\begin{array}{cccccc|cccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
\end{array}$$

REMARK 2.1.3 *The edge-edge intersection matrix can describe more complex graph, for example, a graph has loops and parallel edges.*

THEOREM 2.1.1 *Let* G *be a directed graph without insolated vertex, and* Ψ *be the intersection matrix of edge-edge of* G*. Then the following statements are true.*

1) Ψ *is a non-negative matrix;*

2) the number of positive eigenvalue of Ψ *(taking multiplicity into account) is equal to the number of vertices of* G*, each positive eigenvalue is the degree of some vertex;*

3) the eigenvectors of Ψ *corresponding to positive eigenvalues are the column vector of* Ψ*, they form an orthogonal group in* $\mathbb{R}^{2n}$*;*

4) Using the linearly independent group of eigenvector of Ψ *one can reconstruct the graph* G.

Proof 1) For any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{2n}$,

$$\begin{aligned}(\Psi\xi, \xi)_{\mathbb{R}^{2n}} &= ((\Phi^+)^T\Phi^+\xi_1 + (\Phi^+)^T\Phi^-\xi_2, \xi_1)_{\mathbb{R}^n} + (\Phi^-)^T\Phi^+\xi_1 + (\Phi^-)^T\Phi^-\xi_2, \xi_2)_{\mathbb{R}^n} \\ &= (\Phi^+\xi_1, \Phi^+\xi_1)_{\mathbb{R}^n} + 2(\Phi^-\xi_2, \Phi^+\xi_1)_{\mathbb{R}^n} + (\Phi^-\xi_2, \Phi^-\xi_2)_{\mathbb{R}^n} \geq 0.\end{aligned}$$

2) and 3). Since Ψ is a non-negative matrix, its eigenvalues are nonnegative. Let Ψ_j denote the j^{th} column vector of Ψ, i.e., $\Psi_j = (\psi_{1j}, \psi_{2j}, \cdots, \psi_{2n,j})^T$. Then using (2.1.19) one gets that

$$\Psi\Psi_j = ((\Psi_1, \Psi_j), (\Psi_2, \Psi_j), \cdots, (\Psi_{2n}, \Psi_j))^T = \sum_{i=1}^{2n} \psi_{ij}\,(\psi_{1j}, \psi_{2j}, \cdots, \psi_{2n,j})^T = \sum_{i=1}^{2n} \psi_{ij}\Psi_j.$$

According to Proposition 2.1.4, $\sum\limits_{i=1}^{2n} \psi_{ij}$ is the degree of some vertex of G. So, $\lambda = \sum\limits_{i=1}^{2n} \psi_{ij}$ is an eigenvalue of Ψ, and Ψ_j is an eigenvector.

4) Let $\Psi_{j_1}, \Psi_{j_2}, \cdots, \Psi_{jm}$ be the linearly independent row vectors. One decomposes $\Psi_{j_k} = (\Psi^+_{j_k}, \Psi^-_{j_k})$, where $\Psi^+_{j_k}$ and $\Psi^-_{j_k}$ are n-dimensional vectors.

Define a new incidence matrix by

$$\begin{array}{c} \\ a'_1 \\ a'_2 \\ \vdots \\ \vdots \\ a'_m \end{array} \begin{pmatrix} e_1^+ & e_2^+ & \cdots & e_n^+ & e_1^- & e_2^- & \cdots & e_n^- \\ \psi_{j_1,1} & \psi_{j_1,2} & \cdots & \psi_{j_1,n} & \psi_{j_1,n+1} & \psi_{j_1,n+2} & \cdots & \psi_{j_1,2n} \\ \psi_{j_2,1} & \psi_{j_2,2} & \cdots & \psi_{j_2,n} & \psi_{j_2,n+1} & \psi_{j_1,n+2} & \cdots & \psi_{j_2,2n} \\ \vdots & \cdots & \ddots & \vdots & \cdots & \cdots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \vdots & \cdots & \cdots & \ddots & \vdots \\ \psi_{j_m,1} & \psi_{j_m,2} & \cdots & \psi_{j_m,n} & \psi_{j_m,n+1} & \psi_{j_m,n+2} & \cdots & \psi_{j_m,2n} \end{pmatrix}.$$

Set

$$\begin{aligned}\widehat{\Phi}^+ &= \begin{pmatrix} \psi_{j_1,1} & \psi_{j_1,2} & \cdots & \psi_{j_1,n} \\ \psi_{j_2,1} & \psi_{j_2,2} & \cdots & \psi_{j_2,n} \\ \vdots & \cdots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \psi_{j_m,1} & \psi_{j_m,2} & \cdots & \psi_{j_m,n} \end{pmatrix}_{m\times n} \\ \widehat{\Phi}^- &= \begin{pmatrix} \psi_{j_1,n+1} & \psi_{j_1,n+2} & \cdots & \psi_{j_1,2n} \\ \psi_{j_2,n+1} & \psi_{j_2,n+2} & \cdots & \psi_{j_2,2n} \\ \vdots & \cdots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ \psi_{j_m,n+1} & \psi_{j_m,n+2} & \cdots & \psi_{j_m,2n} \end{pmatrix}_{m\times n}.\end{aligned}$$

Then the connection relation of vertex of G is given by $\widehat{\Phi}^+(\widehat{\Phi}^-)^T$:

$$\begin{array}{cc} & \begin{array}{ccccc} a'_1 & a'_2 & \cdots & \cdots & a'_m \end{array} \\ \begin{array}{c} a'_1 \\ a'_2 \\ \vdots \\ \vdots \\ a'_m \end{array} & \begin{pmatrix} \sum\limits_{k=1}^{n} \psi_{j_1,k}\psi_{j_1,n+k} & \sum\limits_{k=1}^{n} \psi_{j_1,k}\psi_{j_2,n+k} & \cdots & \cdots & \sum\limits_{k=1}^{n} \psi_{j_1,k}\psi_{j_m,n+k} \\ \sum\limits_{k=1}^{n} \psi_{j_2,k}\psi_{j_1,n+k} & \sum\limits_{k=1}^{n} \psi_{j_2,k}\psi_{j_2,n+k} & \cdots & \cdots & \sum\limits_{k=1}^{n} \psi_{j_2,k}\psi_{j_m,n+k} \\ \vdots & \cdots & \ddots & \vdots & \\ \vdots & \cdots & \ddots & \vdots & \\ \sum\limits_{k=1}^{n} \psi_{j_m,k}\psi_{j_1,n+k} & \sum\limits_{k=1}^{n} \psi_{j_m,k}\psi_{j_2,n+k} & \cdots & \cdots & \sum\limits_{k=1}^{n} \psi_{j_m,k}\psi_{j_m,n+k} \end{pmatrix}_{m\times m} \end{array}$$

this is not a symmetric matrix. The vertex adjacency matrix of G is

$$\widehat{\Phi}^{+}(\widehat{\Phi}^{-})^{T} + \widehat{\Phi}^{-}(\widehat{\Phi}^{+})^{T}.$$

This is symmetric matrix. $\square$

2.2 A function defined on graph

Let G be a metric graph with the vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and the edge set $E = \{e_1, e_2, \cdots, e_n\}$ with edge length set $\{\ell_1, \ell_2, \cdots, \ell_n\}$.

Let $u(x)$ be a function defined on E. For each $e_j \in E$, we define the parameterization realization of $u(x)$ on e_j by

$$u_j(s) = u(x(s)), \quad s \in (0, \ell_j).$$

If limit $\lim_{s\to 0} u_j(s)$ ($\lim_{s\to \ell_j} u_j(s)$) exists, denote it by $u_j(0)$($u_j(\ell_j)$, respectively).

Define the function spaces $L^2(E)$ and $H^k(E)$ by

$$L^2(E) = \{f(x) \mid f_j(s) \in L^2(0, \ell_j)\}$$

$$H^k(E) = \{f(x) \in L^2(E) \mid f_j(s) \in H^k(0, \ell_j)\}.$$

Let $u(x)$ be a function defined on E. We can define a function

$$w(x) = u(x\ell_j), \quad x \in (0, 1),\ x\ell_j \in e_j.$$

$w(x)$ is said to be a normalized function on E. In the sequel, one always uses the normalized function. The restriction of w on e_j denotes w_j. For $a \in V$,

$$\lim_{s\to 1} w_j(s) = w_j(1), \quad \text{if } j \in J^{+}(a),$$
$$\lim_{s\to 0} w_j(s) = w_j(0), \quad \text{if } j \in J^{-}(a).$$

Define a mapping w from V to $\mathbb{C}^m (m = \#V)$ by

$$w : (a_1, a_2, \cdots, a_m) \to (w(a_1), w(a_2) \cdots, w(a_m)) \in \mathbb{C}^m.$$

DEFINITION 2.2.1 *Let G be a metric graph with vertex set V and edge set E. $u(x)$ is said to be a function defined on G if $u(x)$ has definition on E and V, i.e., $u : G = V \cup E \to \mathbb{R}$.*

2.2.1 Continuity condition

Let $u(x)$ be a function defined on E and w be a function defined on V. Set

$$U(x) = (u_1(x), u_2(x), \cdots, u_n(x))^T, \quad W(v) = (w(a_1), w(a_2), \cdots, w(a_m))^T.$$

Consider the relation

$$\begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ \vdots \\ u_n(0) \end{bmatrix} = \begin{bmatrix} \phi_{11}^- & \phi_{21}^- & \cdots & \phi_{m1}^- \\ \phi_{12}^- & \phi_{22}^- & \cdots & \phi_{m2}^- \\ \phi_{13}^- & \phi_{23}^- & \cdots & \phi_{m3}^- \\ \vdots & \ddots & \ddots & \vdots \\ \phi_{1n}^- & \phi_{2n}^- & \cdots & \phi_{mn}^- \end{bmatrix} \begin{bmatrix} w(a_1) \\ w(a_2) \\ \vdots \\ w(a_m) \end{bmatrix}$$

which means that the outgoing are equal to the value of w at vertices, i.e.,

$$u_j(0) = w(a_k), \quad \forall j \in J^-(a_k), \quad k = 1, 2, \cdots, m.$$

So one can rewrite it into $U(0) = [\Phi^-]^T W(v)$.

The relation

$$\begin{bmatrix} u_1(1) \\ u_2(1) \\ u_3(1) \\ \vdots \\ u_n(1) \end{bmatrix} = \begin{bmatrix} \phi_{11}^+ & \phi_{12}^+ & \cdots & \phi_{1m}^+ \\ \phi_{21}^+ & \phi_{22}^+ & \cdots & \phi_{2m}^+ \\ \phi_{31}^+ & \phi_{32}^+ & \cdots & \phi_{3m}^+ \\ \vdots & \ddots & \ddots & \vdots \\ \phi_{n1}^+ & \phi_{n2}^+ & \cdots & \phi_{nm}^+ \end{bmatrix} \begin{bmatrix} w(a_1) \\ w(a_2) \\ \vdots \\ w(a_m) \end{bmatrix}$$

means that the incoming are equal to the value of w at vertices, i.e.,

$$u_j(1) = w(a_k), \quad \forall j \in J^+(a_k), \quad k = 1, 2, \cdots, m.$$

This also is a continuity condition. It can be written into $U(1) = [\Phi^+]^T W(v)$.

DEFINITION 2.2.2 *A function $u(x)$ defined on G is said to be the incoming continuous at $a \in V$ if $u(x)$ is continuous on E and has limits at two endpoints of each edge in E, moreover it satisfies*

$$u_j(1) = u(a), \quad \forall\, j \in J^+(a).$$

It is said to be the outgoing continuous at a if $u(x)$ is continuous on E and has limits at two endpoints of each edge in E, and

$$u_i(0) = u(a), \quad \forall\, i \in J^-(a).$$

For a multiple node a, $u(x)$ is said to be continuous at a if $\lim_{x \to a} u(x) = u(a)$ or equivalently

$$u(a) = u_j(1) = u_i(0), \quad \forall i \in J^-(a),\ j \in J^+(a).$$

A function u defined on G is said to be a continuous function if it is continuous on E, and continuous at very interior vertex $a \in V_{int}$, and at each boundary vertex $a_i \in \partial G$, it holds that

$$\lim_{s\to 1} u(s) = u_j(1) = u(a_i), \quad \text{if } j \in J^+(a_i),$$
$$\lim_{s\to 0} u(s) = u_k(0) = u(a_i), \quad \text{if } k \in J^-(a_i).$$

One denotes the set of all continuous function on G by $C(G)$.

If $u(x)$ is continuous on G, then one has

$$U(1) = (\Phi^+)^T U(v), \quad U(0) = (\Phi^-)^T U(v). \tag{2.2.1}$$

For simplicity, one can write the continuity of u as there is a vector $d \in \mathbb{C}^m$ such that

$$U(1) = (\Phi^+)^T d, \quad U(0) = (\Phi^-)^T d. \tag{2.2.2}$$

In order to obtain a direct relation between $U(1)$ and $U(0)$, from equality (2.2.1) one gets that

$$\Phi^+U(1) = \Phi^+(\Phi^+)^T U(v) = D_+U(v), \quad \Phi^-U(0) = \Phi^-(\Phi^-)^T U(v) = D_-U(v)$$

and hence

$$U(v) = D^{-1}[\Phi^+U(1) + \Phi^-U(0)].$$

Substituting it into (2.2.1) leads to

$$\begin{cases} [(\Phi^+)^T D^{-1}\Phi^+ - I]U(1) + (\Phi^+)^T D^{-1}\Phi^- U(0) = 0, \\ (\Phi^-)^T D^{-1}\Phi^+ U(1) + [(\Phi^-)^T D^{-1}\Phi^- - I]U(0) = 0. \end{cases} \tag{2.2.3}$$

For simplification, one denotes $D = \operatorname{diag}(d_1, d_2, \cdots, d_m)$. Now one calculates the product of the structure matrices of G

$$\begin{aligned}
(\Phi^+)^T D^{-1}\Phi^+ &= \begin{pmatrix} \phi^+_{11} & \phi^+_{21} & \cdots & \cdots & \phi^+_{m1} \\ \phi^+_{12} & \phi^+_{22} & \cdots & \cdots & \phi^+_{m2} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \phi^+_{1n} & \phi^+_{2n} & \cdots & \cdots & \phi^+_{mn} \end{pmatrix} \begin{pmatrix} d_1^{-1}\phi^+_{11} & d_1^{-1}\phi^+_{12} & \cdots & \cdots & d_1^{-1}\phi^+_{1n} \\ d_2^{-1}\phi^+_{21} & d_2^{-1}\phi^+_{22} & \cdots & \cdots & d_2^{-1}\phi^+_{2n} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ d_m^{-1}\phi^+_{m1} & d_m^{-1}\phi^+_{m2} & \cdots & \cdots & d_m^{-1}\phi^+_{mn} \end{pmatrix} \\
&= \begin{pmatrix} \sum\limits_{j=1}^m \phi^+_{j1} d_j^{-1} & \sum\limits_{j=1}^m \phi^+_{j1}\phi^+_{j2} d_j^{-1} & \cdots & \cdots & \sum\limits_{j=1}^m \phi^+_{j1}\phi^+_{jn} d_j^{-1} \\ \sum\limits_{j=1}^m \phi^+_{j2}\phi^+_{j1} d_j^{-1} & \sum\limits_{j=1}^m \phi^+_{j2}\phi^+_{j2} d_j^{-1} & \cdots & \cdots & \sum\limits_{j=1}^m \phi^+_{j2}\phi^+_{jn} d_j^{-1} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \sum\limits_{j=1}^m \phi^+_{jn}\phi^+_{j1} d_j^{-1} & \sum\limits_{j=1}^m \phi^+_{jn}\phi^+_{j2} d_j^{-1} & \cdots & \cdots & \sum\limits_{j=1}^m \phi^+_{jn}\phi^+_{jn} d_j^{-1} \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \sum_{j=1}^m \phi_{j1}^+ d_j^{-1}\psi_{11} & \sum_{j=1}^m \phi_{j1}^+ d_j^{-1}\psi_{12} & \cdots & \cdots & \sum_{j=1}^m \phi_{j1}^+ d_j^{-1}\psi_{1n} \\ \sum_{j=1}^m \phi_{j2}^+ d_j^{-1}\psi_{21} & \sum_{j=1}^m \phi_{j2}^+ d_j^{-1}\psi_{22} & \cdots & \cdots & \sum_{j=1}^m \phi_{j2}^+ d_j^{-1}\psi_{2n} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \sum_{j=1}^m \phi_{jn}^+ d_j^{-1}\psi_{n1} & \sum_{j=1}^m \phi_{jn}^+ d_j^{-1}\psi_{n21} & \cdots & \cdots & \sum_{j=1}^m \phi_{jn}^+ d_j^{-1}\psi_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^m \phi_{j1}^+ d_j^{-1} & 0 & \cdots & \cdots & 0 \\ 0 & \sum_{j=1}^m \phi_{j2}^+ d_j^{-1} & \cdots & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \sum_{j=1}^m \phi_{jn}^+ d_j^{-1} \end{pmatrix} (\Phi^+)^T\Phi^+$$

$$(\Phi^-)^T D^{-1}\Phi^- = \begin{pmatrix} \phi_{11}^- & \phi_{21}^- & \cdots & \cdots & \phi_{m1}^- \\ \phi_{12}^- & \phi_{22}^- & \cdots & \cdots & \phi_{m2}^- \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \phi_{1n}^- & \phi_{2n}^- & \cdots & \cdots & \phi_{mn}^- \end{pmatrix} \begin{pmatrix} \phi_{11}^- d_1^{-1} & \phi_{12}^- d_1^{-1} & \cdots & \cdots & \phi_{1n}^- d_1^{-1} \\ \phi_{21}^- d_2^{-1} & \phi_{22}^- d_2^{-1} & \cdots & \cdots & \phi_{2n}^- d_2^{-1} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \phi_{m1}^- d_m^{-1} & \phi_{m2}^- d_m^{-1} & \cdots & \cdots & \phi_{mn}^- d_m^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^m \phi_{j1}^- d_j^{-1} & \sum_{j=1}^m \phi_{j1}^-\phi_{j2}^- d_j^{-1} & \cdots & \cdots & \sum_{j=1}^m \phi_{j1}^-\phi_{jn}^- d_j^{-1} \\ \sum_{j=1}^m \phi_{j2}^-\phi_{j1}^- d_j^{-1} & \sum_{j=1}^m \phi_{j2}^-\phi_{j2}^- d_j^{-1} & \cdots & \cdots & \sum_{j=1}^m \phi_{j2}^-\phi_{jn}^- d_j^{-1} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \sum_{j=1}^m \phi_{jn}^-\phi_{j1}^- d_j^{-1} & \sum_{j=1}^m \phi_{jn}^-\phi_{j2}^- d_j^{-1} & \cdots & \cdots & \sum_{j=1}^m \phi_{jn}^-\phi_{jn}^- d_j^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^m \phi_{j1}^- d_j^{-1}\psi_{n+1,n+1} & \sum_{j=1}^m \phi_{j1}^- d_j^{-1}\psi_{n+1,n+2} & \cdots & \cdots & \sum_{j=1}^m \phi_{j1}^- d_j^{-1}\psi_{n+1,2n} \\ \sum_{j=1}^m \phi_{j2}^- d_j^{-1}\psi_{n+2,n+1} & \sum_{j=1}^m \phi_{j2}^- d_j^{-1}\psi_{n+2,n+2} & \cdots & \cdots & \sum_{j=1}^m \phi_{j2}^- d_j^{-1}\psi_{n+2,2n} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \sum_{j=1}^m \phi_{jn}^- d_j^{-1}\psi_{2n,n+1} & \sum_{j=1}^m \phi_{jn}^- d_j^{-1}\psi_{2n,n+2} & \cdots & \cdots & \sum_{j=1}^m \phi_{jn}^- d_j^{-1}\psi_{2n,2n} \end{pmatrix}$$

$$
= \begin{pmatrix}
\sum_{j=1}^{m} \phi_{j1}^{-} d_j^{-1} & 0 & \cdots & \cdots & 0 \\
0 & \sum_{j=1}^{m} \phi_{j2}^{-} d_j^{-1} & \cdots & \cdots & 0 \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \sum_{j=1}^{m} \phi_{jn}^{-} d_j^{-1}
\end{pmatrix} (\Phi^{-})^T \Phi^{-}
$$

$$
\begin{aligned}
(\Phi^{+})^T D^{-1} \Phi^{-} &= \begin{pmatrix}
\phi_{11}^{+} & \phi_{21}^{+} & \cdots & \cdots & \phi_{m1}^{+} \\
\phi_{12}^{+} & \phi_{22}^{+} & \cdots & \cdots & \phi_{m2}^{+} \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
\phi_{1n}^{+} & \phi_{2n}^{+} & \cdots & \cdots & \phi_{mn}^{+}
\end{pmatrix}
\begin{pmatrix}
\phi_{11}^{-} d_1^{-1} & \phi_{12}^{-} d_1^{-1} & \cdots & \cdots & \phi_{1n}^{-} d_1^{-1} \\
\phi_{21}^{-} d_2^{-1} & \phi_{22}^{-} d_2^{-1} & \cdots & \cdots & \phi_{2n}^{-} d_2^{-1} \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
\phi_{m1}^{-} d_m^{-1} & \phi_{m2}^{-} d_m^{-1} & \cdots & \cdots & \phi_{mn}^{-} d_m^{-1}
\end{pmatrix} \\
&= \begin{pmatrix}
\sum_{j=1}^{m} \phi_{j1}^{+} \phi_{j1}^{-} d_j^{-1} & \sum_{j=1}^{m} \phi_{j1}^{+} \phi_{j2}^{-} d_j^{-1} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j1}^{+} \phi_{jn}^{-} d_j^{-1} \\
\sum_{j=1}^{m} \phi_{j2}^{+} \phi_{j1}^{-} d_j^{-1} & \sum_{j=1}^{m} \phi_{j2}^{+} \phi_{j2}^{-} d_j^{-1} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j2}^{+} \phi_{jn}^{-} d_j^{-1} \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
\sum_{j=1}^{m} \phi_{jn}^{+} \phi_{j1}^{-} d_j^{-1} & \sum_{j=1}^{m} \phi_{jn}^{+} \phi_{j2}^{-} d_j^{-1} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{jn}^{+} \phi_{jn}^{-} d_j^{-1}
\end{pmatrix} \\
&= \begin{pmatrix}
\sum_{j=1}^{m} \phi_{j1}^{+} d_j^{-1} \psi_{1,n+1} & \sum_{j=1}^{m} \phi_{j1}^{+} d_j^{-1} \psi_{1,n+2} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j1}^{+} d_j^{-1} \psi_{1,2n} \\
\sum_{j=1}^{m} \phi_{j2}^{+} d_j^{-1} \psi_{2,n+1} & \sum_{j=1}^{m} \phi_{j2}^{+} d_j^{-1} \psi_{2,n+2} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{j2}^{+} d_j^{-1} \psi_{2,2n} \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
\sum_{j=1}^{m} \phi_{jn}^{+} d_j^{-1} \psi_{n,n+1} & \sum_{j=1}^{m} \phi_{jn}^{+} d_j^{-1} \psi_{n,n+2} & \cdots & \cdots & \sum_{j=1}^{m} \phi_{jn}^{+} d_j^{-1} \psi_{n,2n}
\end{pmatrix} \\
&= \begin{pmatrix}
\sum_{j=1}^{m} \phi_{j1}^{+} d_j^{-1} & 0 & \cdots & \cdots & 0 \\
0 & \sum_{j=1}^{m} \phi_{j2}^{+} d_j^{-1} & \cdots & \cdots & 0 \\
\vdots & \cdots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \sum_{j=1}^{m} \phi_{jn}^{+} d_j^{-1}
\end{pmatrix} (\Phi^{+})^T \Phi^{-}
\end{aligned}
$$

and

$$
\begin{aligned}
(\Phi^-)^T D^{-1}\Phi^+ &= \begin{pmatrix} \phi^-_{11} & \phi^-_{21} & \cdots & \cdots & \phi^-_{m1} \\ \phi^-_{12} & \phi^-_{22} & \cdots & \cdots & \phi^-_{m2} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \phi^-_{1n} & \phi^-_{2n} & \cdots & \cdots & \phi^-_{mn} \end{pmatrix} \begin{pmatrix} \phi^+_{11}d_1^{-1} & \phi^+_{12}d_1^{-1} & \cdots & \cdots & \phi^+_{1n}d_1^{-1} \\ \phi^+_{21}d_2^{-1} & \phi^+_{22}d_2^{-1} & \cdots & \cdots & \phi^+_{2n}d_2^{-1} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \phi^+_{m1}d_m^{-1} & \phi^+_{m2}d_m^{-1} & \cdots & \cdots & \phi^+_{mn}d_m^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^m \phi^-_{j1}d_j^{-1}\psi_{n+1,1} & \sum_{j=1}^m \phi^-_{j1}d_j^{-1}\psi_{n+1,2} & \cdots & \cdots & \sum_{j=1}^m \phi^-_{j1}d_j^{-1}\psi_{n+1,n} \\ \sum_{j=1}^m \phi^-_{j2}d_j^{-1}\psi_{n+2,1} & \sum_{j=1}^m \phi^-_{j2}d_j^{-1}\psi_{n+2,2} & \cdots & \cdots & \sum_{j=1}^m \phi^-_{j2}d_j^{-1}\psi_{n+2,n} \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \sum_{j=1}^m \phi^-_{jn}d_j^{-1}\psi_{2n,1} & \sum_{j=1}^m \phi^-_{jn}d_j^{-1}\psi_{2n,2} & \cdots & \cdots & \sum_{j=1}^m \phi^-_{jn}d_j^{-1}\psi_{2n,n} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^m \phi^-_{j1}d_j^{-1} & 0 & \cdots & \cdots & 0 \\ 0 & \sum_{j=1}^m \phi^-_{j2}d_j^{-1} & \cdots & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \sum_{j=1}^m \phi^-_{jn}d_j^{-1} \end{pmatrix} (\Phi^-)^T\Phi^+
\end{aligned}
$$

Note that the matrices

$$
\begin{pmatrix} \sum_{j=1}^m \phi^\pm_{j1}d_j^{-1} & 0 & \cdots & \cdots & 0 \\ 0 & \sum_{j=1}^m \phi^\pm_{j2}d_j^{-1} & \cdots & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \sum_{j=1}^m \phi^\pm_{jn}d_j^{-1} \end{pmatrix}
$$

are invertible, in particular, their inverse are given by

$$\begin{pmatrix} \sum_{j=1}^{m} \phi_{j1}^{\pm} d_j & 0 & \cdots & \cdots & 0 \\ 0 & \sum_{j=1}^{m} \phi_{j2}^{\pm} d_j & \cdots & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \sum_{j=1}^{m} \phi_{jn}^{\pm} d_j \end{pmatrix} := \Phi_{\pm}(D).$$

So formula (2.2.3) can rewritten into

$$\begin{pmatrix} (\Phi^+)^T \Phi^+ - \Phi_+(D) & (\Phi^+)^T \Phi^- \\ (\Phi^-)^T \Phi^+ & (\Phi^-)^T \Phi^- - \Phi_-(D) \end{pmatrix} \begin{pmatrix} U(1) \\ U(0) \end{pmatrix} = 0. \tag{2.2.4}$$

The coefficients matrix gives the connection relation between both ends of the vector-valued function $U(x)$.

EXAMPLE 2.2.1 *Herein we consider a continuous function $y(x)$ defined on a graph G given by Example 2.1.1,*

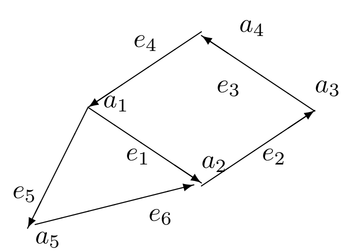

Fig. 2.2.1. A directed graph without boundary

The connective conditions are

$$y(a_1) = y_1(0) = y_4(1) = y_5(0); \quad y(a_2) = y_2(0) = y_6(1) = y_1(1); \quad y(a_3) = y_2(1) = y_3(0);$$
$$y(a_4) = y_3(1) = y_4(0); \qquad y(a_5) = y_5(1) = y_6(0).$$

The incidence matrix Φ

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

The connective conditions are represented into the matrices form

$$\begin{bmatrix} y_1(1) \\ y_2(1) \\ y_3(1) \\ y_4(1) \\ y_5(1) \\ y_6(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(a_1) \\ y(a_2) \\ y(a_3) \\ y(a_4) \\ y(a_5) \end{bmatrix}$$

and

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \\ y_5(0) \\ y_6(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y(a_1) \\ y(a_2) \\ y(a_3) \\ y(a_4) \\ y(a_5) \end{bmatrix}.$$

The matrix D is given by

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

and hence

$$\Phi_+(D) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad \Phi_-(D) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Further one has

$$(\Phi^+)^T\Phi^+ = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(\Phi^-)^T\Phi^- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$(\Phi^-)^T\Phi^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus the boundary relation is given by

$$\left[\begin{array}{cccccc|cccccc} -2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \begin{bmatrix} y_1(1) \\ y_2(1) \\ y_3(1) \\ y_4(1) \\ y_5(1) \\ y_6(1) \\ y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \\ y_5(0) \\ y_6(0) \end{bmatrix} = 0.$$

$\square$

2.2.2 Flow continuous condition

If $u(x)$ is a function on E, then the relation

$$\begin{bmatrix} w(a_1) \\ w(a_2) \\ w(a_3) \\ \vdots \\ w(a_m) \end{bmatrix} = \begin{bmatrix} \phi_{11}^- & \phi_{12}^- & \cdots & \phi_{1n}^- \\ \phi_{21}^- & \phi_{22}^- & \cdots & \phi_{2n}^- \\ \phi_{31}^- & \phi_{32}^- & \cdots & \phi_{3n}^- \\ \vdots & \ddots & \ddots & \vdots \\ \phi_{m1}^- & \phi_{m2}^- & \cdots & \phi_{mn}^- \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ \vdots \\ u_n(0) \end{bmatrix}$$

denotes the outgoing at each a_j, i.e.,

$$w(a_j) = \sum_{k=1}^{n} \phi_{jk}^- u_k(0) = \sum_{k\in J^-(a_j)} u_k(0);$$

and the relation

$$\begin{bmatrix} w(a_1) \\ w(a_2) \\ w(a_3) \\ \vdots \\ w(a_m) \end{bmatrix} = \begin{bmatrix} \phi_{11}^+ & \phi_{12}^+ & \cdots & \phi_{1n}^+ \\ \phi_{21}^+ & \phi_{22}^+ & \cdots & \phi_{2n}^+ \\ \phi_{31}^+ & \phi_{32}^+ & \cdots & \phi_{3n}^+ \\ \vdots & \ddots & \ddots & \vdots \\ \phi_{m1}^+ & \phi_{m2}^+ & \cdots & \phi_{mn}^+ \end{bmatrix} \begin{bmatrix} u_1(1) \\ u_2(1) \\ u_3(1) \\ \vdots \\ u_n(1) \end{bmatrix}$$

denotes the incoming at a_j, i.e.,

$$w(a_j) = \sum_{k=1}^{n} \phi_{jk}^+ u_k(1) = \sum_{k\in J^+(a_j)} u_k(1).$$

DEFINITION 2.2.3 *Let $u(x)$ be a function defined on G and continuous on E(include end-points of each edge). Set*

$$U(x) = (u_1(x), u_2(x), \cdots, u_n(x))^T, \quad U(v) = (u(a_1), u(a_2), \cdots, u(a_m))^T.$$

If $u(x)$ satisfies condition

$$u(a_j) = \sum_{k=1}^{n} \phi_{jk}^+ u_k(1) = \sum_{k\in J^+(a_j)} u_k(1), \tag{2.2.5}$$

then $u(x)$ is said to be the incoming flow continuous at vertex a_j(Kirchhoff law); if for each $a_j \in V_{int}$, it holds that

$$u(a_j) = \sum_{k\in J^+(a_j)} u_k(1), \quad \forall a_j \in V_{int} \tag{2.2.6}$$

then $u(x)$ is called the incoming flow continuous (Kirchhoff law) on V_{int}.

If $u(x)$ at vertex a_j satisfies condition

$$u(a_j) = \sum_{k=1}^{n} \phi_{jk}^- u_k(0) = \sum_{k\in J^-(a_j)} u_k(0), \tag{2.2.7}$$

then $u(x)$ is said to be the outgoing flow continuous at a_j(Kirchhoff law); If $u(x)$ at each $a_j \in V_{int}$ satisfies

$$u(a_j) = \sum_{i \in J^-(a_j)} u_i(0), \quad \forall a_j \in V_{int} \tag{2.2.8}$$

then $u(x)$ is called the outgoing flow continuous (Kirchhoff law) on V_{int}.

If $u(x)$ satisfies the condition at vertex $a_j \in V_{int}$

$$u(a_j) = \sum_{i \in J^-(a_j)} u_i(0) = \sum_{i \in J^+(a_j)} u_i(1), \tag{2.2.9}$$

then $u(x)$ is said to be the flow continuous (Kirchhoff law) at a_j.

If $u(x)$ satisfies the condition

$$u(a_i) = u_i(0),\ i \in J^-(a_i), \quad or\,(= u_i(1), i \in J^+(a_i)), \quad \forall a_i \in \partial G \tag{2.2.10}$$

then $u(x)$ is said to be continuous on the boundary ∂G.

REMARK 2.2.1 *The flow continuous condition has obviously physical meaning. If a is a multiple node, $u(a) = \sum_{j \in J^+(a)} u_j(1)$ means that the amount of flow at the node is equal to the total incoming flow. If there is no sink and source at the node, then the total outgoing flow is $u(a) = \sum_{j \in J^-(a)} u_j(0)$.*

Here one should mention that the flow continuous condition on V_{int} is in fact only defined on its interior nodes, at the boundary ∂G it does not satisfy the flow continuous condition, this is because it has only incoming or outgoing flow continuous condition. Therefore, the incoming flow continuity on V_{int} is not equivalent to $U(v) = \Phi^+ U(1)$, also the outgoing flow condition on V_{int} is not equivalent to $U(v) = \Phi^- U(0)$ if G has nonempty boundary. If G has no boundary, and for each $a_j \in V$ there are at least one incoming edge and one outgoing edge, then $U(v) = \Phi^+ U(1)$ and $U(v) = \Phi^- U(0)$ denote the incoming and outgoing flow continuity, respectively.

The following example shows that if G has nonempty boundary, then flow continuous condition is not satisfied at the boundary.

EXAMPLE 2.2.2 *Let G be a planar graph that has structure shown as Fig. 2.2.2*

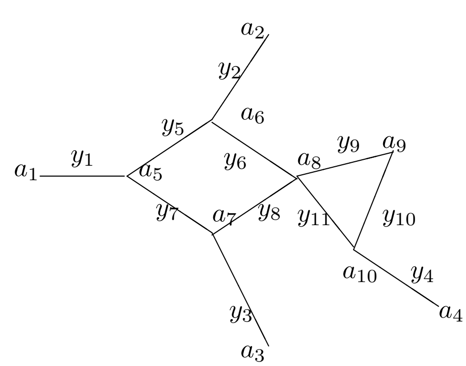

Fig. 2.2.2. A network with nonempty boundary

The directed edges are defined by

$$\begin{array}{llll} e_1 = (a_1, a_5) & e_2 = (a_2, a_6) & e_3 = (a_3, a_7) & e_4 = (a_{10}, a_4) \\ e_5 = (a_5, a_6) & e_6 = (a_6, a_8) & e_7 = (a_5, a_7) & e_8 = (a_7, a_8) \\ e_9 = (a_8, a_9) & e_{10} = (a_9, a_{10}) & e_{11} = (a_8, a_{10}) & \end{array}$$

Obviously, the boundary of G is $\partial G = \{a_1, a_2, a_3, a_4\}$.

Let $y(x)$ be a function defined on G, whose restriction on e_j be $y_j(x)$. Assume that $y_j(x)$ is continuous on e_j,$j = 1, 2, \cdots, n$. The conditions

$$y_1(0) = y(a_1), \quad y_2(0) = y(a_2), \quad y_3(0) = y(a_3), \quad y_4(1) = y(a_4)$$

mean that $y(x)$ is continuous on boundary of G; and the connective conditions are

$$\begin{array}{ll} y_1(1) = y(a_5) = y_5(0) + y_7(0), & y_6(0) = y(a_6) = y_5(1) + y_2(1), \\ y_7(1) + y_3(1) = y(a_7) = y_8(0), & y_8(1) + y_6(1) = y(a_8) = y_9(0) + y_{11}(0), \\ y_9(1) = y(a_9) = y_{10}(0), & y_{11}(1) + y_{10}(1) = y(a_{10}) = y_4(0). \end{array}$$

If one uses the representation of the incidence matrix, then one has

$$\begin{bmatrix} y(a_1) \\ y(a_2) \\ y(a_3) \\ y(a_4) \\ y(a_5) \\ y(a_6) \\ y(a_7) \\ y(a_8) \\ y(a_9) \\ y(a_{10}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \\ y_5(0) \\ y_6(0) \\ y_7(0) \\ y_8(0) \\ y_9(0) \\ y_{10}(0) \\ y_{11}(0) \end{bmatrix}$$

and

$$\begin{bmatrix} y(a_1) \\ y(a_2) \\ y(a_3) \\ y(a_4) \\ y(a_5) \\ y(a_6) \\ y(a_7) \\ y(a_8) \\ y(a_9) \\ y(a_{10}) \end{bmatrix} = \begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1(1) \\ y_2(1) \\ y_3(1) \\ y_4(1) \\ y_5(1) \\ y_6(1) \\ y_7(1) \\ y_8(1) \\ y_9(1) \\ y_{10}(1) \\ y_{11}(1) \end{bmatrix}$$

where the row including $*$ denotes that this can not represent. If one supposes that $y(x)$ takes zero values on ∂G, then $* = 0$, the flow continuous conditions on G can write into

$$U(v) = \Phi^+ U(1) = \Phi^- U(0).$$

If G has no boundary, this always is true. $\square$

Now one assumes that a function $u(x)$ defined on G is continuous on boundary ∂G, and flow continuous at all interior nodes V_{int}. In order to give a matrix representation of relationship between $U(1), U(0)$ and $U(v)$, one calculates the following matrices

$$(\Phi^+)^T \Phi^+ = \begin{pmatrix} \sum_{j=1}^m \phi_{j1}^+ \phi_{j1}^+ & \sum_{j=1}^m \phi_{j1}^+ \phi_{j2}^+ & \cdots & \sum_{j=1}^m \phi_{j1}^+ \phi_{jn}^+ \\ \sum_{j=1}^m \phi_{j2}^+ \phi_{j1}^+ & \sum_{j=1}^m \phi_{j2}^+ \phi_{j2}^+ & \cdots & \sum_{j=1}^m \phi_{j2}^+ \phi_{jn}^+ \\ \cdots & \ddots & \ddots & \vdots \\ \sum_{j=1}^m \phi_{jn}^+ \phi_{j1}^+ & \sum_{j=1}^m \phi_{jn}^+ \phi_{j2}^+ & \cdots & \sum_{j=1}^m \phi_{jn}^+ \phi_{jn}^+ \end{pmatrix}$$

then

$$(\Phi^+)^T \Phi^+ U(1) = \begin{pmatrix} \sum_{k=1}^n \sum_{j=1}^m \phi_{j1}^+ \phi_{jk}^+ u_k(1) \\ \sum_{k=1}^n \sum_{j=1}^m \phi_{j2}^+ \phi_{jk}^+ u_k(1) \\ \vdots \\ \sum_{k=1}^n \sum_{j=1}^m \phi_{jn}^+ \phi_{jk}^+ u_k(1) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m \phi_{j1}^+ u(a_j) \\ \sum_{j=1}^m \phi_{j2}^+ u(a_j) \\ \vdots \\ \sum_{j=1}^m \phi_{jn}^+ u(a_j) \end{pmatrix} = (\Phi^+)^T U(v).$$

where one has used the incoming flow continuous condition $u(a_j) = \sum_{k=1}^n \phi_{jk}^+ u_k(1)$ at all vertices of G (if there is no incoming flow at a_j, one always has $\sum_{k=1}^n \phi_{jk}^+ u_k(1) = 0$).

Similarly, one has

$$(\Phi^-)^T\Phi^- = \begin{pmatrix} \sum_{j=1}^m \phi^-_{j1}\phi^-_{j1} & \sum_{j=1}^m \phi^-_{j1}\phi^+_{j2} & \cdots & \sum_{j=1}^m \phi^-_{j1}\phi^-_{jn} \\ \sum_{j=1}^m \phi^-_{j2}\phi^-_{j1} & \sum_{j=1}^m \phi^-_{j2}\phi^-_{j2} & \cdots & \sum_{j=1}^m \phi^-_{j2}\phi^-_{jn} \\ \cdots & \ddots & \ddots & \vdots \\ \sum_{j=1}^m \phi^-_{jn}\phi^-_{j1} & \sum_{j=1}^m \phi^-_{jn}\phi^-_{j2} & \cdots & \sum_{j=1}^m \phi^-_{jn}\phi^-_{jn} \end{pmatrix}$$

and

$$(\Phi^-)^T\Phi^-U(0) = \begin{pmatrix} \sum_{k=1}^n\sum_{j=1}^m \phi^-_{j1}\phi^-_{jk}u_k(0) \\ \sum_{k=1}^n\sum_{j=1}^m \phi^-_{j1}\phi^-_{jk}u_k(0) \\ \vdots \\ \sum_{k=1}^n\sum_{j=1}^m \phi^-_{j1}\phi^-_{jk}u_k(0) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m \phi^-_{j1}u(a_j) \\ \sum_{j=1}^m \phi^-_{j1}u(a_j) \\ \vdots \\ \sum_{j=1}^m \phi^-_{j1}u(a_j) \end{pmatrix} = (\Phi^-)^TU(v)$$

where one has used the outgoing flow continuous condition $u(a_j) = \sum_{k=1}^n \phi^-_{jk}u_k(0)$ at all vertices of G (if there is no outgoing flow at a_j, one always has $\sum_{k=1}^n \phi^-_{jk}u_k(0) = 0$). Therefore one has the following result.

PROPOSITION 2.2.1 *Let G be a directed graph, and $u(x)$ be a function defined on G with the normalized parametrization. If $u(x)$ is continuous on boundary and flow continuous on V_{int}, then one has*

$$(\Phi^+)^T\Phi^+U(1) = (\Phi^+)^TU(v), \quad (\Phi^-)^T\Phi^-U(0) = (\Phi^-)^TU(v). \tag{2.2.11}$$

If G has no boundary , then $\Phi^+(\Phi^+)^T$ and $\Phi^-(\Phi^-)^T$ are full rank diagonal matrix, one can deduces from (2.2.11) that $\Phi^+U(1) = U(v)$ and $\Phi^-U(0) = U(v)$.

In order to obtain a direct relation between $U(1)$ and $U(0)$, from (2.2.11) one gets that

$$\Phi^+(\Phi^+)^T\Phi^+U(1) = D_+U(v), \quad \Phi^-(\Phi^-)^T\Phi^-U(0) = D_-U(v)$$

and hence

$$D_-D_+\Phi^+U(1) - D_+D_-\Phi^-U(0) = 0$$

and

$$U(v) = D^{-1}[D_+\Phi^+U(1) + D_-\Phi^-U(0)].$$

Substituting them into (2.2.11) yield

$$(\Phi^+)^T\Phi^+U(1) = (\Phi^+)^TD^{-1}[D_+\Phi^+U(1) + D_-\Phi^-U(0)],$$
$$(\Phi^-)^T\Phi^-U(0) = (\Phi^-)^TD^{-1}[D_+\Phi^+U(1) + D_-\Phi^-U(0)].$$

A calculation similar to previous shows that

$$\begin{aligned}
(\Phi^+)^T D^{-1} D_+ \Phi^+ &= \Phi_+(D^{-1}D_+)(\Phi^+)^T\Phi^+,\\
(\Phi^+)^T D^{-1} D_- \Phi^- &= \Phi_+(D^{-1}D_-)(\Phi^+)^T\Phi^-,\\
(\Phi^-)^T D^{-1} D_+ \Phi^+ &= \Phi_-(D^{-1}D_+)(\Phi^-)^T\Phi^+,\\
(\Phi^-)^T D^{-1} D_- \Phi^- &= \Phi_-(D^{-1}D_-)(\Phi^-)^T\Phi^-.
\end{aligned}$$

Thus one gets that

$$\begin{aligned}
&[\Phi_+(D^{-1}D_+) - I](\Phi^+)^T\Phi^+U(1) + \Phi_+(D^{-1}D_-)(\Phi^+)^T\Phi^-U(0) = 0,\\
&\Phi_-(D^{-1}D_+)(\Phi^-)^T\Phi^+U(1) + (\Phi_-(D^{-1}D_-) - 1)(\Phi^-)^T\Phi^-U(0) = 0.
\end{aligned}$$

Note that

$$\Phi_+(D^{-1}D_+) = \Phi_+(D^{-1})\Phi_+(D_+), \quad \Phi_+(D^{-1}D_-) = \Phi_+(D^{-1})\Phi_+(D_-),$$

$$\Phi_-(D^{-1}D_+) = \Phi_-(D^{-1})\Phi_-(D_+), \quad \Phi_-(D^{-1}D_-) = \Phi_-(D^{-1})\Phi_-(D_-).$$

Therefore, one has

$$\begin{aligned}
&\Phi_+(D_-)(\Phi^+)^T\Phi^+U(1) + \Phi_+(D_-)(\Phi^+)^T\Phi^-U(0) = 0,\\
&\Phi_-(D_+)(\Phi^-)^T\Phi^+U(1) + \Phi_-(D_+)(\Phi^-)^T\Phi^-U(0) = 0,
\end{aligned}$$

i.e.,

$$\begin{bmatrix} \Phi_+(D_-)(\Phi^+)^T\Phi^+ & \Phi_+(D_-)(\Phi^+)^T\Phi^- \\ \Phi_-(D_+)(\Phi^-)^T\Phi^+ & \Phi_-(D_+)(\Phi^-)^T\Phi^- \end{bmatrix} \begin{bmatrix} U(1) \\ U(0) \end{bmatrix} = 0 \tag{2.2.12}$$

2.2.3 Weighted flow condition

In the definition of flow continuous condition at node a_i, one sees that the value of function at the node is equal to the total incoming flow:

$$u(a_i) = \sum_{j\in J^+(a_i)} u_j(1).$$

However, for the outgoing flow $u_k(0)$, one can assign a weighted w_{ik} on it, for example,

$$\phi^-_{ik}u_k(0) = w^-_{ik}u(a_i) = w^-_{ik}\sum_{j\in J^+(a_i)} u_j(1). \tag{2.2.13}$$

Let w^-_{ij} satisfy the following conditions

$$0 \le w^-_{ij} \le 1, \quad w^-_{ij} = \phi^-_{ij}w^-_{ij}, \quad \sum_{j=1}^{n} w^-_{ij} = 1, \tag{2.2.14}$$

then w^-_{ij} expresses a proportion of the flow leaving the vertex a_i into the edge e_j. In this case, one still has

$$\sum_{k\in J^-(a_i)} u_k(0) = u(a_i) = \sum_{j\in J^+(a_i)} u_j(1).$$

Denote this weighted matrix by Φ_w^-, it is said to be the weighted outgoing incidence matrix, which is column stochastic.

Here one calculates the $m \times m$ matrix $\Phi^-(\Phi_w^-)^T$,

$$
\begin{aligned}
\Phi^-(\Phi_w^-)^T &= \begin{pmatrix} \phi_{11}^- & \phi_{12}^- & \cdots & \phi_{1n}^- \\ \phi_{21}^- & \phi_{22}^- & \cdots & \phi_{2n}^- \\ \cdots & \ddots & \ddots & \vdots \\ \phi_{m1}^- & \phi_{m2}^- & \cdots & \phi_{mn}^- \end{pmatrix} \begin{pmatrix} w_{11} & w_{21} & \cdots & w_{m1} \\ w_{12} & w_{22} & \cdots & w_{m2} \\ \cdots & \ddots & \ddots & \vdots \\ w_{1n} & w_{2n} & \cdots & w_{mn} \end{pmatrix} \\
&= \begin{pmatrix} \sum\limits_{j=1}^n \phi_{1j}^- w_{1j} & \sum\limits_{j=1}^n \phi_{1j}^- w_{2j} & \cdots & \sum\limits_{j=1}^n \phi_{1j}^- w_{mj} \\ \sum\limits_{j=1}^n \phi_{2j}^- w_{1j} & \sum\limits_{j=1}^n \phi_{2j}^- w_{2j} & \cdots & \sum\limits_{j=1}^n \phi_{2j}^- w_{mj} \\ \cdots & \ddots & \ddots & \vdots \\ \sum\limits_{j=1}^n \phi_{mj}^- w_{1j} & \sum\limits_{j=1}^n \phi_{mj}^- w_{2j} & \cdots & \sum\limits_{j=1}^n \phi_{mj}^- w_{mj} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_m,
\end{aligned}
$$

that is,

$$\Phi^-(\Phi_w^-)^T = I_m. \tag{2.2.15}$$

More generally one always assigns a matrix associated with the incidence matrix, $W^- = (w_{ij}^-)$ and $W^+ = (w_{ij}^+)$, respectively, by

$$w_{ij}^{\pm} = \phi_{ij}^{\pm} w_{ij}^{\pm}. \tag{2.2.16}$$

If $A = (a_{ij})$ is a $m \times n$ matrix, then the $m \times n$ matrix $\Phi_+ \bullet A$ has such a property, where $A \bullet B$ denotes the Hadamard product of both matrices, which is defined by $A \bullet B = (a_{ij}b_{ij})$.

2.2.4 Linearly nodal condition

Let $u(x)$ be a function defined on a graph G. For each $a \in V_{int}$, when $\#J(a) \geq 3$, the connection condition of $u(x)$ at a may be very complicated. Here one still considers the linearly connective condition. One can decompose theses edges jointed a into several groups: the edges in each group have one continuity—-the flow continuous condition or continuous condition, these groups have one of flow continuous and continuity condition.

EXAMPLE 2.2.3 *we consider a function $y(x)$ defined on a graph G that is given by*

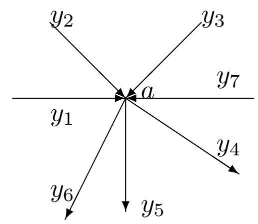

Fig. 2.2.3. A star-shape graph

with connective conditions at a

$$y_1(1) + y_3(1) + y_7(1) = y(a), \quad y_2(1) + y_3(1) + y_7(1) = y(a),$$
$$y_5(0) + y_6(0) = y(a), \qquad y_4(0) = y(a)$$

From connective conditions one can see that: in incoming direction

1) the group $\{y_1(1), y_2(1)\}$ is continuous connection;

2) the group $\{y_3(1), y_7(1)\}$ is the incoming flow continuous connection;

3) the groups $\{y_1(1), y_2(1)\}$ and $\{y_3(1), y_7(1)\}$ form the incoming flow continuous at node a.

And in the outgoing direction,

4) $\{y_5(0), y_6(0)\}$ forms the outgoing flow continuous condition; and

5) $\{y_5(0), y_6(0)\}$ and $y_4(0)$ form the continuity condition. □

In preceding treatment, one always seeks for the certain continuity at the node of graph G in the sense of flow continuity or classical continuity. Now one finds out the other form of the connection conditions.

For an interior node $a \in V_{int}$, when the function $u(x)$ satisfies the flow continuous condition, one has two equations

$$u(a) = \sum_{j\in J^+(a)} u_j(1), \quad u(a) = \sum_{s\in J^-(a)} u_s(0),$$

eliminating $u(a)$ one gets that $\sum_{j\in J^+(a)} u_j(1) - \sum_{s\in J^-(a)} u_s(0) = 0$. When the function satisfies the continuous condition, one has $\#J(a) = \#J^+(a) + \#J^-(a) = p + q$ equations, i.e.,

$$u_j(1) = u(a), j \in J^+(a), \quad u_i(0) = u(a), i \in J^-(a).$$

Eliminating the mid-variable $u(a)$ leads to $p+q-1$ many linearly independent equations. When the node has the connection condition of other types, for instance, the condition described as in Example2.2.3, the number of linear equations is an integer in $[2, p+q]$.

In general, let $a \in V_{int}$, $p = \#J^+(a)$ and $q = \#J^-(a)$ be the degree of incoming and outgoing connection respectively, the value of $u_j(1); j \in J^+(a), u_i(0); i \in J^-(a)$ forms a vector in $\mathbb{R}^{p+q}$. Assume that they have a linear relation, then the linearly nodal condition can write into $HU = eu(a)$ where H is a $(p+q) \times (p+q)$ matrix and $U = (u_j(a); j \in J(a))$ is a vector,

i.e.,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} & b_{11} & \cdots & b_{1q} \\ a_{21} & a_{22} & \cdots & a_{2p} & b_{21} & \cdots & b_{2q} \\ \vdots & \cdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} & b_{p1} & \cdots & b_{pq} \\ c_{11} & c_{12} & \cdots & c_{1p} & d_{11} & \cdots & d_{1q} \\ c_{21} & c_{22} & \cdots & c_{2p} & d_{21} & \cdots & d_{2q} \\ \vdots & \cdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ c_{q1} & c_{q2} & \cdots & c_{qp} & d_{q1} & \cdots & d_{qq} \end{pmatrix} \begin{pmatrix} u_{j_1}(1) \\ u_{j_2}(1) \\ \vdots \\ u_{j_p}(1) \\ u_{i_1}(0) \\ u_{i_2}(0) \\ \vdots \\ u_{i_q}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} u(a) \tag{2.2.17}$$

The matrix

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has rank $rank(H) \in [2, p+q]$.

REMARK 2.2.2 *Let G be a directed metric graph, and $u(x)$ denote a liquid distributed on G. In this sense, the edges are viewed as branches of stream, $u_j(x)$ denotes the distribution of the liquid on the edge e_j, $u(a)$ denotes the source or sink flow at interior node a. Assume that the liquid is continuous on the boundary, i.e., $u_j(1) = u(a), a \in \partial G, j \in J^+(a)$ which means that the amount of incoming flow is equal to the sink flow at the boundary, or $u_i(1) = u(a), a \in \partial G, i \in J^-(a)$ which means that the amount of outgoing flow is equal to the one of the source. At an interior node $a_k \in V_{int}$, the linearly nodal condition means that the difference amount between incoming and outgoing is equal to that of the source or sink flow, i.e.,*

$$\sum_{j \in J^+(a)} c_{kj}\phi^+_{kj}u_j(1) - \sum_{i \in J^-(a)} c_{ki}\phi^-_{ki}u_i(0) = u(a_k).$$

Note that this equality can not give a more detail relation among the branches. So as shown in (2.2.17) one needs some more relations to determine which branch is the mainly contributing one to source or a sink.

Let G be a directed graph and Ψ be the intersection matrix of edge-edge of G. Define the matrices by

$$\Psi_+ = \begin{pmatrix} (\Phi^+)^T\Phi^+ \\ (\Phi^-)^T\Phi^+ \end{pmatrix}_{2n\times n}, \quad \Psi_- = \begin{pmatrix} (\Phi^+)^T\Phi^- \\ (\Phi^-)^T\Phi^- \end{pmatrix}_{2n\times n}. \tag{2.2.18}$$

Let $\mathbb{M}_{2n\times n}(\Psi_+)$ be the set of matrices that has pattern Ψ_+, i.e.,

$$\mathbb{M}_{2n\times n}(\Psi_+) = \{A \in \mathbb{M}_{2n\times n} \mid A \bullet \Psi_+ = A\}. \tag{2.2.19}$$

Similarly, $\mathbb{M}_{2n\times n}(\Psi_-)$ is the set of matrices that has pattern Ψ_-, i.e.,

$$\mathbb{M}_{2n\times n}(\Psi_-) = \{B \in \mathbb{M}_{2n\times n} \mid B \bullet \Psi_- = B\}. \tag{2.2.20}$$

Now let $u(x)$ be a function defined on the graph G with normalized parameterization. One coincides the function u with a vector-valued function by

$$U(x) = [u_1(x), u_2(x), \cdots u_n(x)]^T, \quad U(v) = [u(a_1), u(a_2), \cdots u(a_m)]^T.$$

Then the linearly nodal connection condition can be written as

$$(C_1, C_2)\begin{pmatrix} U(1) \\ U(0) \end{pmatrix} = \begin{pmatrix} (\Phi^+)^T \\ (\Phi^-)^T \end{pmatrix} U(v), \quad C_1 \in \mathbb{M}_{2n\times n}(\Psi_+), C_2 \in \mathbb{M}_{2n\times n}(\Psi_-) \tag{2.2.21}$$

where C_1 and C_2 are called the connection matrix of the function u. Sometimes $C = (C_1, C_2)$ is said to be the connection matrix.

PROPOSITION 2.2.2 *Let G be a directed graph and Ψ be the intersection matrix of edge-edge of G. Let $C = (C_1, C_2) \in \mathbb{M}_{2n\times 2n}(\Psi)$, then C is a connection matrix of $u(x)$ defined on G if and only if the following conditions satisfied*

1) If $\sum_{j=1}^{2n} \psi_{ij} = 1$, then $\sum_{j=1}^{2n} c_{ij} \neq 0$, this means that at the boundary point $a_i \in \partial G$, the coefficient is not zero;

2) $\#\partial G + 2\#V_{int} \leq rank(C) \leq 2n$;

REMARK 2.2.3 *The matrix Ψ presents a connection relation of edge-edge in G, it does not denote relation between the edges and the vertices; The connection matrix C gives the relation between the vertices and the edges jointed with vertex. So the matrix C gives more detail structure of the graph G.*

2.2.5 Nonlinear node condition

Let $u(x)$ be a function defined on a graph G. In previous subsections one always seeks for certain continuity at a node of graph G in the sense of classical function. However, it is just such a continuity so that one losses some important property of the network—node dynamics.

Let

$$U(s) = (u_1(s), u_2(s), \cdots, u_n(s))^T, \quad W(v) = (w(a_1), w(a_2), \cdots, w(a_m))^T.$$

The vector-valued function $U(s)$ denotes the dynamic behavior of edges of the graph and $W(v)$ denotes the behavior of vertices of the graph.

The behavior of a vertex depends upon the incidence edges, usually, it depends nonlinearly on the values of incoming and outgoing, one can write it into

$$F(u_j(1); j \in J^+(a), u_i(0); i \in J^-(a), w(a)) = 0. \tag{2.2.22}$$

The flow continuous, weighted flow continuous and continuity are the special case of (2.2.22), they are linearly dependence of the incoming and outgoing.

Now let $w(a)$ be the value at vertex a. One considers the outgoing at node a,

$$u_i(0) = \frac{\beta_i}{2}(|w(a) + p_i| - |w(a) - p_i|), \quad i \in J^-(a), \tag{2.2.23}$$

this is a piecewise linear function, where β_i is the transmission rate.

If the value $w(a)$ depends on the incoming and outgoing, then one can write it into

$$g(w(a)) + \sum_{j \in J^+(a)} \alpha_j f(u_j(1)) + \sum_{i \in J^-(a)} \beta_i u_i(0) + z(a) = 0. \tag{2.2.24}$$

the first term is the dynamic characteristic of the vertex, the second is the input template, the third is the output template, the final term is a constant value dependent the vertex a.

Chapter 3

Ordinary Differential Equations on Graphs

Let G be a directed metric graph with vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and edge set $E = \{e_1, e_2, \cdots, e_n\}$, and let $u(x)$ be a function defined on G, $u_j(s)$ be its normalized realization on the edge $e_j \in E$. If $u(x)$ satisfies the differential equation on each $e_j \in E$

$$L(u_j) = p_{j,0}u_{j,s^k}(s) + p_{j,1}u_{j,s^{k-1}}(s) + \cdots + p_{j,k}u_j(s) = f_j(s), \quad s \in (0,1)$$

where u_{s^k} denotes $\frac{d^k u(s)}{ds^k}$, and $p_{j,k}$ are the functions defined on the edge e_j with appropriate continuity conditions, $f_j(s)$ are given functions, then $u(x)$ is called satisfying the differential equation on E.

The differential equations are always defined on E, one needs some connection and boundary conditions to determine uniquely a solution. Due to restriction of graph G, the connection conditions at the vertices become an important component to solve the differential equations.

3.1 First order linear differential equation

Let us consider first order differential equation

$$p(x)u'(x) + q(x)u(x) = f(x), \quad x \in E$$

or equivalently, on each edge $e_j, j = 1, 2, \cdots, n$,

$$p_j(s)u_j'(s) + q_j(s)u_j(s) = f_j(s), \quad s \in (0,1), \quad p_j(s) > 0. \tag{3.1.1}$$

Obviously, the general solutions to (3.1.1) are given by

$$u_j(s) = u_j(0)\exp\left\{-\int_0^s \frac{q_j(r)}{p_j(r)}dr\right\} + \int_0^s \frac{f_j(t)}{p_j(t)}\exp\left\{-\int_t^s \frac{q_j(r)}{p_j(r)}dr\right\}dt, \quad j = 1, 2, \cdots, n. \tag{3.1.2}$$

It is well known that, for first order ordinary equation on finite interval $[a,b]$, one can determine uniquely a solution provided that one gives a value of the function at some point in $[a,b]$, for example, the boundary condition $u(a)=\xi$. Obviously, if one gives a value at each edge e_k, one also determines uniquely a solution of (3.1.1). The question is that combination of these solutions needs not to be a solution on G, this is because there are some restrictions at interior node of G. Therefore, one must study the solvability of first order differential equation on a graph G.

Define the diagonal matrices $P(s)$ and $Q(s)$ by

$$P(s)=\mathrm{diag}(p_1(s),p_2(s),\cdots,p_n(s)),\quad Q(s)=\mathrm{diag}(q_1(s),q_2(s),\cdots,q_n(s))$$

and define vector-valued functions

$$U(s)=(u_1(s),u_2(s),\cdots,u_n(s))^T,\quad F(s)=(f_1(s),f_2(s),\cdots,f_n(s))^T.$$

Then the first-order differential equations on E can be rewritten into a vector-valued differential equation

$$P(s)U'(s)+Q(s)U(s)=F(s)$$

and the formal solution is given by

$$U(s)=S(s,0)U(0)+\int_0^s S(s,r)P^{-1}(r)F(r)dr \tag{3.1.3}$$

where $S(s,r)=\mathrm{diag}\left(\exp\left\{-\int_r^s \frac{q_j(t)}{p_j(t)}dt\right\}\right)$ is the fundamental matrix of the first-order differential equation.

3.1.1 Continuous solution

Here one considers the case that $u(x)$ is continuous function on G, i.e., there exists a vector $d\in\mathbb{C}^m$ such that

$$U(1)=(\Phi^+)^Td,\quad U(0)=(\Phi^-)^Td.$$

Substituting the formal solution into above yields

$$[(\Phi^+)^T-S(1,0)(\Phi^-)^T]d=\widehat{F}(1) \tag{3.1.4}$$

where $\widehat{F}(1)=\int_0^1 S(1,r)P^{-1}(r)F(r)dr$. Therefore, the differential equation has a continuous solution on G if and only if the algebraic equation (3.1.4) is solvable, this requires $\widehat{F}(1)\in\mathcal{R}([(\Phi^+)^T-S(1,0)(\Phi^-)^T])$.

Note that $[(\Phi^+)^T-S(1,0)(\Phi^-)^T]$ is a $n\times m$ matrix, $\widehat{F}(1)\in\mathbb{C}^n$. One discusses the solvability of (3.1.4) according to relation between n and m.

1) If $m<n$, one has that $rank[(\Phi^+)^T-S(1,0)(\Phi^-)^T]\le m$, then (3.1.4) is not solvable for any $\widehat{F}(1)\in\mathbb{C}^n$;

2) If $m\ge n$, i.e., $\#V\ge\#E$, then (3.1.4) is solvable for any $\widehat{F}(1)\in\mathbb{C}^n$ if and only if $rank([(\Phi^+)^T-S(1,0)(\Phi^-)^T])=n$.

Let (3.1.4) be solvable, and d_0 be its a particular solution. Then the general solutions are of the form

$$d = d_0 + \sum_{j=1}^{k} c_j d_j, \quad d_j \in \mathcal{N}[(\Phi^+)^T - S(1,0)(\Phi^-)^T] \tag{3.1.5}$$

where $k = \dim \mathcal{N}[(\Phi^+)^T - S(1,0)(\Phi^-)^T]$ and $\{d_j\}_{j=1}^k$ is an independent group of the subspace $\mathcal{N}[(\Phi^+)^T - S(1,0)(\Phi^-)^T]$. And hence the general solution of (3.1.3) is given by

$$U(s) = S(s,0)(\Phi^-)^T d_0 + \sum_{j=1}^{k} c_j S(s,0)(\Phi^-)^T d_j + \int_0^s S(s,r)P^{-1}(r)F(r)dr.$$

In this case, one needs k many conditions to determine uniquely a solution of (3.1.1). Therefore, one has the following result.

THEOREM 3.1.1 *Let G be a directed metric graph. The differential equations*

$$P(s)U'(s) + Q(s)U(s) = F(s), \quad s \in (0,1) \tag{3.1.6}$$

has the continuous solution on G for any $F \in L^2(G)$ if and only if

$$rank([(\Phi^+)^T - S(1,0)(\Phi^-)^T]) = n. \tag{3.1.7}$$

When (3.1.7) holds, the general solution of the homogeneous equations are given by

$$U(s) = \sum_{j=1}^{k} c_j S(s,0)(\Phi^-)^T d_j, \quad d_j \in \mathcal{N}([(\Phi^+)^T - S(1,0)(\Phi^-)^T]) \tag{3.1.8}$$

where $c_j \in \mathbb{C}$ and $k = \dim \mathcal{N}([(\Phi^+)^T - S(1,0)(\Phi^-)^T])$.

Observing that the solvability of differential equations defined on E is independent of directions of the edges, herein the directions are only used to parameterize the function $u(x)$. So G can be anyone graph.

One now assumes that G is a connected graph. According to Theorem 3.1.1, the necessary condition for the differential equations having a continuous solution for any $F \in L^2(G)$ is $m \geq n$, i.e., $\#V \geq \#E$. In this case, G is a tree or a cycle since G is connected.

THEOREM 3.1.2 *If G is a tree, given a value of u at the boundary point a, the differential equation*

$$p(x)u'(x) + q(x)u(x) = f(x), \quad f \in L^2(G), \quad u(a) = \xi, \quad a \in \partial G$$

has uniquely a continuous solution on G.

Proof Suppose that G is a tree, a is the root of G, then one has $m = n + 1$ and $(\Phi^+)^T - S(1,0)(\Phi^-)^T$ is a $n \times m$ matrix and $rank((\Phi^+)^T - S(1,0)(\Phi^-)^T) = n$. Thus there exists an

$n \times n$ invertible matrix K such that

$$K[(\Phi^+)^T - S(1,0)(\Phi^-)^T] = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & k_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & k_2 \\ 0 & 0 & 1 & 0 & \cdots & 0 & k_3 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & k_n \end{pmatrix}$$

Thus from the equality

$$[(\Phi^+)^T - S(1,0)(\Phi^-)^T]d = \widehat{F}(1)$$

one gets that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & k_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & k_2 \\ 0 & 0 & 1 & 0 & \cdots & 0 & k_3 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & k_n \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_m \end{pmatrix} = \begin{pmatrix} \widetilde{F}_1 \\ \widetilde{F}_2 \\ \widetilde{F}_3 \\ \vdots \\ \vdots \\ \widetilde{F}_n \end{pmatrix}$$

i.e.,

$$\begin{cases} d_1 + k_1 d_m = \widetilde{F}_1 \\ d_2 + k_2 d_m = \widetilde{F}_2 \\ d_3 + k_3 d_m = \widetilde{F}_3 \\ \quad \cdots \quad \cdots \\ d_n + k_n d_m = \widetilde{F}_n \end{cases}$$

Obviously, if $d_m = \xi$, then $d_j, j = 1, 2, \cdots, n$ are determined via d_m and $\widetilde{F}_j$. $\square$

Now one supposes that G is a cycle, then $(\Phi^+)^T - S(1,0)(\Phi^-)^T$ is a $n \times n$ matrix. In this case one needs some restriction conditions on functions $p_j(s)$ and $q_j(s)$ so that the matrix satisfies $rank((\Phi^+)^T - S(1,0)(\Phi^-)^T) = n$. When it holds, $(\Phi^+)^T - S(1,0)(\Phi^-)^T$ is invertible matrix, hence the algebraic equations

$$[(\Phi^+)^T - S(1,0)(\Phi^-)^T]d = \widehat{F}(1)$$

has unique a solution $d \in \mathbb{C}^n$

$$d = [(\Phi^+)^T - S(1,0)(\Phi^-)^T]^{-1}\widehat{F}(1).$$

This means that one need not any initial condition for solving equation (3.1.1). Thus the solution is given by

$$U(s) = S(s,0)(\Phi^-)^T[(\Phi^+)^T - S(1,0)(\Phi^-)^T]^{-1}\widehat{F}(1) + \int_0^s S(s,r)P^{-1}(r)F(r)dr.$$

In order to show the solvability condition of differential equation on a cycle, one considers the following example.

EXAMPLE 3.1.1 *For the sake of simplicity, one considers a cycle with four edges, see Fig. 3.1.1*

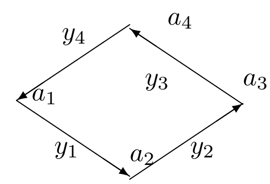

Fig. 3.1.1. A continuous network on a cycle

the differential equations are given by

$$y_j'(s) - \alpha_j y(s)_j = f_j(s), \quad j = 1, 2, 3, 4.$$

A direct calculation shows

$$(\Phi^+)^T - S(1,0)(\Phi^-)^T = \begin{pmatrix} -e^{\alpha_1} & 1 & 0 & 0 \\ 0 & -e^{\alpha_2} & 1 & 0 \\ 0 & 0 & -e^{\alpha_3} & 1 \\ 1 & 0 & 0 & -e^{\alpha_4} \end{pmatrix}$$

and hence

$$\det \begin{pmatrix} -e^{\alpha_1} & 1 & 0 & 0 \\ 0 & -e^{\alpha_2} & 1 & 0 \\ 0 & 0 & -e^{\alpha_3} & 1 \\ 1 & 0 & 0 & -e^{\alpha_4} \end{pmatrix} = e^{\sum_{j=1}^{4} \alpha_j} - 1.$$

Therefore, the solvability condition is $\sum_{j=1}^{4} \alpha_j \neq 0$. □

REMARK 3.1.1 *For first-order differential equations on a graph G, if one wants to solve uniquely these equations, one needs n many conditions. If the solution is continuous on G, then number of continuity condition of linearly independent in the interior nodes is*

$$\sum_{k=1}^{m} [\#J^+(a_k) + \#J^-(a_k) - 1] = \sum_{k=1}^{m} \#J(a_k) - m.$$

Note that $\sum_{j=1}^{m} \#J(a_j) = 2n$. When $m < n$, one has $\sum_{k=1}^{m} \#J(a_k) - m > n$, this means that there exist so many restrictions on the equations. Therefore, the equations may have no solution. If $m \geq n$, one has

$$\sum_{k=1}^{m} \#J(a_k) - m \leq n.$$

So one needs $m - n$ many initial (or boundary) conditions to determine uniquely a solution.

3.1.2 Flow continuous solution

Here one considers the function satisfying flow continuous condition at all interior nodes and continuity conditions on boundary of G. In this case, the connection relation are given by

$$(\Psi^+)^T\Psi^+U(1) = (\Psi^+)^T U(v), \quad (\Psi^-)^T\Psi^- U(0) = (\Psi^-)^T U(v)$$

or equivalently,

$$\sum_{j\in J^+(a_k)} u_j(1) - \sum_{i\in J^-(a_k)} u_i(0) = 0, \quad \forall a_k \in V_{int}.$$

Since, for each $j \in \{1, 2\cdots, n\}$,

$$u_j(1) = \exp\{\int_0^1 \frac{q_j(s)}{p_j(s)}ds\}u_j(0) + \int_0^1 \frac{f_j(s)}{p_j(s)} \exp\{\int_0^s \frac{q_j(r)}{p_j(r)}dr\}ds = s_j(1,0)u_j(0) + \widehat{f}_j,$$

substituting into the connection conditions lead to

$$\sum_{j\in J^+(a_k)} s_j(1,0)u_j(0) - \sum_{i\in J^-(a_k)} u_i(0) = \sum_{j\in J^+(a_k)} \widehat{f}_j, \quad a_k \in V_{int}$$

or

$$\sum_{j=1}^n [\phi^+_{kj} s_j(1,0) - \phi^-_{kj}]u_j(0) = \sum_{j=1}^n \phi^+_{kj}\widehat{f}_j, \quad a_k \in V_{int}.$$

It has $\#V_{int} = h$ many linearly independent conditions. Without loss of generality one can assume that the first h vertices are the interior nodes, then one has

$$\sum_{j=1}^n [\phi^+_{kj} s_j(1,0) - \phi^-_{kj}]u_j(0) = \sum_{j=1}^n \phi^+_{kj}\widehat{f}_j, \quad k = 1, 2, \cdots, h. \tag{3.1.9}$$

Let $\widehat{U}(0) = (\widehat{u}_1(0), \widehat{u}_2(0), \cdots, \widehat{u}_n(0))$ be a particular solution to (3.1.9), and $U_{i0} = (v_{i1}, v_{i2}, \cdots, v_{in}), i = 1, 2, \cdots, n-h$ be the linearly independent solutions to the homogeneous equations

$$\sum_{j=1}^n [\phi^+_{kj} s_j(1,0) - \phi^-_{kj}]v_j = 0, \quad k = 1, 2, \cdots, h,$$

then the general flow continuous solution to the equation is

$$U(s) = S(s,0)\widehat{U}(0) + \sum_{i=1}^{n-h} c_i S(s,0)U_{i0} + \int_0^s S(s,t)P^{-1}(t)F(t)dt.$$

Therefore, one needs $(n-h)$ many initial conditions if one wants to determine uniquely a solution of flow continuity.

EXAMPLE 3.1.2 *Consider a tree with seven edges as shown in Fig. 3.1.2*

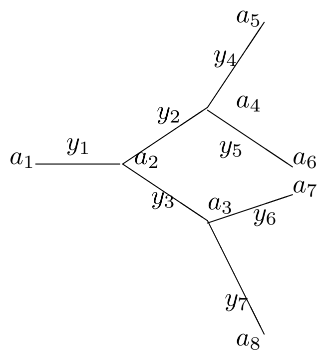

Fig. 3.1.2. A tree-shaped flow network

the differential equation on each edge e_j *is*

$$y_j'(s) - \alpha_j y_j(s) = f_j(s), \quad j \in \{1, 2, \cdots, 7\}$$

By directly solving the equations, one gets the general solution of the form

$$y(x) = \begin{cases} y_1(s) = y_1(0)e^{\alpha_1 s} + \int_0^s e^{\alpha_1(s-t)} f_1(s)ds, & s \in (0,1) \\ y_2(s) = \alpha y_1(1)e^{\alpha_2 s} + \int_0^s e^{\alpha_2(s-t)} f_2(s)ds, & s \in (0,1) \\ y_3(s) = (1-\alpha)y_1(1)e^{\alpha_3 s} + \int_0^s e^{\alpha_3(s-t)} f_3(s)ds, & s \in (0,1) \\ y_4(s) = \beta y_2(1)e^{\alpha_4 s} + \int_0^s e^{\alpha_4(s-t)} f_4(s)ds, & s \in (0,1) \\ y_5(s) = (1-\beta)y_2(1)e^{\alpha_5 s} + \int_0^s e^{\alpha_5(s-t)} f_5(s)ds, & s \in (0,1) \\ y_6(s) = \gamma y_3(1)e^{\alpha_6 s} + \int_0^s e^{\alpha_6(s-t)} f_6(s)ds, & s \in (0,1) \\ y_7(s) = (1-\gamma)y_3(1)e^{\alpha_7 s} + \int_0^s e^{\alpha_7(s-t)} f_7(s)ds, & s \in (0,1) \end{cases}$$

where $y_1(0)$ is an inflow condition, and α, β and γ are stochastic parameters, they present a distribution of the flow at the interior nodes. $\square$

3.1.3 Solution to linearly nodal condition

In this subsection one considers the more complex case that there are many restrictive conditions at each vertex. For each vertex $a \in V_{int}$, one takes the linear nodal conditions as follows

$$\begin{cases} \sum_{j \in J^+(a)} \alpha_{1j} u_j(1) - \sum_{i \in J^-(a)} \beta_{1i} u_i(0) = 0 \\ \sum_{j \in J^+(a)} \alpha_{2j} u_j(1) - \sum_{i \in J^-(a)} \beta_{2i} u_i(0) = 0 \\ \cdots \quad \cdots, \\ \sum_{j \in J^+(a)} \alpha_{hj} u_j(1) - \sum_{i \in J^-(a)} \beta_{hi} u_i(0) = 0 \end{cases} \tag{3.1.10}$$

where $h = h(a)$ satisfying $1 \leq h \leq \#J(a) - 1$.

To obtain a simple form, one rewrites (3.1.10) into following form for $a = a_k$,

$$\begin{cases} \sum_{j=1}^n \alpha_{1j}\phi_{kj}^+ u_j(1) - \sum_{i=1}^n \beta_{1i}\phi_{ki}^- u_i(0) = 0 \\ \sum_{j=1}^n \alpha_{2j}\phi_{kj}^+ u_j(1) - \sum_{i=1}^n \beta_{2i}\phi_{ki}^- u_i(0) = 0 \\ \cdots \quad \cdots, \\ \sum_{j=1}^n \alpha_{hj}\phi_{kj}^+ u_j(1) - \sum_{i=1}^n \beta_{hj}\phi_{ki}^- u_i(0) = 0 \end{cases} \tag{3.1.11}$$

Define matrices

$$A_{a_k} = (\alpha_{rj}\phi_{kj}^+), \quad B_{a_k} = (\beta_{ri}\phi_{ki}^+), \quad A_{a_k} \in \mathbb{M}_{h(a_k)\times n}, B_{a_k} \in \mathbb{M}_{h(a_k)\times n}$$

Then the linear nodal conditions (3.1.11) can be written as

$$A_{a_k}U(1) - B_{a_k}U(0) = 0, \quad a_k \in V_{int}$$

where $rank(A_{a_k}, B_{a_k}) = h(a_k)$. Using representation of the general solution

$$U(s) = S(s,0)U(0) + \int_0^s S(s,t)P^{-1}(t)F(t)dt,$$

one gets

$$A_{a_k}S(1,0)U(0) - B_{a_k}U(0) = A_{a_k}\widehat{F}(1), \quad a_k \in V_{int}. \tag{3.1.12}$$

Thus one has $p = \sum_{a_k \in V_{int}} h(a_k)$ many number of independent equations. Note that $1 \leq h(a_k) \leq \#J(a_k) - 1$, so

$$\#V_{int} \leq p = \sum_{a_k \in V_{int}} h(a_k) \leq \sum_{a_k \in V_{int}} [\#J(a_k) - 1] = 2n - m.$$

Let $q = \#V_{int}$. Without loss of generality one can assume that $a_k, k = 1, 2, \cdots, q$ are the interior nodes. Set

$$C = \begin{pmatrix} A_{a_1}S(1,0) - B_{a_1} \\ A_{a_2}S(1,0) - B_{a_2} \\ \cdots\cdots \\ A_{a_q}S(1,0) - B_{a_q} \end{pmatrix}_{p\times n}, \quad C_1 = \begin{pmatrix} A_{a_1} \\ A_{a_2} \\ \cdots \\ A_{a_q} \end{pmatrix}_{p\times n},$$

one has algebraic equations

$$CU(0) = C_1\widehat{F}(1).$$

Clearly, if $p > n$, then the algebraic equations might have no solution due to $rank(C) \leq \min\{p, n\}$. This means that there exist too many restrictions on the equations. If $p = n$, then the algebraic equations have unique a solution

$$U(0) = C^{-1}C_1\widehat{F}(1).$$

This shows that one need not have any initial or boundary condition. In this case, the unique a solution is given by

$$U(s) = S(s,0)C^{-1}C_1\widehat{F}(1) + \int_0^s S(s,t)P^{-1}(t)F(t)dt.$$

If $q \le p < n$, then the homogeneous algebraic equations have $(n-p)$ many linearly independent solutions. Let $\widehat{U}$ be a particular solution to (3.1.12), and $U_{i0}, i = 1, 2 \cdots n-p$ be the independent solutions to the homogeneous equations, i.e., $U_{i0} \in \mathcal{N}(C)$, then the general solution of the algebraic equations (3.1.12) are given by

$$U(0) = \widehat{U} + \sum_{j=1}^{n-p} c_j U_{j0}.$$

Therefore, the general solution to (3.1.1) satisfying the equations (3.1.10) are

$$U(s) = S(s,0)\widehat{U} + \sum_{i=1}^{n-p} c_j S(s,0) U_{i0} + \int_0^s S(s,t) P^{-1}(t) F(t) dt.$$

To determine uniquely a solution to (3.1.1) under the condition (3.1.10), one needs $(n-p)$ many boundary conditions.

REMARK 3.1.2 *The results in this section show that the solvability of first-order ordinary differential equation on graph G is essentially different from that on an interval. The solvability is strongly dependent upon the structure of the graph G and continuity assumption of the solution at the junction. Therefore, one must treat carefully the differential equations on a graph.*

3.2 Second order differential equations

Let G be a metric graph with vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and edge set $E = \{e_1, e_2, \cdots, e_n\}$. In this section one considers second-order ordinary differential equations on G. For each edge $e_j \in E$, the differential equation is given by

$$u_{j,ss}(s) + p_j(s) u_{j,s} + q_j(s) u_j(s) = f_j(s), \quad s \in (0,1). \tag{3.2.1}$$

Let $\varphi_j(s)$ and $\psi_j(s)$ be the fundamental solutions to homogeneous equation

$$u_{j,ss}(s) + p_j(s) u_{j,s}(s) + q_j(s) u_j(s) = 0$$

satisfying conditions $\varphi_j(0) = 1, \varphi_j'(0) = 0$ and $\psi_j(0) = 0, \psi_j'(0) = 1$, respectively. Then the general solution to (3.2.1) and its differential are given by

$$u_j(s) = u_j(0)\varphi_j(s) + u_j'(0)\psi_j(s) - \int_0^s S_j(s,r) f_j(r) dr$$

$$u_j'(s) = u_j(0)\varphi_j'(s) + u_j'(0)\psi_j'(s) - \int_0^s \partial_s S_j(s,r) f_j(r) dr$$

where

$$S_j(s,t) = \frac{\psi_j(s)\varphi_j(t) - \varphi_j(s)\psi_j(t)}{\psi_j'(t)\varphi_j(t) - \varphi_j'(t)\psi_j(t)}.$$

Obviously, one needs $2n$ many conditions to determine uniquely a solution to (3.2.1).

3.2.1 Continuous solution on G

Suppose that $u(x)$ is continuous on G, i.e., $u \in C(G)$, then one has

$$u_j(a) = u_i(a) = u(a), \quad \forall i \in J^-(a), j \in J^+(a), \quad a \in V_{int}.$$

The continuity implies the condition number

$$\sum_{j=1}^{m}[\#J(a) - 1] = 2n - m,$$

it remains m many conditions. Therefore, one proposes at most one condition for its derivative at each vertex $a \in V$.

In order to obtain the connective condition of derivative functions at each vertex $a \in V_{int}$, let $O(a)$ be a neighborhood of a. We define the function class $C_0^\infty(O(a))$ with support set in $O(a)$ by

$$v(x) = \begin{cases} 1, & |x-a| \leq \varepsilon, \\ \hat{v}(x), & \varepsilon \leq |x-a| \leq 2\varepsilon \\ 0, & |x-a| \geq 2\varepsilon. \end{cases}$$

where $\hat{v}(x)$ is a C^∞ function on $O(a) \cap e_k$ provided that e_k is an edge jointed a. Now let $v(x)$ be a function in $C_0^\infty(O(a))$. Then integrating over $O(a)$ yields

$$\begin{aligned} & \int_{O(a)\cap G} (u_{xx}(x)) + p(x)u_x(x) + q(x)u(x))v(x)dx \\ = & \sum_{j\in J^+(a)} [u_{js}(a)v_j(a)] - \sum_{j\in J^-(a)} [u_{j,s}(a)v_j(a)] \\ & - \int_{O(a)\cap G} [u_x(x)v_x(x) - p(x)u_x(x)v(x) - q(x)u(x)v(x)]dx \\ = & v(a)\left[\sum_{j\in J^+(a)} u_{js}(a) - \sum_{i\in J^-(a)} u_{i,s}(a)\right] \\ & - \int_{O(a)\cap G} [u_x(x)v_x(x) - p(x)u_x(x)v(x) - q(x)u(x)v(x)]dx. \end{aligned}$$

So a natural condition at each $a \in V_{int}$ is

$$\sum_{j\in J^+(a)} u_{js}(a) - \sum_{i\in J^-(a)} u_{i,s}(a) = 0.$$

If $\#V_{int} \neq m$, then one only needs to impose $m - \#V_{int} = \#\partial G$ many conditions at the boundary of G.

The discussion above gives merely a correct condition number for the differential equations (3.2.1) having a continuous solution. However, it is not a sufficient condition for the solvability. In what follows, one will discuss the solvability of the differential equations.

More generally one assumes that the node conditions are

$$\sum_{j\in J^+(a_k)} \alpha_{k,j}\phi_{kj}^+ u_{j,s}(1) - \sum_{i\in J^-(a_k)} \beta_{k,i}\phi_{ki}^- u_{i,s}(0) + \gamma_k u(a_k) = 0, \quad k = 1, 2, \cdots, m \qquad (3.2.2)$$

where $\alpha_{k,j}$,$\beta_{k,j}$ and γ_k are constants.

Now let

$$U(s) = (u_1(s), u_2(s), \cdots, u_n(s)), \quad U(v) = (u(a_1), u(a_2), \cdots, u(a_m)).$$

Then the continuity conditions are

$$U(1) = (\Phi^+)^T U(v), \quad U(0) = (\Phi^-)^T U(v),$$

and the conditions (3.2.2) can be written into

$$AU'(1) - BU'(0) + \Gamma U(v) = 0, \quad A \in \mathbb{M}_{m\times n}(\Phi^+), B \in \mathbb{M}_{m\times n}(\Phi^-) \tag{3.2.3}$$

where Γ is the $m \times m$ diagonal matrix, and $rank(A, B, \Gamma) = m$.

Note that the general solutions of (3.2.1) can rewrite into the vector form

$$U(s) = \Phi(s)U(0) + \Psi(s)U'(0) - \int_0^s \mathcal{S}(s,t)F(t)dt$$

where

$$\Phi(s) = \mathrm{diag}(\varphi_1(s), \varphi_2(s), \cdots, \varphi_n(s)), \quad \Psi(s) = \mathrm{diag}(\psi_1(s), \psi_2(s), \cdots, \psi_n(s))$$

and

$$\mathcal{S}(s,t) = \mathrm{diag}(S_1(s,t), S_2(s,t), \cdots, S_n(s,t)).$$

Thus one has

$$\begin{cases} [(\Phi^+)^T - \Phi(1)(\Phi^-)^T]U(v) - \Psi(1)U'(0) = \widehat{S}(1) \\ [A\Phi'(1)(\Phi^-)^T + \Gamma]U(v) + [A\Psi'(1) - B]U'(0) = A\widehat{S}'(1) \end{cases} \tag{3.2.4}$$

where $\Phi^\pm$ are the incidence matrices of G, and

$$\widehat{S}(1) = \int_0^1 S(1,t)F(t)dt, \quad \widehat{S}'(1) = \int_0^1 \partial_s S(1,t)F(t)dt.$$

Therefore, the differential equations have unique a solution if and only if the determinant of coefficients matrix of (3.2.4) is not vanishing, i.e.,

$$\begin{vmatrix} [(\Phi^+)^T - \Phi(1)(\Phi^-)^T] & -\Psi(1) \\ [A\Phi'(1)(\Phi^-)^T + \Gamma] & [A\Psi'(1) - B] \end{vmatrix} \neq 0 \tag{3.2.5}$$

Therefore, one has the following result.

THEOREM 3.2.1 *Let differential equations be given as in (3.2.1). Then the equations have a continuous solution on G satisfying the condition (3.2.3) if and only if (3.2.5) is satisfied.*

3.2.2 Linearly nodal conditions

Here one considers more general connection conditions for the equations (3.2.1). For each node $a \in V$, there are $\#J(a)$ many edges jointed it. For each $f \in H^2(E)$, the function at vertex a forms a vector

$$\widehat{f}(a) = (f_{j_1}(a), f_{j_2}, \cdots, f_{j_d}(a); f'_{j_1}(a), f'_{j_2}, \cdots, f'_{j_d}(a))^T$$

where $j_k \in J(a)$, $d = \#J(a)$ is the degree of vertex a and $f^{(i)}_{j_k}(a) = \lim_{s\to 1} f^{(i)}_{j_k}(s)$ if $j_k \in J^+(a)$ or $f^{(i)}_{j_k}(a) = \lim_{s\to 0} f^{(i)}_{j_k}(s)$ if $j_k \in J^-(a)$. These vectors form a $2d(a)$-dimensional linear space. One can define a functional on $H^2(E)$, by

$$\beta_a(f) = \sum_{r=1}^{d} (\alpha_r f_{j_r}(a) + \beta_r f'_{j_r}(a)), \quad f \in H^2(E).$$

If one distinguishes the incoming edges and the outgoing edges, then it can be rewritten as

$$\beta_a(f) = \sum_{r=1}^{p} \alpha_r f_{j_r}(1) + \sum_{s=1}^{q} b_s f_{j_s}(0) + \sum_{r=1}^{p} c_r f'_{j_r}(1) + \sum_{s=1}^{q} \beta_s f'_{j_s}(0)$$

where $p = \#J^+(a)$ and $q = \#J^-(a)$. There are at most $2d(a)$ many linearly independent functionals at the vertex $a \in V$. From discussion of continuous solution one sees that one needs at most $\#J(a) = p + q$ many conditions to determine uniquely a solution. So one can choose $\#J(a) = d$ many linearly independent functionals $\beta_{a,1}, \beta_{a,2}, \cdots, \beta_{a,d}$ as the restriction conditions:

$$\beta_{a,k}(f) = 0, \quad k = 1, 2, \cdots, d(a)$$

or simply write into an operator from $H^2(E)$ to $\mathbb{C}^{d(a)}$

$$\mathcal{B}_a(f) = B_a\widehat{f}(a) = B_{1,a}\widehat{f}(a) + B_{2,a}\widehat{f}'(a) = 0.$$

Further, one uses the vector-valued form:

$$F(s) = (f_1(s), f_2(s), \cdots, f_n(s))^T,$$

the restrictive condition can be written into

$$\mathcal{B}_a(f) = A_aF(1) + B_aF(0) + C_aF'(1) + D_aF'(0) = 0, \tag{3.2.6}$$

where A_a, B_a, C_a, D_a are the $d(a) \times n$ matrices and $rank(A_a, B_a, C_a, D_a) = d(a) = \#J(a)$. Therefore, one gets $\sum_k^m \#J(a_k) = 2n$ many linearly independent conditions:

$$\mathcal{B}_{a_k}(f) = 0, \quad k = 1, 2, \cdots, m. \tag{3.2.7}$$

Therefore, the equations (3.2.1) with restrictive conditions (3.2.7) have unique a solution for each $f \in H^2(E)$ if and only if the algebraic equations

$$[A_a\Phi(1) + C_a\Phi'(1) + B_a]U(0) + [A_a\Psi(1) + C_a\Psi'(1) + D_a]U'(0) = 0, \quad \forall a \in V. \tag{3.2.8}$$

have unique an zero solution. Therefore, one has the following result.

THEOREM 3.2.2 *The second-order differential equations (3.2.1) defined on E under the restrictive conditions*

$$\mathcal{B}_a(f) = A_a U(1) + B_a U(0) + C_a U'(1) + D_a U'(0) = 0, \quad rank(A_a, B_a, C_a, D_a) = \#J(a)$$

have unique a solution for each $f \in H^2(E)$ if and only if the algebraic equations

$$[A_a \Phi(1) + C_a \Phi'(1) + B_a]U(0) + [A_a \Psi(1) + C_a \Psi'(1) + D_a]U'(0) = 0, \quad \forall a \in V.$$

have unique an zero solution.

Now let us recall the structural matrix Ψ of the graph G,

$$\Psi = \begin{pmatrix} (\Phi^+)^T \Phi^+ & (\Phi^+)^T \Phi^- \\ (\Phi^-)^T \Phi^+ & (\Phi^-)^T \Phi^- \end{pmatrix}_{2n \times 2n}.$$

One decomposes the matrix Ψ as $\Psi = (\Psi^+, \Psi^-)$, where Ψ^+ and Ψ^- are the $2n \times n$ matrices. In order to normalize the conditions (3.2.7), one arranges the conditions in the following way

$$\sum_{r=1}^{n} \alpha_{j,r} f_r(1) + \sum_{r=1}^{r} b_{j,s} f'_r(1) + \sum_{s=1}^{n} c_{j,r} f_r(0) + \sum_{s=1}^{n} d_{j,s} f'_s(0) = 0, \quad j = 1, 2, \cdots, n, n+1, \cdots, 2n$$

such that they are embedded the structure matrix pattern

$$\begin{pmatrix} (\Phi^+)^T \Phi^+ & (\Phi^+)^T \Phi^+ & (\Phi^+)^T \Phi^- & (\Phi^+)^T \Phi^- \\ (\Phi^-)^T \Phi^+ & (\Phi^-)^T \Phi^+ & (\Phi^-)^T \Phi^+ & (\Phi^-)^T \Phi^- \end{pmatrix} = (\Psi^+, \Psi^+, \Psi^-, \Psi^-).$$

Setting

$$A = (\alpha_{j,r})_{2n \times n}, \quad B = (b_{j,r})_{2n \times n}, \quad C = (c_{j,s})_{2n \times n}, \quad D = (d_{j,s})_{2n \times n},$$

one has

$$A, B \in \mathbb{M}_{2n \times n}(\Psi^+) = \{A = (a_{ji}) \in \mathbb{M}_{2n \times n} \mid \Psi^+ \bullet A = A\}$$

and

$$C, D \in \mathbb{M}_{2n \times n}(\Psi^-) = \{C = (c_{ji}) \in \mathbb{M}_{2n \times n} \mid \Psi^- \bullet C = C\}$$

where $A \bullet B$ denotes the Hadamard product which is defined by $A \bullet B = (a_{ij} b_{ij})$. Thus the conditions (3.2.7) can be rewritten into

$$(A\ B\ C\ D) \begin{pmatrix} F(1) \\ F'(1) \\ F(0) \\ F'(0) \end{pmatrix} = 0$$

and $rank(A, B, C, D) = 2n$. This form is said to be the normalized the connected condition.

3.3 Second-order differential operator and its adjoint

It is well known that the solvability of the second-order differential equation on finite interval is closely related to the linear operator determined by the equation. The Fredholm theory of linear differential operators defined on the finite interval says that the solvability determined via its adjoint operator. In developing physical models one often needs to know its adjoint state when a differential operator is defined on a given graph, for instance, quantum-mechanical problems associated with advances in micro-electronic fabrication [2][36][37][40]. This section provides a description of adjoint operator of second-order differential operator defined on a graph G.

Let G be a metric graph with vertex set V and the edge set $E=\{e_1,e_2,\cdots,e_n\}$. Suppose that the formal differential operator on each e_j is defined by

$$\mathcal{L}u_j=u_{j,ss}(s)+p_j(s)u_{j,s}(s)+q_j(s)u_j(s),\quad s\in(0,1),\quad j\in\{1,2,\cdots,n\} \tag{3.3.1}$$

where p_j and q_j are real continuous functions.

For each $u\in H^2(E)$, one coincides it with the vector-valued function

$$U(s)=(u_1(s),u_2(s),\cdots,u_n(s)),\quad s\in(0,1).$$

Define diagonal matrices

$$P(s)=\mathrm{diag}(p_1(s),p_2(s),\cdots,p_n(s)),\quad Q(s)=\mathrm{diag}(q_1(s),q_2(s),\cdots,q_n(s))$$

Then the formal differential operator $\mathcal{L}$ can be rewritten into

$$\mathcal{L}U(s)=U''(s)+P(s)U'(s)+Q(s)U(s), \tag{3.3.2}$$

and the space $L^2(G)$ is equivalent to $L^2[(0,1),\mathbb{C}^n]$.

Suppose that the vertex connection conditions are given by

$$AU(1)+BU'(1)+CU(0)+DU'(0)=0,\quad rank(A,B,C,D)=2n, \tag{3.3.3}$$

this is a normalized connection condition. Thus the formal differential operator (3.3.2) together with (3.3.3) define an operator on G:

$$D(\mathcal{L})=\{U\in H^2((0,1),\mathbb{C}^n)\bigm| AU(1)+BU'(1)+CU(0)+DU'(0)=0\}. \tag{3.3.4}$$

For any $U\in D(\mathcal{L})$, $F\in H^2((0,1),\mathbb{C}^n)$, we have

$$\begin{aligned}
(\mathcal{L}U,F)_{L^2} &= \int_0^1 (U''(s)+P(s)U'(s)+Q(s)U(s),F(s))_{\mathbb{C}^n}ds\\
&= (U'(1),F(1))_{\mathbb{C}^n}-(U'(0),F(0))_{\mathbb{C}^n}-(U(1),F'(1))_{c^n}+(U(0),F'(0))_{\mathbb{C}^n}\\
&\quad +(P(1)U(1),F(1))_{\mathbb{C}^n}-(P(0)U(0),F(0))_{\mathbb{C}^n}\\
&\quad +\int_0^1 (U(s),F''(s)-(P(s)F(s))'+Q(s)F(s))_{\mathbb{C}^n}ds\\
&= -(U(1),F'(1)-P(1)F(1))_{\mathbb{C}^n}+(U'(1),F(1))_{\mathbb{C}^n}\\
&\quad +(U(0),F'(0)-P(0)F(0))_{\mathbb{C}^n}-(U'(0),F(0))_{\mathbb{C}^n}\\
&\quad +\int_0^1 (U(s),F''(s)-(P(s)F(s))'+Q(s)F(s))_{\mathbb{C}^n}ds.
\end{aligned}$$

Obviously, $\mathcal{L}^*$ on E is of the form

$$\mathcal{L}^* F = F''(s) - (P(s)F(s))' + Q(s)F(s) \tag{3.3.5}$$

and $F \in D(\mathcal{L}^*)$ if and only if

$$\begin{aligned} 0 &= (U(1), -F'(1) + P(1)F(1))_{\mathbb{C}^n} + (U'(1), F(1))_{\mathbb{C}^n} \\ &\quad + (U(0), F'(0) - P(0)F(0))_{\mathbb{C}^n} + (U'(0), -F(0))_{\mathbb{C}^n}, \quad \forall U \in D(\mathcal{L}). \end{aligned}$$

On the other hand, we get from $U \in D(\mathcal{L})$ that

$$(U(1), A^* X)_{\mathbb{C}^n} + (U'(1), B^* X)_{\mathbb{C}^n} + (U(0), C^* X)_{\mathbb{C}^n} + (U'(0), D^* X)_{\mathbb{C}^n} = 0, \quad \forall X \in \mathbb{C}^{2n}.$$

Thus there is an $X \in \mathbb{C}^{2n}$ such that

$$A^* X = -F'(1) + P(1)F(1), \quad B^* X = F(1), \quad C^* X = F'(0) - P(0)F(0), \quad D^* X = -F(0).$$

Suppose that the restriction condition in domain of $\mathcal{L}^*$ is of the form

$$\widehat{A}F(1) + \widehat{B}F'(1) + \widehat{C}F(0) + \widehat{D}F'(0) = 0$$

where $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D} \in \mathbb{M}_{2n \times n}$, then

$$\begin{aligned} 0 &= \widehat{A}F(1) + \widehat{B}F'(1) + \widehat{C}F(0) + \widehat{D}F'(0) \\ &= \widehat{A}[B^* X] + \widehat{B}[-A^* X + P(1)B^* X] + \widehat{C}[-D^* X] + \widehat{D}[C^* X - P(0)D^* X] \\ &= [\widehat{A}B^* - \widehat{B}A^* + \widehat{B}P(1)B^* + \widehat{D}C^* - \widehat{C}D^* - \widehat{D}P(0)D^*]X. \end{aligned}$$

Therefore, A, B, C, D and $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ satisfy relation

$$\widehat{A}B^* - \widehat{B}A^* + \widehat{B}P(1)B^* + \widehat{D}C^* - \widehat{C}D^* - \widehat{D}P(0)D^* = 0. \tag{3.3.6}$$

Note that if we denote the formal differential operator (3.3.5) by $\mathcal{L}^+$, then we have

$$(\mathcal{L}U, F)_{L^2} - (U, \mathcal{L}^+ F)_{L^2} = [U, F]$$

where $[U, F]$ is a non-degenerate quadratic form involving the boundary values and first-order derivatives of functions U and F. Formula (3.3.6) gives a condition such that $[U, F] \equiv 0$. The following example shows an expression of non-degenerate quadratic form $[U, F]$ in single interval.

EXAMPLE 3.3.1 *Consider second order differential operator defined on* (a, b) *by*

$$\mathcal{L}f = f''(x) + p(x)f'(x) + q(x)f(x), \quad x \in (a, b)$$

where $p(x), q(x)$ *are real continuous functions on* $[a, b]$*. For any* $f, g \in H^2[a, b]$*, one has*

$$\begin{aligned} &\int_a^b [\mathcal{L}f(x)\overline{g(x)} - f(x)\overline{\mathcal{L}^+ g(x)}]dx \\ = \; & f'(b)\overline{g(b)} - f'(a)\overline{g(a)} + f(a)\overline{g'(a)} - f(b)\overline{g'(b)} \\ & + f(b)p(b)\overline{g(b)} - f(a)p(a)\overline{g(a)} \\ = \; & \left\langle \begin{pmatrix} p(b) & 1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} f(b) \\ f'(b) \end{bmatrix}, \begin{bmatrix} g(b) \\ g'(b) \end{bmatrix} \right\rangle_{\mathbb{C}^2} - \left\langle \begin{pmatrix} p(a) & 1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} f(a) \\ f'(a) \end{bmatrix}, \begin{bmatrix} g(a) \\ g'(a) \end{bmatrix} \right\rangle_{\mathbb{C}^2} \end{aligned}$$

so we have

$$[f,g] = \left\langle \begin{pmatrix} p(b) & 1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} f(b) \\ f'(b) \end{bmatrix}, \begin{bmatrix} g(b) \\ g'(b) \end{bmatrix} \right\rangle_{\mathbb{C}^2} - \left\langle \begin{pmatrix} p(a) & 1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} f(a) \\ f'(a) \end{bmatrix}, \begin{bmatrix} g(a) \\ g'(a) \end{bmatrix} \right\rangle_{\mathbb{C}^2}.$$

Note that the matrices

$$S(x) = \begin{pmatrix} p(x) & 1 \\ -1 & 0 \end{pmatrix}, \quad x \in (a,b)$$

are skew-adjoint on $\mathbb{C}^2$*. So* $[f,g]$ *is also skew-symmetric quadratic form.*

Note that the connective conditions (3.3.3) do not obviously depend on the structure of G. Sometimes one use the following notion to distinguish property of $\mathcal{L}$ at each vertex.

DEFINITION 3.3.1 *Let* G *be a metric graph and let* $\phi : G \to \mathbb{C}$ *denote a* C^∞ *function which has compact support in* G *and is constant in an open neighborhood of each vertex. Let* $\mathcal{L}$ *be a formal differential on the edges of* G*.* $\mathcal{L}$ *is said to be local operator if for every* ϕ*,* ϕf *is in the domain of* $\mathcal{L}$ *whenever* f *is.*

When G is a finite graph, the continuity conditions and linearly nodal conditions at internal nodes are the local property. Hence the operator determined by these conditions are local operators. As shown in subsection 3.2.1, if $\mathcal{L}$ is a local operator on the graph G, then one can treat it in small open neighborhood of vertex a as an operator defined on a star-shaped graph with zero boundary conditions.

EXAMPLE 3.3.2 Schrödinger operators on graphs *Let* $G = (V,E)$ *be a metric graph. On each edge* $e_j \in E$*, Schrödinger operator is defined as*

$$\mathcal{L}f_j(x) = -f_j''(x) + p_j(x)f_j(x), \quad x \in (0,1)$$

where we assume the continuous vertex conditions, i.e., there are $\#J(a) - 1$ *independent conditions at each vertex* a*,*

$$f_j(a) = f_i(a), \quad j \in J^+(a),\ i \in J^-(a)$$

and an independent dynamic condition

$$\sum_{j \in J^+(a)} f_j'(1) - \sum_{k \in J^-(a)} f_k'(0) = 0. \tag{3.3.7}$$

$\mathcal{L}$ *is a self-adjoint operator in* $L^2(G)$*. In particular,* L *is a local operator on* G*.*

REMARK 3.3.1 *More discussion about Schrödinger operator on graphs (Quantum graphs), we refer to Kuchment's works, e.g., see [64] and [65].*

REMARK 3.3.2 *There is a classical description of adjoint operator (involving self-adjoint extensions) in terms of boundary conditions [25, pp.284–297] for regular ordinary differential operators acting on* $L^2[a,b]$*. This theory has a close connection with the abstract treatment of self-adjoint extensions of symmetric operators [101, pp. 140–141]. Note that the differential* *operator $\mathcal{L}$ defined on a graph G is actually a group of ordinary differential operators on intervals whose lengths are given by the arc lengths of the edges in graph G. The general treatment is somewhat deficient for differential operators on graphs, since the role of the vertices of the graph G is unclear. In particular, when G is an infinite graph, the description of the differential operator defined on it appears particularly awkward. Usually one need to impose additional restrictions on the domain of $\mathcal{L}$. Under the assumption that the edge lengths of G have a positive lower bound and each vertex receives at most finite many edges, Robert Carlson in [20] gave a complete character for the domain of the adjoint of a local operator $\mathcal{L}$ when the coefficients of the operator are bounded and satisfy some mild additional regularity assumptions.*

Chapter 4

Partial Differential Equations on Graphs

In this chapter one discusses three classes of elastic systems on metric graphs: one dimensional wave equation, Euler-Bernoulli beam and Timoshenko beam. For these systems, our purpose is to find out reasonable vertex conditions so that these systems describe the physical phenomena, which means that these systems are the energy conservation without exterior disturbance and forces. In this chapter, one will give a correct number of independent conditions at each vertex and the most general form of the local vertex conditions and whole vertex conditions according to the nodal equilibrium and structure equilibrium of the systems, respectively.

4.1 wave equation on graph

Let G be a metric graph with vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and the edge set $E = \{e_1, e_2, \cdots, e_n\}$ with $|e_j| = \ell_j$. Let $u(x,t)$ be a function defined on $G \times \mathbb{R}_+$, $u_j(x,t)$ be its parameterization realization on $e_j \times \mathbb{R}_+$. If $u_j(x,t), j \in \{1, 2, \cdots, n\}$, satisfy the partial differential equation

$$m_j(x)\frac{\partial^2 u_j(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(T_j(x)\frac{\partial u_j(x,t)}{\partial x}\right) - q_j(x)u_j(x,t), \quad x \in (0, \ell_j), \tag{4.1.1}$$

where $m_j(x)$, $T_j(x)$ are positive continuous function and $q_j(x)$ are nonnegative continuous functions, then $u(x,t)$ is called satisfying the wave equation on E.

REMARK 4.1.1 *For a function $u(x,t)$ satisfied the wave equation, we can define its normalized realization on e_j by*

$$w_j(s,t) = u_j(s\ell_j, t), \quad s \in (0,1).$$

Then we have

$$\frac{\partial w_j(s,t)}{\partial s} = \ell_j \frac{\partial u_j(s\ell_j, t)}{\partial x}$$

$$\frac{\partial}{\partial x}\left(T_j(s\ell_j)\frac{\partial u_j(s\ell_j,t)}{\partial x}\right) = \frac{1}{\ell_j}\frac{\partial(T_j(s\ell_j)u_{j,x}(s\ell_j,t))}{\partial s} = \frac{1}{\ell_j^2}\frac{\partial}{\partial s}\left(T_j(s\ell_j)\frac{\partial w_j(s,t)}{\partial s}\right).$$

So without loss of generality we can assume that $w_j(s,t)$ satisfies the wave equation

$$m_j(s)\frac{\partial^2 w_j(s,t)}{\partial t^2} = \frac{\partial}{\partial s}\left(T_j(s)\frac{\partial w_j(s,t)}{\partial s}\right) - q_j(s)w_j(s,t), \quad s\in(0,1). \tag{4.1.2}$$

The function $w_j(s,t)$ is called the normalized realization of $u(x,t)$. In the sequel, we always use the normalized realization of a function.

The partial differential equations are always defined on E, we need some connective and boundary conditions and initial data to determine uniquely a solution.

4.1.1 Nodal condition

Let G be a metric graph with vertex set $V=\{a_1,a_2,\cdots,a_m\}$ and edge set $E=\{e_1,e_2,\cdots,e_n\}$. We consider the self-adjoint property of the differential operator in $L^2(G)$

$$\mathcal{L}w_j = (T_j(s)w_{j,s}(s))_s - q_j(s)w_j(s), \quad s\in(0,1), \quad j=1,2,\cdots,n; \tag{4.1.3}$$

For any $F\in H^2(E)$, $W\in H^2(E)$,

$$(\mathcal{L}W,F)_{L^2} - (W,\mathcal{L}F)_{L^2} = \sum_{j=1}^n\int_{e_j}(T_j(s)w_{j,s}(s))_s\overline{f_j(s)}ds - \sum_{k=1}^n\int_{e_k} w_k(s)\overline{(T_k(s)f_{k,s})_s}ds$$

$$= \sum_{j=1}^n T_j(1)[w_{j,s}(1)\overline{f_j(1)} - w_j(1)\overline{f_{j,s}(1)}] - \sum_{j=1}^n T_j(0)[w_{j,s}(0)\overline{f_j(s)} - w_j(0)\overline{f_{j,s}(0)}]$$

$$= \sum_{i=1}^m\left(\sum_{j\in J^+(a_i)} T_j(1)[w_{j,s}(1)\overline{f_j(1)} - w_j(1)\overline{f_{j,s}(1)}] - \sum_{k\in J^-(a_i)} T_k(0)[w_{k,s}(0)\overline{f_k(0)} - w_k(0)\overline{f_{k,s}(0)}]\right)$$

Set $J^+(a_i)=\{j_1,j_2,\cdots,j_p\}$, $J^-(a_i)=\{k_1,k_2,\cdots,k_q\}$. For each $W\in H^2(E)$, one define local column vectors at a_i by

$$W_+(a_i) = [w_{j_1}(1),w_{j_2}(1),\cdots,w_{j_p}(1)]^T, \quad W_-(a_i) = [w_{k_1}(0),w_{k_2}(0),\cdots,w_{k_q}(0)]^T,$$

and

$$W'_+(a_i) = [w_{j_1,s}(1),w_{j_2,s}(1),\cdots,w_{j_p,s}(1)]^T, \quad W'_-(a_i) = [w_{k_1,s}(0),w_{k_2,s}(0),\cdots,w_{k_q,s}(0)]^T.$$

Define the diagonal matrices

$$\begin{cases} T_+(a_i) = \mathrm{diag}(T_{j_1}(1),T_{j_2}(1),\cdots,T_{j_p}(1)),\\ T_-(a_i) = \mathrm{diag}(T_{k_1}(0),T_{k_2}(0),\cdots,T_{k_q}(0))\end{cases} \tag{4.1.4}$$

and the bond matrices

$$\mathcal{Q}_+(a_i) = \begin{bmatrix} 0 & T_+(a_i)\\ -T_+(a_i) & 0\end{bmatrix}_{2p\times 2p} \tag{4.1.5}$$

$$\mathcal{Q}_-(a_i) = \begin{bmatrix} 0 & T_-(a_i) \\ -T_-(a_i) & 0 \end{bmatrix}_{2q\times 2q}. \tag{4.1.6}$$

With help of these notations, we can write the vertex condition into following

$$\begin{aligned}
&\left(\sum_{j\in J^+(a_i)} T_j(1)[w_{j,s}(1)\overline{f_j(1)} - w_j(1)\overline{f_{j,s}(1)}] - \sum_{k\in J^-(a_i)} T_k(0)[w_{k,s}(0)\overline{f_k(0)} - w_k(0)\overline{f_{k,s}(0)}]\right)\\
=\;& \big(T_+(a_i)W'_+(a_i), F_+(a_i)\big)_{\mathbb{C}^p} - \big(T_+(a_i)W_+(a_i), F'_+(a_i)\big)_{\mathbb{C}^p}\\
&- \big(T_-(a_i)W'_-(a_i), F_-(a_i)\big)_{\mathbb{C}^q} + \big(T_-(a_i)W_-(a_i), F'_-(a_i)\big)_{\mathbb{C}^q}\\
=\;& \left(\begin{bmatrix} 0 & T_+(a_i) \\ -T_+(a_i) & 0 \end{bmatrix}\begin{bmatrix} W_+(a_i) \\ W'_+(a_i)\end{bmatrix}, \begin{bmatrix} F_+(a_i) \\ F'_+(a_i)\end{bmatrix}\right)_{\mathbb{C}^{2p}}\\
&- \left(\begin{bmatrix} 0 & T_-(a_i) \\ -T_-(a_i) & 0 \end{bmatrix}\begin{bmatrix} W_-(a_i) \\ W'_-(a_i)\end{bmatrix}, \begin{bmatrix} F_-(a_i) \\ F'_-(a_i)\end{bmatrix}\right)_{C^{2q}}\\
=\;& \big(\mathcal{Q}_+(a_i)[W_+, W'_+]^T(a_i), [F_+, F'_+]^T(a_i)\big)_{\mathbb{C}^{2p}} - \big(\mathcal{Q}_-(a_i)[W_-, W'_-]^T(a_i), [F_-, F'_-]^T(a_i)\big)_{\mathbb{C}^{2q}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\mathcal{L}W, F)_{L^2} - (W, \mathcal{L}F)_{L^2} \;=\;& \sum_{i=1}^m \big(\mathcal{Q}_+(a_i)[W_+, W'_+]^T(a_i), [F_+, F'_+]^T(a_i)\big)\\
&-\sum_{i=1}^m \big(\mathcal{Q}_-(a_i)[W_-, W'_-]^T(a_i), [F_-, F'_-]^T(a_i)\big). \qquad (4.1.7)
\end{aligned}$$

4.1.2 Nodal condition for the dynamic equilibrium

THEOREM 4.1.1 *Let the formal differential operator $\mathcal{L}$ be defined by (4.1.3), and let $\mathcal{Q}_+(a)$ and $\mathcal{Q}_-(a)$ be defined by (4.1.5) and (4.1.6), respectively. Then the following statements are true*

1) For each node $a \in V$, there are $p + q = \#J(a)$ many linearly independent conditions, which have the form

$$A_a[W_+(a), W'_+(a)]^T + B_a[W_-(a), W'_-(a)]^T = 0 \tag{4.1.8}$$

where $A_a = A_{(p+q)\times 2p}$ and $B_a = B_{(p+q)\times 2q}$;

2) The operator $\mathcal{L}$ with nodal conditions (4.1.8) is nodal equilibrium, i.e.,

$$\left(\mathcal{Q}_+(a)\widehat{W}_+(a), \widehat{F}_+(a)\right)_{\mathbb{C}^{2p}} - \left(\mathcal{Q}_-(a)\widehat{W}_-(a), \widehat{F}_-(a)\right)_{\mathbb{C}^{2q}} = 0 \tag{4.1.9}$$

where $\widehat{W}_+(a) = [W_+(a), W'_+(a)]^T$ and $\widehat{W}_-(a) = [W_-(a), W'_-(a)]^T$, if and only if A_a and B_a satisfy the condition

$$A_a\mathcal{Q}_+^{-1}(a)A_a^* = B_a\mathcal{Q}_-^{-1}(a)B_a^*. \tag{4.1.10}$$

3) If for each $a \in V$ the matrices A_a and B_a satisfy (4.1.10) and $\operatorname{rank}(A_a, B_a) = p+q$, *then $\mathcal{L}$ with nodal conditions (4.1.8) is a self-adjoint operator.*

Proof For each $a \in V$, there are $(p+q)$ many edges jointing it. Since the single vertex condition is a local property, one can regard the vertex as the center of a star shape graph with fixed boundary conditions. Therefore, we have $(p+q)$ many second order differential equations, and hence there are at most $(p+q)$ number the connection conditions at a since they already have $(p+q)$ many Dirichlet boundary conditions. So there are at most $(p+q)$ linearly independent conditions at vertex a.

Suppose that the nodal conditions are given by (4.1.8). Set $\widehat{W}_+(a) = [W_+(a), W'_+(a)]^T$ and $\widehat{W}_-(a) = [W_-(a), W'_-(a)]^T$, (4.1.8) can be rewritten into

$$A_a\widehat{W}_+(a) + B_a\widehat{W}_-(a) = 0.$$

Then for any $X \in \mathbb{C}^{(p+q)}$, it holds that

$$(A_a\widehat{W}_+(a), X)_{\mathbb{C}^{(p+q)}} + (B_a\widehat{W}_-(a), X)_{\mathbb{C}^{(p+q)}} = 0. \tag{4.1.11}$$

If the operator $\mathcal{L}$ with nodal conditions (4.1.8) is nodal equilibrium, i.e., for any $W, F \in H^2(E)$ satisfying (4.1.8), it holds that

$$\Big(\mathcal{Q}_+(a)\widehat{W}_+(a), \widehat{F}_+(a)\Big)_{\mathbb{C}^{2p}} - \Big(\mathcal{Q}_-(a)\widehat{W}_-(a), \widehat{F}_-(a)\Big)_{\mathbb{C}^{2q}} = 0, \tag{4.1.12}$$

comparing (4.1.11) to (4.1.12), we get that there exists an $X \in \mathbb{C}^{(p+q)}$ such that

$$-\mathcal{Q}_+(a)\widehat{F}_+(a) = A_a^*X, \quad \mathcal{Q}_-(a)\widehat{F}_-(a) = B_a^*X$$

where we have used equalities $\mathcal{Q}_\pm^*(a) = -\mathcal{Q}_\pm(a)$, this leads to

$$\widehat{F}_+(a) = -\mathcal{Q}_+^{-1}(a)A_a^*X, \quad \widehat{F}_-(a) = \mathcal{Q}_-^{-1}(a)B_a^*X.$$

Thus, we have

$$0 = A_a\widehat{F}_+(a) + B_a\widehat{F}_-(a) = -A_a\mathcal{Q}_+^{-1}(a)A_a^*X + B_a\mathcal{Q}_-^{-1}(a)B_a^*X, \quad \forall X \in \mathbb{C}^{(p+q)}.$$

So (4.1.10) holds.

Conversely, suppose that A_a and B_a in (4.1.8) satisfy the condition (4.1.10), $W \in H^2(E)$ is a function satisfying the condition $A_a\widehat{W}_+(a) + B_a\widehat{W}_-(a) = 0$. By using condition (4.1.10), there exists some one $X \in \mathbb{C}^{(p+q)}$ such that

$$\widehat{W}_+(a) = \mathcal{Q}_+^{-1}(a)A_a^*X, \quad \widehat{W}_-(a) = -\mathcal{Q}_-^{-1}(a)B_a^*X.$$

Thus we have

$$\mathcal{Q}_+(a)\widehat{W}_+(a) = A_a^*X, \quad -\mathcal{Q}_-(a)\widehat{F}_-(a) = B_a^*X,$$

Let $F \in H^2(E)$ be anyone function satisfying $A_a\widehat{F}_+(a) + B_a\widehat{F}_-(a) = 0$. Then

$$\begin{aligned}
& \Big(\mathcal{Q}_+(a)\widehat{W}_+(a), \widehat{F}_+(a)\Big)_{\mathbb{C}^{2p}} - \Big(\mathcal{Q}_-(a)\widehat{W}_-(a), \widehat{F}_-(a)\Big)_{\mathbb{C}^{2q}} \\
= & \Big(A^*X, \widehat{F}_+(a)\Big)_{\mathbb{C}^{2p}} + \Big(B^*X, \widehat{F}_-(a)\Big)_{\mathbb{C}^{2q}} \\
= & \Big(X, A\widehat{F}_+(a)\Big)_{\mathbb{C}^{(p+q)}} + \Big(X, B\widehat{F}_-(a)\Big)_{\mathbb{C}^{(p+q)}} \\
= & \Big(X, A\widehat{F}_+(a) + B\widehat{F}_-(a)\Big)_{\mathbb{C}^{(p+q)}} = 0,
\end{aligned}$$

this shows that the node is equilibrium.

Now one assumes that for each $a \in V$, the matrices A_a and B_a satisfy the condition (4.1.10). Define a subset of $L^2(G)$ by

$$\mathcal{D}(\mathcal{L}) = \{W \in H^2(E) \mid A_a\widehat{W}_+(a) + B_a\widehat{W}_-(a) = 0, \forall a \in V\}.$$

From (4.1.7) one sees that $\mathcal{L}$ with $\mathcal{D}(\mathcal{L})$ is a symmetric operator. Since, for each $a \in V$, $\operatorname{rank}(A_a, B_a) = p + q = \#J(a)$, this condition implies that if $W \in H^2(E)$ satisfies conditions

$$(iI + \mathcal{L})W = 0, \quad \text{and} \quad A_a\widehat{W}_+(a) + B_a\widehat{W}_-(a) = 0, \quad \forall a \in V,$$

we can deduce that $W \equiv 0$. Therefore, $\mathcal{L}$ is a self adjoint operator. □

THEOREM 4.1.2 *Let $\mathcal{L}$ be defined as (4.1.3) and let $\mathcal{Q}_\pm(a)$ be defined by (4.1.5) and (4.1.6) respectively. Suppose that for each $a \in V$ matrices $A_a = A_{(p+q)\times 2p}$ and $B_a = B_{(p+q)\times 2q}$ satisfy the condition (4.1.10) and* $\operatorname{rank}(A_a, B_a) = p + q$. *Define the operator $\mathcal{L}_0$ by $\mathcal{L}_0 = -\mathcal{L}$ with domain*

$$\mathcal{D}(\mathcal{L}_0) = \{W \in H^2(E) \mid A_a\widehat{W}_+(a) + B_a\widehat{W}_-(a) = 0, \forall a \in V\} \tag{4.1.13}$$

where $\widehat{W}_+(a)$ and $\widehat{W}_-(a)$ for each $W \in H^2(E)$ are defined as before. Then $\mathcal{L}_0$ is a nonnegative operator if and only if A_a and B_a satisfy the condition

$$A_a\mathcal{E}_p\mathcal{Q}_+^{-1}(a)A_a^* = B_a\mathcal{E}_q\mathcal{Q}_-^{-1}(a)B_a^* \tag{4.1.14}$$

where

$$\mathcal{E}_p = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}_{2p\times 2p}, \quad \mathcal{E}_q = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}_{2q\times 2q}.$$

Proof For any $W \in H^2(E)$, we have

$$\begin{aligned}
-(\mathcal{L}W, W)_{L^2} &= -\sum_{j=1}^n \int_{e_j} (T_j(s)w_{j,s}(s))_s\overline{w_j(s)}ds + \sum_{j=1}^n \int_{e_j} q_j(s)|w_j(s)|^2 ds \\
&= -\sum_{j=1}^n T_j(s)[w_{j,s}(s)\overline{w_j(s)}]\big|_0^1 + \sum_{j=1}^m \int_{e_j} [T_j(s)|w_{j,s}(s))|^2 + q_j(s)|w_j(s)|^2]ds \\
&= -\sum_{i=1}^m \left(\sum_{j\in J^+(a_i)} T_j(1)w_{j,s}(1)\overline{w_j(1)} - \sum_{k\in J^-(a_i)} T_k(0)w_{k,s}(0)\overline{w_k(0)} \right) \\
&\quad + \sum_{j=1}^n \int_{e_j} [T_j(s)|w_{j,s}(s)|^2 + q_j(s)|w_j(s)|^2]ds.
\end{aligned}$$

Since

$$\begin{aligned}
&\left(\sum_{j\in J^+(a_i)} T_j(1)w_{j,s}(1)\overline{w_j(1)} - \sum_{k\in J^-(a_i)} T_k(0)w_{k,s}(0)\overline{w_k(0)} \right) \\
&= \big(T_+(a_i)W_+'(a_i), W_+(a_i)\big)_{\mathbb{C}^p} - \big(T_-(a_i)W_-'(a_i), W_-(a_i)\big)_{\mathbb{C}^q}
\end{aligned}$$

$$\begin{aligned}
&= \left(\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & T_+(a_i) \\ -T_+(a_i) & 0 \end{bmatrix} \begin{bmatrix} W_+(a_i) \\ W'_+(a_i) \end{bmatrix}, \begin{bmatrix} W_+(a_i) \\ W'_+(a_i) \end{bmatrix} \right) \\
&\quad - \left(\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & T_-(a_i) \\ -T_-(a_i) & 0 \end{bmatrix} \begin{bmatrix} W_-(a_i) \\ W'_-(a_i) \end{bmatrix}, \begin{bmatrix} W_-(a_i) \\ W'_-(a_i) \end{bmatrix} \right) \\
&= \left(\mathcal{E}_p \mathcal{Q}_+(a_i) \widehat{W}_+(a_i), \widehat{W}_+(a_i) \right) - \left(\mathcal{E}_q \mathcal{Q}_-(a_i) \widehat{W}_-(a_i), \widehat{W}_-(a_i) \right)
\end{aligned}$$

where

$$\mathcal{E}_p = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}_{2p\times 2p}, \quad \mathcal{E}_q = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}_{2q\times 2q},$$

when $W \in \mathcal{D}(\mathcal{L})$, one has

$$\widehat{W}_+(a_i) = \mathcal{Q}_+^{-1}(a_i) A_{a_i}^* X, \quad \widehat{W}_-(a_i) = -\mathcal{Q}_-^{-1}(a_i) B_{a_i}^* X, \quad X \in \mathbb{C}^{(p+q)}$$

and hence

$$\mathcal{E}_p \mathcal{Q}_+(a_i) \widehat{W}_+(a_i) = \mathcal{E}_p A_{a_i}^* X, \quad \mathcal{E}_q \mathcal{Q}_-(a_i) \widehat{W}_-(a_i) = -\mathcal{E}_q B_{a_i}^* X.$$

Thus,

$$\begin{aligned}
&\left(\sum_{j\in J^+(a_i)} T_j(1) w_{j,s}(1) \overline{w_j(1)} - \sum_{k\in J^-(a_i)} T_k(0) w_{k,s}(0) \overline{w_k(0)} \right) \\
&= \left(\mathcal{E}_p \mathcal{Q}_+(a_i) \widehat{W}_+(a_i), \widehat{W}_+(a_i) \right)_{\mathbb{C}^{2p}} - \left(\mathcal{E}_q \mathcal{Q}_-(a_i) \widehat{W}_-(a_i), \widehat{W}_-(a_i) \right)_{\mathbb{C}^{2q}} \\
&= \left(\mathcal{E}_p A_{a_i}^* X, \mathcal{Q}_+^{-1}(a_i) A_{a_i}^* X \right)_{\mathbb{C}^{2p}} - \left(\mathcal{E}_q B_{a_i}^* X, \mathcal{Q}_-^{-1}(a_i) B_{a_i}^* X \right)_{\mathbb{C}^{2q}} \\
&= \left(X, A_{a_i} \mathcal{E}_p \mathcal{Q}_+^{-1}(a_i) A_{a_i}^* X - B_{a_i} \mathcal{E}_q \mathcal{Q}_-^{-1}(a_i) B_{a_i}^* X \right)_{\mathbb{C}^{(p+q)}}.
\end{aligned}$$

Since $A_{a_i} \mathcal{Q}_+^{-1}(a_i) A_{a_i}^* = B_{a_i} \mathcal{Q}_-^{-1}(a_i) B_{a_i}^*$, we have the equality

$$[A_{a_i} \mathcal{E}_p \mathcal{Q}_+^{-1}(a_i) A_{a_i}^* - B_{a_i} \mathcal{E}_q \mathcal{Q}_-^{-1}(a_i) B_{a_i}^*]^* = A_{a_i} \mathcal{E}_p \mathcal{Q}_+^{-1}(a_i) A_{a_i}^* - B_{a_i} \mathcal{E}_q \mathcal{Q}_-^{-1}(a_i) B_{a_i}^*.$$

Therefore, $\mathcal{L}_0$ is nonnegative if and only if

$$A_{a_i} \mathcal{E}_p \mathcal{Q}_+^{-1}(a_i) A_{a_i}^* = B_{a_i} \mathcal{E}_q \mathcal{Q}_-^{-1}(a_i) B_{a_i}^*.$$

The proof is then complete. □

In Theorem 4.1.1, the condition $\text{rank}(A_a, B_a) = p + q$ for any $a \in V$ is a assumption of condition number that is used to ensure that $\mathcal{L}$ has no defect number, while $A_a \mathcal{Q}_+^{-1}(a) A_a^* = B_a \mathcal{Q}_-^{-1}(a) B_a^*, \forall a \in V$ are used to ensure the self-adjoint-ness of $\mathcal{L}$. For operator $\mathcal{L}_0$, the conditions $A_a \mathcal{E}_p \mathcal{Q}_+^{-1}(a) A_a^* = B_a \mathcal{E}_q \mathcal{Q}_-^{-1}(a) B_a^*, \forall a \in V$ are used to ensure the nonnegativity of $\mathcal{L}_0$. In what follows, one will show that condition $A_a \mathcal{E}_p \mathcal{Q}_+^{-1}(a) A_a^* = B_a \mathcal{E}_q \mathcal{Q}_-^{-1}(a) B_a^*$ implies $A_a \mathcal{Q}_+^{-1}(a) A_a^* = B_a \mathcal{Q}_-^{-1}(a) B_a^*$.

Since

$$\mathcal{Q}_+^{-1}(a) = \begin{bmatrix} 0 & -T_+^{-1}(a) \\ T_+^{-1}(a) & 0 \end{bmatrix}_{2p\times 2p}, \quad \mathcal{Q}_-^{-1}(a) = \begin{bmatrix} 0 & -T_-^{-1}(a) \\ T_-^{-1}(a) & 0 \end{bmatrix}_{2q\times 2q}$$

$$\mathcal{E}_p\mathcal{Q}_+^{-1}(a)=\begin{bmatrix}0 & -T_+^{-1}(a)\\ 0 & 0\end{bmatrix}_{2p\times 2p},\quad \mathcal{E}_q\mathcal{Q}_-^{-1}(a)=\begin{bmatrix}0 & -T_-^{-1}(a)\\ 0 & 0\end{bmatrix}_{2q\times 2q}$$

$$A_a=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}_{(p+q)\times 2p},\quad B_a=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}_{(p+q)\times 2q}$$

$$A_a^*=\begin{bmatrix}A_{11}^* & A_{21}^*\\ A_{12}^* & A_{22}^*\end{bmatrix}_{2p\times(p+q)},\quad B_a^*=\begin{bmatrix}B_{11}^* & B_{21}^*\\ B_{12}^* & B_{22}^*\end{bmatrix}_{2q\times(p+q)},$$

a direct calculation gives

$$\begin{aligned}0 &= A_a\mathcal{E}_p\mathcal{Q}_+^{-1}(a)A_a^*-B_a\mathcal{E}_q\mathcal{Q}_-^{-1}(a)B_a^*\\ &= \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}_{(p+q)\times 2p}\begin{bmatrix}0 & -T_+^{-1}(a)\\ 0 & 0\end{bmatrix}_{2p\times 2p}\begin{bmatrix}A_{11}^* & A_{21}^*\\ A_{12}^* & A_{22}^*\end{bmatrix}_{2p\times(p+q)}\\ &\quad -\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}_{(p+q)\times 2q}\begin{bmatrix}0 & -T_-^{-1}(a)\\ 0 & 0\end{bmatrix}_{2q\times 2q}\begin{bmatrix}B_{11}^* & B_{21}^*\\ B_{12}^* & B_{22}^*\end{bmatrix}_{2q\times(p+q)}\\ &= -\begin{bmatrix}A_{11}T_+^{-1}(a)A_{12}^*-B_{11}T_-^{-1}(a)B_{12}^* & A_{11}T_+^{-1}(a)A_{22}^*-B_{11}T_-^{-1}(a)B_{22}^*\\ A_{21}T_+^{-1}(a)A_{12}^*-B_{21}T_-^{-1}(a)B_{12}^* & A_{21}T_+^{-1}(a)A_{22}^*-B_{21}T_-^{-1}(a)B_{22}^*\end{bmatrix}_{(p+q)\times(p+q)}\end{aligned}$$

this gives an explicit expression of (4.1.14). Thus we have

$$\begin{aligned}&A_{11}T_+^{-1}(a)A_{12}^*-B_{11}T_-^{-1}(a)B_{12}^*=0,\quad A_{11}T_+^{-1}(a)A_{22}^*-B_{11}T_-^{-1}(a)B_{22}^*=0\\ &A_{21}T_+^{-1}(a)A_{12}^*-B_{21}T_-^{-1}(a)B_{12}^*=0,\quad A_{21}T_+^{-1}(a)A_{22}^*-B_{21}T_-^{-1}(a)B_{22}^*=0.\end{aligned}$$

Taking the dual operation for above equations leads to

$$\begin{aligned}&A_{12}T_+^{-1}(a)A_{11}^*-B_{12}T_-^{-1}(a)B_{11}^*=0,\quad A_{22}T_+^{-1}(a)A_{11}^*-B_{22}T_-^{-1}(a)B_{11}^*=0\\ &A_{12}T_+^{-1}(a)A_{21}^*-B_{12}T_-^{-1}(a)B_{21}^*=0,\quad A_{22}T_+^{-1}(a)A_{21}^*-B_{22}T_-^{-1}(a)B_{21}^*=0.\end{aligned}$$

A straightforward calculation gives

$$\begin{aligned}&A_a\mathcal{Q}_+^{-1}(a)A_a^*-B_a\mathcal{Q}_-^{-1}(a)B_a^*\\ =&\begin{bmatrix}A_{12}T_+^{-1}(a)A_{11}^*-B_{12}T_-^{-1}(a)B_{11}^* & A_{12}T_+^{-1}(a)A_{21}^*-B_{12}T_-^{-1}(a)B_{21}^*\\ A_{22}T_+^{-1}(a)A_{11}^*-B_{22}T_-^{-1}(a)B_{11}^* & A_{22}T_+^{-1}(a)A_{21}^*-B_{22}T_-^{-1}(a)B_{21}^*\end{bmatrix}_{(p+q)\times(p+q)}=0.\end{aligned}$$

Therefore, the following result is true.

COROLLARY 4.1.1 *Let G be a metric graph and $\mathcal{L}$ be the formal differential operator on E defined by (4.1.3). Set $\mathcal{L}_0=-\mathcal{L}$ with domain*

$$\mathcal{D}(\mathcal{L}_0)=\left\{W\in H^2(E)\;\middle|\;A_a\widehat{W}_+(a)+B_a\widehat{W}_-(a)=0, A_a\mathcal{E}_p\mathcal{Q}_+^{-1}A_a^*=B_a\mathcal{E}_q\mathcal{Q}_-^{-1}(a)B_a^*,\forall a\in V\right\},$$

where $\widehat{W}_+(a)$ and $\widehat{W}_-(a)$ for each $W\in H^2(E)$ are defined as before. Then $\mathcal{L}_0$ is a nonnegative operator on $L^2(G)$. In particular, it is a positive definite operator if $q_j(s)$ are positive functions.

REMARK 4.1.2 *In the nodal equilibrium condition, one requires the condition*

$$(\mathcal{Q}_+(a)\widehat{W}_+(a), \widehat{F}_+(a))_{\mathbb{C}^{2p}} - (\mathcal{Q}_-(a)\widehat{W}_-(a), \widehat{F}_-(a))_{\mathbb{C}^{2q}} = 0$$

at each node a. Usually, in mechanics, $\mathcal{L}_0$ should be a positive operator, for example, in the equation (4.1.2), so the coefficient matrices A_a and B_a at each node a satisfy the nodal conditions $A_a\mathcal{E}_p\mathcal{Q}_+^{-1}(a)A_a^* = B_a\mathcal{E}_q\mathcal{Q}_-^{-1}(a)B_a^*$. Since this relation is independent of the other vertices, it is called the local structure condition (or the local vertex condition).

Now one recalls the structural matrix Ψ of G,

$$\Psi = \begin{pmatrix} (\Phi^+)^T\Phi^+ & (\Phi^+)^T\Phi^- \\ (\Phi^-)^T\Phi^+ & (\Phi^-)^T\Phi^- \end{pmatrix}_{2n\times 2n}$$

the matrix shows the intersection of endpoint of edge-edge. One decomposes the matrix Ψ as $\Psi = (\Psi^+, \Psi^-)$, where Ψ^+ and Ψ^- are the $2n\times n$ matrix. Define the connective pattern matrices

$$\Psi_+ = (\Psi^+, \Psi^+), \quad \Psi_- = (\Psi^-, \Psi^-) \tag{4.1.15}$$

Denote the matrices set by

$$\mathbb{M}_{2n\times 2n}(\Psi_\pm) = \{A = (a_{ij}) \in \mathbb{M}_{2n\times 2n} \mid \Psi_\pm \bullet A = A\}$$

where $A \bullet B$ denotes the Hadamard product which is defined by $A \bullet B = (a_{ij}b_{ij})$.

One coincides a function $W \in H^2(E)$ with a vector-valued function W defined by

$$W(s) = [w_1(s), w_2(s), w_3(s), \cdots, w_n(s)]^T, \quad s \in (0,1).$$

By an appropriate arrangement, the local vertex conditions of graph G are written into the form

$$A\widehat{W}(1) + B\widehat{W}(0) = 0, \quad A \in \mathbb{M}_{2n\times 2n}(\Psi_+), \quad B \in \mathbb{M}_{2n\times 2n}(\Psi_-). \tag{4.1.16}$$

where

$$\widehat{W}(1) = \begin{bmatrix} W(1) \\ W'(1) \end{bmatrix}, \quad \widehat{W}(0) = \begin{bmatrix} W(0) \\ W'(0) \end{bmatrix}$$

4.1.3 The structural equilibrium condition

In this subsection one discusses the case that the node may not be equilibrium but its structure is equilibrium. we called it the structural equilibrium condition (or whole vertex condition).

Let G be a metric graph with edge set $E = \{e_j; j = 1, 2, \cdots, n\}$. For $W \in L^2(E)$, one coincides W with a vector-valued function $W(s) = [w_1(s), w_2(s), \cdots, w_n(s)]^T$.

Let $\mathcal{L}$ be defined by (4.1.3). Define the diagonal matrices

$$\mathbb{T}(s) = \mathrm{diag}\,[T_1(s), T_2(s), \cdots, T_n(s)]\,, \quad \mathbb{Q}(s) = \mathrm{diag}\,[q_1(s), q_2(s), \cdots, q_n(s)]\,,$$

then $\mathcal{L}$ can be rewritten into the matrix form

$$\mathcal{L} = \frac{d}{ds}\left(\mathbb{T}(s)\frac{d}{ds}\right) - \mathbb{Q}(s).$$

For any $W, F \in L^2(G)$,

$$W(s) = [w_1(s), w_2(s), \cdots, w_n(s)]^T, \quad F(s) = [f_1(s), f_2(s), \cdots, f_n(s)]^T, \quad s \in (0,1)$$

one has

$$(W,F)_{L^2} = \int_G W(x)\overline{F(x)}dx = \sum_{j=1}^{n} \int_0^1 w_j(s)\overline{f_j(s)}dx = \int_0^1 (W(s), F(s))_{\mathbb{C}^n} ds.$$

Thus

$$\begin{aligned}
& (\mathcal{L}W, F)_{L^2} - (W, \mathcal{L}F)_{L^2} = \int_0^1 (\mathcal{L}W(s), F(s))_{\mathbb{C}^n} ds - \int_0^1 (W(s), \mathcal{L}F(s))_{\mathbb{C}^n} ds \\
= \; & (\mathbb{T}(1)W'(1), F(1))_{\mathbb{C}^n} - (\mathbb{T}(0)W'(0), F(0))_{\mathbb{C}^n} \\
& -(\mathbb{T}(1)W(1), F'(1))_{\mathbb{C}^n} + (\mathbb{T}(0)W(0), F'(0))_{\mathbb{C}^n} \\
= \; & \left(\begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix} \begin{bmatrix} W(1) \\ W'(1) \end{bmatrix}, \begin{bmatrix} F(1) \\ F'(1) \end{bmatrix} \right)_{\mathbb{C}^{2n}} \\
& - \left(\begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix} \begin{bmatrix} W(0) \\ W'(0) \end{bmatrix}, \begin{bmatrix} F(0) \\ F'(0) \end{bmatrix} \right)_{\mathbb{C}^{2n}}.
\end{aligned}$$

Now let A and B be the $2n \times 2n$ matrices, the connection condition are given by

$$A\widehat{W}(1) + B\widehat{W}(0) = 0, \quad \operatorname{rank}(A, B) = 2n \tag{4.1.17}$$

where $\widehat{W}(1) = [W(1), W'(1)]^T$ and $\widehat{W}(0) = [W(0), W'(0)]^T$.

THEOREM 4.1.3 *Let the formal differential operator $\mathcal{L}$ be defined as (4.1.3) and let $\mathcal{L}$ on graph G have the connective condition (4.1.17). Then $\mathcal{L}$ is structural self-adjoint if only if A and B satisfy the condition*

$$A \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* = B \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^*. \tag{4.1.18}$$

Proof Set

$$\mathcal{T}(s) = \begin{bmatrix} 0 & \mathbb{T}(s) \\ -\mathbb{T}(s) & 0 \end{bmatrix}.$$

Then

$$(\mathcal{L}W, F)_{L^2} - (W, \mathcal{L}F)_{L^2} = (\mathcal{T}(1)\widehat{W}(1), \widehat{F}(1))_{\mathbb{C}^{2n}} - (\mathcal{T}(0)\widehat{W}(0), \widehat{F}(0))_{\mathbb{C}^{2n}}$$

and (4.1.18) becomes

$$A\mathcal{T}^{-1}(1)A^* = B\mathcal{T}^{-1}(0)B^*.$$

Suppose that $\mathcal{L}$ is structural self-adjoint, i.e., for any $W, F \in H^2(E)$ satisfying (4.1.17), it holds that

$$(\mathcal{T}(1)\widehat{W}(1), \widehat{F}(1))_{\mathbb{C}^{2n}} - (\mathcal{T}(0)\widehat{W}(0), \widehat{F}(0))_{\mathbb{C}^{2n}} = 0.$$

For any $X \in \mathbb{C}^{2n}$, from (4.1.17) we get

$$(A\widehat{W}(1), X)_{\mathbb{C}^{2n}} + (B\widehat{W}(0), X)_{\mathbb{C}^{2n}} = 0.$$

Comparing both equalities above we have

$$\mathcal{T}^*(1)\widehat{F}(1) = A^*X, \quad \mathcal{T}^*(0)\widehat{F}(0) = -B^*X.$$

Thus

$$\widehat{F}(1) = -\mathcal{T}^{-1}(1)A^*X, \quad \widehat{F}(0) = \mathcal{T}^{-1}(0)B^*X.$$

Hence

$$0 = A\widehat{F}(1) + B\widehat{F}(0) = -A\mathcal{T}^{-1}(1)A^*X + B\mathcal{T}^{-1}(0)B^*X,$$

(4.1.18) follows from this equality.

Conversely, we suppose that (4.1.18) holds. For any $X \in \mathbb{C}^{2n}$, let $W \in H^2(E)$ satisfy the condition

$$\widehat{W}(1) = \mathcal{T}^{-1}(1)A^*X, \quad \widehat{W}(0) = -\mathcal{T}^{-1}(0)B^*X,$$

then W satisfies (4.1.17). Thus for any $F \in H^2(E)$,

$$\begin{aligned}(\mathcal{L}W, F)_{L^2} - (W, \mathcal{L}F)_{L^2} &= \Big(\mathcal{T}(1)\widehat{W}(1), \widehat{F}(1)\Big)_{\mathbb{C}^{2n}} - \Big(\mathcal{T}(0)\widehat{W}(0), \widehat{F}(0)\Big)_{\mathbb{C}^{2n}} \\ &= \Big(X, A\widehat{F}(1) + B\widehat{F}(0)\Big)_{\mathbb{C}^{2n}}.\end{aligned}$$

Therefore, $(\mathcal{L}W, F) - (W, \mathcal{L}F) = 0$ if and only if $A\widehat{F}(1) + B\widehat{F}(0) = 0$. □

THEOREM 4.1.4 *Let $\mathcal{L}$ be defined as before and let the connective condition be given by (4.1.18). Then $-\mathcal{L}$ is a structural positive operator if only if $A, B \in \mathbb{M}_{2n\times 2n}$ satisfy the conditions rank$(A, B) = 2n$ and*

$$A\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* = B\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^* \tag{4.1.19}$$

where

$$\mathcal{E}_n = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}_{2n\times 2n}.$$

Proof Let $A, B \in \mathbb{M}_{2n\times 2n}$ satisfy the conditions $rank(A, B) = 2n$ and (4.1.19). Firstly we prove that A and B also satisfy the condition (4.1.18).

Note that

$$\begin{aligned}&A\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* - B\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^* \\ = \;& A \begin{bmatrix} 0 & -\mathbb{T}^{-1}(1) \\ 0 & 0 \end{bmatrix} A^* - B \begin{bmatrix} 0 & -\mathbb{T}^{-1}(0) \\ 0 & 0 \end{bmatrix} B^*\end{aligned}$$

and

$$\begin{aligned}
0 &= \left[A\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* - B\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^*\right]^* \\
&= A \begin{bmatrix} 0 & 0 \\ -\mathbb{T}^{-1}(1) & 0 \end{bmatrix} A^* - B \begin{bmatrix} 0 & 0 \\ -\mathbb{T}^{-1}(0) & 0 \end{bmatrix} B^* \\
&= -A(I-\mathcal{E}_n) \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* + B(I-\mathcal{E}_n) \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^*.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
& A \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* - B \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^* \\
= \; & A\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* - B\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^* \\
& + A(I-\mathcal{E}_n) \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix} A^* - B(I-\mathcal{E}_n) \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0)0 & 0 \end{bmatrix}^{-1} B^* = 0.
\end{aligned}$$

The condition (4.1.18) follows.

Let $\mathcal{L}$ have the connective condition

$$A \begin{bmatrix} W(1) \\ W'(1) \end{bmatrix} + B \begin{bmatrix} W(0) \\ W'(0) \end{bmatrix} = 0, \quad A, B \in \mathbb{M}_{2n\times 2n},$$

then it is a self-adjoint operator according to Theorem 4.1.3. So there exists an $X \in \mathbb{C}^{2n}$ such that

$$\begin{bmatrix} W(1) \\ W'(1) \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* X,$$

and

$$\begin{bmatrix} W(0) \\ W'(0) \end{bmatrix} = - \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^* X.$$

Thus one has

$$\begin{aligned}
& (\mathcal{L}W, W) + \int_0^1 (\mathbb{T}(s)W'(s), W'(s))_{\mathbb{C}^n} + (\mathbb{Q}(s)W(s), W(s))_{\mathbb{C}^n} ds \\
= \; & (\mathbb{T}(1)W'(1), W(1))_{\mathbb{C}^n} - (\mathbb{T}(0)W'(0), W(0))_{\mathbb{C}^n} \\
= \; & \left(\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix} \begin{bmatrix} W(1) \\ W'(1) \end{bmatrix}, \begin{bmatrix} W(1) \\ W'(1) \end{bmatrix}\right)_{\mathbb{C}^{2n}}
\end{aligned}$$

$$
\begin{aligned}
&-\left(\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix} \begin{bmatrix} W(0) \\ W'(0) \end{bmatrix}, \begin{bmatrix} W(0) \\ W'(0) \end{bmatrix}\right)_{\mathbb{C}^{2n}} \\
= &\left(\mathcal{E}_n A^* X, \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* X\right)_{\mathbb{C}^{2n}} - \left(\mathcal{E}_n B^* X, \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^* X\right)_{\mathbb{C}^{2n}} \\
= &\left(X, A\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* X - B\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^* X\right)_{\mathbb{C}^{2n}}
\end{aligned}
$$

Therefore, $-\mathcal{L}$ is structural positive if and only if

$$-(\mathcal{L}W, W) = \int_0^1 (\mathbb{T}(s)W'(s), W'(s))_{\mathbb{C}^n} + (\mathbb{Q}(s)W(s), W(s))_{\mathbb{C}^n} ds$$

which is equivalent to

$$A\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* = B\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^*.$$

The desired result follows. □

REMARK 4.1.3 *The structural equilibrium condition is whole vertex condition. If $\mathcal{L}$ has local vertex conditions, then it also satisfies the structural equilibrium condition. But the structural equilibrium need not to be the nodal equilibrium.*

Note that the structural equilibrium condition requires the function values and its derivative at different vertices, so the connection condition usually can not be represented by the structural pattern matrix $\Psi_{\pm}$.

EXAMPLE 4.1.1 *Let G be a metric graph, whose structure is shown as Fig.4.1.1.*

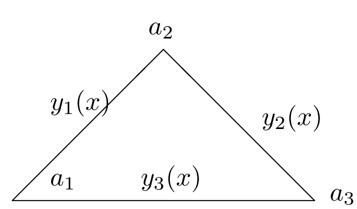

Fig. 4.1.1 A triangle circuit

Let $\mathcal{L}$ be the formal second-order differential operator on G defined as

$$\mathcal{L}y_j(x) = y_j''(x) - q_j(x)y_j(x), \quad x \in (0,1), \quad j = 1,2,3$$

where $q_j(x)$ are nonnegative continuous functions.

Suppose that $\mathcal{D}(\mathcal{L})$ consists of all function $y(x) \in H^2(E)$ satisfy the following conditions

$$y_1(0) = y_3(1), \quad y_1(1) = y_2(0), \quad y_2(1) = y_3(0)$$

and

$$\begin{cases} y_1(0) = \alpha[y_1'(1) - y_2'(0)] - \beta[y_2'(1) - y_3'(0)], \\ y_2(0) = -\alpha[y_3'(1) - y_1'(0)] + \gamma[y_2'(1) - y_3'(0)], \\ y_3(0) = \beta[y_3'(1) - y_1'(0)] - \gamma[y_1'(1) - y_2'(0)]. \end{cases}$$

Then $-\mathcal{L}$ is a structural positive operator.

This is because

$$\begin{aligned}
& (\mathcal{L}y, y)_{L^2} + \sum_{j=1}^{3} \int_0^1 [|y_j'(s)|^2 + q_j(s)|y_j(s)|^2] ds \\
= \; & \sum_{j=1}^{3} \int_0^1 y_j''(s)\overline{y(s)} ds + \sum_{j=1}^{3} \int_0^1 |y_j'(s)|^2 ds \\
= \; & \sum_{j=1}^{3} y_j'(1) y_j(1) - \sum_{j=1}^{3} y_j'(0) y_j(0) \\
= \; & [y_3'(1)y_3(1) - y_1'(0)y_1(0)] + [y_2'(1)y_2(1) - y_3'(0)y_3(0)] + [y_1'(1)y_1(1) - y_2'(0)y_2(0)] \\
= \; & [y_3'(1) - y_1'(0)]y_1(0) + [y_2'(1) - y_3'(0)]y_3(0) + [y_1'(1) - y_2'(0)]y_2(0) \\
= \; & [y_3'(1) - y_1'(0)](\alpha[y_1'(1) - y_2'(0)] - \beta[y_2'(1) - y_3'(0)]) \\
& + [y_2'(1) - y_3'(0)](\beta[y_3'(1) - y_1'(0)] - \gamma[y_1'(1) - y_2'(0)]) \\
& + [y_1'(1) - y_2'(0)](-\alpha[y_3'(1) - y_1'(0)] + \gamma[y_2'(1) - y_3'(0)]) = 0.
\end{aligned}$$

Obviously, $\mathcal{L}$ is the structural equilibrium, but not the nodal equilibrium. □

4.1.4 Wave equation on metric graphs

Now let us return to the wave equations on the metric graph $G = (V, E)$. Note that if the local vertex conditions (4.1.14) hold, then the whole vertex condition (4.1.19) also is fulfilled. Therefore, to avoid some technical details, one restricts oneself here to the whole vertex condition.

Let $w(x,t)$ be a function defined on $E \times \mathbb{R}_+$, $w_j(s,t)$ be its normalized realization on $e_j \times \mathbb{R}_+$ and satisfy the wave equations

$$m_j(s) \frac{\partial^2 w_j(s,t)}{\partial t^2} = \frac{\partial}{\partial s}\left(T_j(s) \frac{\partial w_j(s,t)}{\partial s} \right) - q_j(s) w_j(s,t), \quad s \in (0,1),$$

where $m_j(s)$ and $T_j(s)$ are positive continuous functions, $q_j(s)$ are nonnegative continuous functions. Using the vector-valued function $W(s,t) = [w_1(s,t), w_2(s,t), \cdots, w_n(s,t)]$ on $[0,1] \times \mathbb{R}_+$, one can rewrite the wave equations as

$$\mathbb{M}(s) \frac{\partial^2 W(s,t)}{\partial t^2} = \frac{\partial}{\partial s}\left(\mathbb{T}(s) \frac{\partial W(s,t)}{\partial s} \right) - \mathbb{Q}(s) W(s,t), \quad s \in (0,1), \tag{4.1.20}$$

Suppose that $W(s,t)$ satisfies the vertex conditions

$$A[W(1,t), W_s(1,t)]^T + B[W(0,t), W_s(0,t)]^T = 0 \tag{4.1.21}$$

where the real matrices

$$A = \begin{pmatrix} A_{11} & A_{12}, \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad A, B \in M_{2n\times 2n}$$

satisfy the condition (4.1.19). Thus one has the following result.

THEOREM 4.1.5 *The wave system (4.1.20) with the vertex conditions (4.1.21) is the energy conservation.*

Proof The energy function of the wave system (4.1.20) is defined by

$$\mathcal{E}(t) = \frac{1}{2}\int_0^1 (\mathbb{T}(s)W_s(s,t), W_s(s,t))_{\mathbb{C}^n} + (\mathbb{M}(s)W_t(s,t), W_t(s,t))_{\mathbb{C}^n} + (\mathbb{Q}(s)W(s,t), W(s,t))_{\mathbb{C}^n} ds$$

Let $W(s,t)$ be a real solution of (4.1.20) satisfying (4.1.21). Then one has

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} &= \int_0^1 (\mathbb{T}(s)W_s(s,t), W_{st}(s,t))_{\mathbb{R}^n} ds + \int_0^1 (\mathbb{M}(s)W_{tt}(s,t), W_t(s,t))_{\mathbb{R}^n} ds \\ &\quad + \int_0^1 (\mathbb{Q}(s)W(s,t), W_t(s,t))_{\mathbb{R}^n} ds \\ &= (\mathbb{T}(1)W_s(1,t), W_t(1,t))_{\mathbb{R}^n} - (\mathbb{T}(0)W_s(0,t), W_t(0,t))_{\mathbb{R}^n} \\ &= \left(\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix} \begin{bmatrix} W(1,t) \\ W_s(1,t) \end{bmatrix}, \begin{bmatrix} W_t(1,t) \\ W_{st}(1,t) \end{bmatrix}\right)_{\mathbb{R}^{2n}} \\ &\quad - \left(\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix} \begin{bmatrix} W(0,t) \\ W_s(0,t) \end{bmatrix}, \begin{bmatrix} W_t(0) \\ W_{st}(0,t) \end{bmatrix}\right)_{\mathbb{R}^{2n}}. \end{aligned}$$

The condition (4.1.21) is equivalent to

$$A \begin{bmatrix} W(1,t) \\ W_s(1,t) \end{bmatrix} + B \begin{bmatrix} W(0,t) \\ W_s(0,t) \end{bmatrix} = 0.$$

Note that the condition (4.1.19) implies that there exists $X(t) \in \mathbb{R}^{2n}$ such that

$$\begin{bmatrix} W(1,t) \\ W_s(1,t) \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* X(t),$$

$$\begin{bmatrix} W(0,t) \\ W_s(0,t) \end{bmatrix} = -\begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^* X(t).$$

Thus

$$\frac{d\mathcal{E}(t)}{dt} = \left(\mathcal{E}_n A^* X(t), \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* X'(t)\right)_{\mathbb{R}^{2n}}$$

$$
\begin{aligned}
& -\left(\mathcal{E}_n B^* X(t), \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^* X'(t)\right)_{\mathbb{R}^{2n}} \\
= & \left(X(t), A\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(1) \\ -\mathbb{T}(1) & 0 \end{bmatrix}^{-1} A^* X'(t) - B\mathcal{E}_n \begin{bmatrix} 0 & \mathbb{T}(0) \\ -\mathbb{T}(0) & 0 \end{bmatrix}^{-1} B^* X'(t)\right)_{\mathbb{R}^{2n}} \\
= & \ 0.
\end{aligned}
$$

So the energy of the system is conservation. □

REMARK 4.1.4 *For the local vertex conditions, by appropriate arrangement, the matrices A and B will satisfy $A \in \mathbb{M}_{2n\times 2n}(\Psi_+)$ and $B \in \mathbb{M}_{2n\times 2n}(\Psi_-)$. The conditions in (4.1.21) still hold. Therefore, for $A, B \in \mathbb{M}_{2n\times 2n}$ satisfying (4.1.19), the condition (4.1.21) is not only a sufficient but also a necessary for describing the physical models.*

In [125], the authors gave a whole vertex condition as follows

$$
\begin{cases} W(0,t) = CW(1,t) \\ \mathbb{T}(1)W_s(1,t) - C^T\mathbb{T}(0)W_s(0,t) = 0, \end{cases}
$$

this is a special case of (4.1.21) in which A and B are the block diagonal matrices, and hence the geometric conditions and dynamic conditions are separated.

We observe that the separability of the geometric conditions and dynamical conditions does not imply that these conditions have simple form. Even at local vertex condition, it may have very complicated form. Here we consider a simple star-shaped graph.

EXAMPLE 4.1.2 *Let G be a simple graph, whose structure is shown as Fig. 4.1.2*

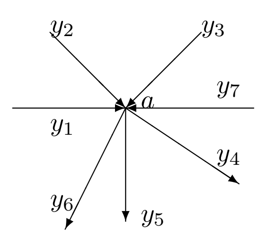

Fig. 4.1.2 A star-shape graph

Let $y(x,t)$ be a function defined on graph G and satisfy the wave equation

$$
m_j(s)y_{j,tt}(s,t) = (T_j(s)y_{j,s}(s,t))_s - q_j(s)y_j(s,t), \quad s \in (0,1), t > 0
$$

with connective conditions at a

$$
\begin{aligned}
& y_1(1,t) + y_3(1,t) + y_7(1,t) = y(a,t), \quad && y_2(1,t) + y_3(1,t) + y_7(1,t) = y(a,t), \\
& y_5(0,t) + y_6(0,t) = y(a,t), && y_4(0,t) = y(a,t)
\end{aligned}
$$

and the boundary conditions

$$y_1(0,t) = y_2(0,t) = y_3(0,t) = y_7(0,t) = 0, \quad y_4(1,t) = y_5(1,t) = y_6(1,t) = 0.$$

The dynamic condition at a is given by

$$T_3(1)y_{3,s}(1,t) = T_7(1)y_{7,s}(1,t) = [T_1(1)y_{1,s}(1,t) + T_2(1)y_{2,s}(1,t)] = T_4(0)y_{4,s}(0,t) + T_5(0)y_{5,s}(0,t)$$
$$T_5(0)y_{5,s}(0,t) = T_6(0)y_{6,s}(0,t)$$

This wave system is the energy conservation

4.1.5 Some classical vertex conditions

In this subsection we give some classical local vertex conditions. Let $w(x,t)$ be a function defined on $G \times \mathbb{R}_+$, $w_j(s,t)$ be its normalized parameterization realization on $e_j \times \mathbb{R}_+$ and satisfy the wave equations

$$m_j(s)\frac{\partial^2 w_j(s,t)}{\partial t^2} = \frac{\partial}{\partial s}\left(T_j(s)\frac{\partial w_j(s,t)}{\partial s}\right) - q_j(s)w_j(s,t), \quad j \in \{1,2,\cdots,n\}.$$

§1. δ-type vertex condition

Let $w(x,t)$ be a function defined on $G \times \mathbb{R}_+$. At the interior node $a \in V_{int}$, $w_j(s,t), j \in \{1,2,\cdots,n\}$ satisfy the geometric conditions: *the nodal continuity conditions*

$$w_j(1,t) = w_i(0,t) = w(a,t), \quad \forall j \in J^+(a),\ i \in J^-(a)$$

and *the dynamic equilibrium condition (Kirchhoff law)*

$$\sum_{j\in J^+(a)} T_j(1)w_{j,s}(1,t) - \sum_{i\in J^-(a)} T_i(0)w_{i,s}(0,t) = 0$$

this condition is also said to be the Neumann-Kirchhoff condition.

The geometric continuity conditions together with the Neumann-Kirchhoff condition are called the δ-type vertex condition at a, i.e.,

$$\begin{cases} w_j(1,t) = w_i(0,t) = w(a,t), \quad \forall j \in J^+(a),\ i \in J^-(a) \\ \sum_{j\in J^+(a)} T_j(1)w_{j,s}(1,t) - \sum_{i\in J^-(a)} T_i(0)w_{i,s}(0,t) = 0. \end{cases} \tag{4.1.22}$$

§2. δ'-type vertex conditions

Let $w(x,t)$ be a function defined on $E \times \mathbb{R}_+$. At the interior node $a \in V_{int}$, $w_j(s,t), j \in \{1,2,\cdots,n\}$ satisfy the Dynamic conditions: *the moment continuity conditions*

$$T_j(1)w_{j,s}(1,t) = T_i(0)w_{i,s}(0,t) = v(a,t), \quad \forall j \in J^+(a),\ i \in J^-(a)$$

and flow equilibrium condition *(Kirchhoff law)*

$$\sum_{j\in J^+(a)} w_j(1,t) - \sum_{i\in J^-(a)} w_i(0,t) = 0$$

i.e.,

$$\begin{cases} T_j(1)w_{j,s}(1,t) = T_i(0)w_{i,s}(0,t) = v(a,t), \quad \forall j \in J^+(a),\ i \in J^-(a) \\ \sum_{j\in J^+(a)} w_j(1,t) - \sum_{i\in J^-(a)} w_i(0,t) = 0. \end{cases} \tag{4.1.23}$$

The connective condition (4.1.23) is said to be the δ'-type vertex condition.

4.2 Euler-Bernoulli beam equations on graphs

Let G be a metric graph with vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and edge set $E = \{e_1, e_2, \cdots, e_n\}$. Let $u(x,t)$ be a function defined on $G \times \mathbb{R}_+$, $u_j(s,t)$ be its normalized realization on $e_j \times \mathbb{R}_+$. If $u_j(s,t), j \in \{1, 2, \cdots, n\}$ satisfy the partial differential equation

$$m_j(s)\frac{\partial^2 u_j(s,t)}{\partial t^2} = -\frac{\partial^2}{\partial s^2}\left(E_j(s)\frac{\partial^2 u_j(s,t)}{\partial s^2}\right) + \frac{\partial}{\partial s}\left(T_j(s)\frac{\partial u_j(s,t)}{\partial s}\right) - p_j(s)u_j(s,t), \quad s \in (0,1), \tag{4.2.1}$$

where $m_j(s)$, $E_j(s)$ and $T_j(s)$ are positive continuous differentiable functions, and $p_j(s)$ are nonnegative continuous functions, then $u(x,t)$ is called satisfying the Euler-Bernoulli beam equation on G.

The partial differential equations are defined on E, we need some vertex restriction conditions and initial data to determine uniquely a solution.

4.2.1 Nodal condition for self-adjoint operator

We consider the self-adjoint property of the differential operator in $L^2(G)$ defined by

$$\mathcal{L}w_j = (E_j(s)w_{j,ss}(s))_{ss} - (T_j(s)w_{j,s}(s))_s + p_j(s)w_j(s), \quad s \in (0,1), \quad j \in \{1, 2, \cdots, n\}. \tag{4.2.2}$$

For each $a_i \in V$, we set $J^+(a_i) = \{j_1, j_2, \cdots, j_p\}$ and $J^-(a_i) = \{k_1, k_2, \cdots, k_q\}$. We define the local matrices at a_i by

$$\mathbb{T}_+(a_i) = \mathrm{diag}[T_{j_1}(1), T_{j_2}(1), \cdots, T_{j_p}(1)], \quad \mathbb{T}_-(a_i) = \mathrm{diag}[T_{k_1}(0), T_{k_2}(0), \cdots, T_{k_q}(0)], \tag{4.2.3}$$

$$\mathbb{E}_+(a_i) = \mathrm{diag}[E_{j_1}(1), E_{j_2}(1), \cdots, E_{j_p}(1)], \quad \mathbb{E}_-(a_i) = \mathrm{diag}[E_{k_1}(0), E_{k_2}(0), \cdots, E_{k_q}(0)], \tag{4.2.4}$$

$$\mathbb{E}'_+(a_i) = \mathrm{diag}[E'_{j_1}(1), E'_{j_2}(1), \cdots, E'_{j_p}(1)], \quad \mathbb{E}'_-(a_i) = \mathrm{diag}[E'_{k_1}(0), E'_{k_2}(0), \cdots, E'_{k_q}(0)] \tag{4.2.5}$$

and the bond matrices

$$\mathcal{Q}_+(a_i) = \begin{bmatrix} O_p & -\mathbb{T}_+(a_i) & \mathbb{E}'_+(a_i) & \mathbb{E}_+(a_i) \\ \mathbb{T}_+(a_i) & O_p & -\mathbb{E}_+(a_i) & O_p \\ -\mathbb{E}'_+(a_i) & \mathbb{E}_+(a_i) & O_p & O_p \\ -\mathbb{E}_+(a_i) & O_p & O_p & O_p \end{bmatrix}_{4p\times 4p} \tag{4.2.6}$$

$$\mathcal{Q}_-(a_i) = \begin{bmatrix} O_q & -\mathbb{T}_-(a_i) & \mathbb{E}'_-(a_i) & \mathbb{E}_-(a_i) \\ \mathbb{T}_-(a_i) & O_q & -\mathbb{E}_-(a_i) & O_q \\ -\mathbb{E}'_-(a_i) & \mathbb{E}_+(a_i) & O_q & O_q \\ -\mathbb{E}_-(a_i) & O_q & O_q & O_q \end{bmatrix}_{4q\times 4q} \tag{4.2.7}$$

where O_p denotes the $p \times p$ zero matrix.

For each $w \in H^4(E)$, we define the column vectors at a_i by

$$W_+^h(a_i) = [w_{j_1}^h(1), w_{j_2}^h(1), \cdots, w_{j_p}^h(1)]^T, \quad W_-^h(a_i) = [w_{k_1}^h(0), w_{k_2}^h(0), \cdots, w_{k_q}^h(0)]^T$$

and set

$$\widehat{W}_+(a_i) = \begin{bmatrix} W_+(a_i) \\ W'_+(a_i) \\ W''_+(a_i) \\ W'''_+(a_i) \end{bmatrix}_{1\times 4p}, \quad \widehat{W}_-(a_i) = \begin{bmatrix} W_-(a_i) \\ W'_-(a_i) \\ W''_-(a_i) \\ W'''_-(a_i) \end{bmatrix}_{1\times 4q}.$$

Then, for any $F, W \in H^4(E)$, we have

$$\begin{aligned}
& (\mathcal{L}W, F)_{L^2} - (W, \mathcal{L}F)_{L^2} \\
= \; & \sum_{j=1}^n \int_{e_j} (E_j(s) w_{j,ss}(s))_{ss} \overline{f_j(s)} ds - \sum_{k=1}^n \int_{e_k} w_k(s) \overline{(E_k(s) f_{k,ss})_{ss}} ds \\
& + \sum_{j=1}^n \int_{e_j} (T_j(s) w_{j,s}(s))_s \overline{f_j(s)} ds - \sum_{k=1}^n \int_{e_k} w_k(s) \overline{(T_k(s) f_{k,s})_s} ds \\
= \; & \sum_{j=1}^n (E_j(s) w_{j,ss}(s))_s \overline{f_j(s)}\big|_{s=1} - \sum_{j=1}^n (E_j(s) w_{j,ss}(s))_s \overline{f_j(s)}\big|_{s=0} \\
& - \sum_{j=1}^n (E_j(s) w_{j,ss}(s)) \overline{f_{j,s}(s)}\big|_{s=1} + \sum_{j=1}^n (E_j(s) w_{j,ss}(s)) \overline{f_{j,s}(s)}\big|_{s=0} \\
& - \sum_{j=1}^n T_j(s) w_{j,s}(s) \overline{f_j(s)}\big|_{s=1} + \sum_{j=1}^n T_j(s) w_{j,s}(s) \overline{f_j(s)}\big|_{s=0} \\
& - \sum_{j=1}^n w_j(s) \overline{(E_j(s) f_{j,ss})_s}(s)\big|_{s=1} + \sum_{k=1}^n w_k(s) \overline{(E_k(s) f_{k,ss})_s}(s)\big|_{s=0} \\
& + \sum_{j=1}^n E_j(s) w_{j,s}(s) \overline{f_{j,ss}(s)}\big|_{s=1} - \sum_{j=1}^n E_j(s) w_{j,s}(s) \overline{f_{j,ss}}(s)\big|_{s=0} \\
& + \sum_{j=1}^n T_j(s) w_j(s) \overline{f_{j,s}(s)}(s)\big|_{s=1} - \sum_{k=1}^n T_k(s) w_k(s) \overline{f_{k,s}}(s)\big|_{s=0} \\
= \; & \sum_{i=1}^m \sum_{j \in J^+(a_i)} [E_{j,s}(s) w_{j,ss}(s) + E_j(s) w_{j,sss}(s) - T_j(s) w_{j,s}(s)] \overline{f_j(s)}\big|_{s=1} \\
& + \sum_{i=1}^m \sum_{j \in J^+(a_i)} [-E_j(s) w_{j,ss}(s) + T_j(s) w_j(s)] \overline{f_{j,s}(s)}\big|_{s=1} \\
& + \sum_{i=1}^m \sum_{j \in J^+(a_i)} (-E_{j,s}(s) w_j(s) + E_j(s) w_{j,s}(s)) \, \overline{f_{j,ss}(s)}\big|_{s=1} \\
& + \sum_{i=1}^m \sum_{j \in J^+(a_i)} (-E_j(s) w_j(s)) \, \overline{f_{j,sss}(s)}\big|_{s=1}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\sum_{j\in J^-(a_i)}[E_{j,s}(s)w_{j,ss}(s)+E_jw_{j,sss}(s)-T_j(s)w_{j,s}(s)]\overline{f_j(s)}\big|_{s=0}\\
&+\sum_{i=1}^{m}\sum_{j\in J^-(a_i)}[-E_j(s)w_{j,ss}(s)+T_j(s)w_j(s)]\overline{f_{j,s}(s)}\big|_{s=0}\\
&+\sum_{i=1}^{m}\sum_{j\in J^-(a_i)}(-E_{j,s}w_j(s)+E_j(s)w_{j,s}(s))\,\overline{f_{j,ss}}\big|_{s=0}\\
&+\sum_{i=1}^{m}\sum_{j\in J^-(a_i)}(-E_j(s)w_k(s))\,\overline{f_{j,sss}(s)}\big|_{s=0}\\
= &\sum_{i=1}^{m}[-\mathbb{T}_+(a_i)W_+'(a_i)+\mathbb{E}_+'(a_i)W_+''(a_i)+\mathbb{E}_+(a_i)W_+'''(a_i)]\cdot\overline{F_+(a_i)}\\
&+\sum_{i=1}^{m}[\mathbb{T}_+(a_i)W_+(a_i)-\mathbb{E}_+(a_i)W_+''(a_i)]\cdot\overline{F_+'(a_i)}\\
&+\sum_{i=1}^{m}[-\mathbb{E}_+'(a_i)W_+(a_i)+\mathbb{E}_+(a_i)W_+'(a_i)]\cdot\overline{F_+''(a_i)}+\sum_{i=1}^{m}[-\mathbb{E}_+(a_i)W_+(a_i)]\cdot\overline{F_+'''(a_i)}\\
&+\sum_{i=1}^{m}[-\mathbb{T}_-(a_i)W_-'(a_i)+\mathbb{E}_-'(a_i)W_-''(a_i)+\mathbb{E}_-(a_i)W_-'''(a_i)]\cdot\overline{F_-(a_i)}\\
&+\sum_{i=1}^{m}[\mathbb{T}_-(a_i)W_-(a_i)-\mathbb{E}_-(a_i)W_-''(a_i)]\cdot\overline{F_-'(a_i)}\\
&+\sum_{i=1}^{m}[-\mathbb{E}_-'(a_i)W_-(a_i)+\mathbb{E}_-(a_i)W_+'(a_i)]\cdot\overline{F_-''(a_i)}+\sum_{i=1}^{m}[-\mathbb{E}_-(a_i)W_-(a_i)]\cdot\overline{F_-'''(a_i)}\\
= &\sum_{i=1}^{m}\left[\left(\mathcal{Q}_+(a_i)\widehat{W}_+(a_i),\widehat{F}_+(a_i)\right)_{\mathbb{C}^{4p}}-\left(\mathcal{Q}_-(a_i)\widehat{W}_-(a_i),\widehat{F}_-(a_i)\right)_{\mathbb{C}^{4q}}\right]
\end{aligned}
$$

Therefore, we have

$$
(\mathcal{L}W,F)_{L^2}-(W,\mathcal{L}F)_{L^2}=\sum_{i=1}^{m}\left(\left(\mathcal{Q}_+(a_i)\widehat{W}_+(a_i),\widehat{F}_+(a_i)\right)_{\mathbb{C}^{4p}}-\left(\mathcal{Q}_-(a_i)\widehat{W}_-(a_i),\widehat{F}_-(a_i)\right)_{\mathbb{C}^{4q}}\right) \tag{4.2.8}
$$

THEOREM 4.2.1 *Let the formal differential operator $\mathcal{L}$ be defined by (4.2.2), and let $\mathcal{Q}_\pm(a)$ be defined by (4.2.6) and (4.2.7), respectively. Then the following statements are true:*

1) For each node $a\in V$, there are at most $2(p+q)=2\#J(a)$ many linearly independent conditions, they are of the form

$$
A_a\widehat{W}_+(a)+B_a\widehat{W}_-(a)=0 \tag{4.2.9}
$$

where $A_a=A_{2(p+q)\times 4p}$ and $B_a=B_{2(p+q)\times 4q}$ and $rank(A_a,B_a)=2(p+q)$;

2) The operator $\mathcal{L}$ with nodal conditions (4.2.9) is nodal equilibrium, i.e.,

$$
\left(\mathcal{Q}_+(a)\widehat{W}_+(a),\widehat{F}_+(a)\right)_{\mathbb{C}^{4p}}-\left(\mathcal{Q}_-(a)\widehat{W}_-(a),\widehat{F}_-(a)\right)_{\mathbb{C}^{4q}}=0 \tag{4.2.10}
$$

if and only if A_a and B_a satisfy the condition

$$
A_a\mathcal{Q}_+^{-1}(a)A_a^*=B_a\mathcal{Q}_-^{-1}(a)B_a^*. \tag{4.2.11}
$$

If for each $a \in V$ the nodal conditions (4.2.11) hold, then $\mathcal{L}$ with restriction (4.2.9) is a self-adjoint operator.

Proof For each $a \in V$, there are $(p+q)$ many edges jointing it. The local vertex condition can be regard as a star shape graph with fixed boundary conditions. Therefore, $(p+q)$ many Four-order differential equations, which have already $2(p+q)$ many boundary conditions, have at most $2(p+q)$ number connection conditions at the common node a. So there are at most $2(p+q)$ linearly independent conditions at each $a \in V$.

Suppose that the nodal conditions are given by (4.2.9), then for any $X \in \mathbb{C}^{2(p+q)}$, it holds that

$$(A_a\widehat{W}_+(a), X)_{\mathbb{C}^{2(p+q)}} + (B_a\widehat{W}_-(a), X)_{\mathbb{C}^{2(p+q)}} = 0. \tag{4.2.12}$$

If $\mathcal{L}$ is the nodal equilibrium, i.e., for any $W, F \in H^4(E)$ satisfying (4.2.9), it holds that

$$\Big(\mathcal{Q}_+(a)\widehat{W}_+(a), \widehat{F}_+(a)\Big)_{\mathbb{C}^{4p}} - \Big(\mathcal{Q}_-(a)\widehat{W}_-(a), \widehat{F}_-(a)\Big)_{\mathbb{C}^{4q}} = 0. \tag{4.2.13}$$

Comparing (4.2.12) to (4.2.13), we get the relation between $F_\pm(a)$ and X

$$-\mathcal{Q}_+(a)\widehat{F}_+(a) = A_a^*X, \quad \mathcal{Q}_-(a)\widehat{F}_-(a) = B_a^*X$$

where we have used equalities $\mathcal{Q}_\pm^*(a) = -\mathcal{Q}_\pm(a)$, and hence

$$\widehat{F}_+(a) = -\mathcal{Q}_+^{-1}(a)A_a^*X, \quad \widehat{F}_-(a) = \mathcal{Q}_-^{-1}(a)B_a^*X.$$

Thus,

$$0 = A_a\widehat{F}_+(a) + B_a\widehat{F}_-(a) = -A_a\mathcal{Q}_+^{-1}(a)A_a^*X + B_a\mathcal{Q}_-^{-1}(a)B_a^*X, \quad \forall X \in \mathbb{C}^{2(p+q)}.$$

So (4.2.11) holds.

Conversely, suppose that A_a and B_a in (4.2.9) satisfy the condition (4.2.11), $W \in H^4(E)$ is a function satisfying the condition (4.2.9). By (4.2.11), there exists someone $X \in \mathbb{C}^{2(p+q)}$ such that

$$\widehat{W}_+(a) = \mathcal{Q}_+^{-1}(a)A_a^*X, \quad \widehat{W}_-(a) = -\mathcal{Q}_-^{-1}(a)B_a^*X.$$

Thus we have

$$\mathcal{Q}_+(a)\widehat{W}_+(a) = A^*X, \quad -\mathcal{Q}_-(a)\widehat{F}_-(a) = B^*X.$$

Let $F \in H^4(E)$ satisfy the condition $A_a\widehat{F}_+(a) + B_a\widehat{F}_-(a) = 0$. Then we have

$$\begin{aligned}
& \Big(\mathcal{Q}_+(a)\widehat{W}_+(a), \widehat{F}_+(a)\Big)_{\mathbb{C}^{4p}} - \Big(\mathcal{Q}_-(a)\widehat{W}_-(a), \widehat{F}_-(a)\Big)_{\mathbb{C}^{4q}} \\
= \ & \Big(A_a^*X, \widehat{F}_+(a)\Big)_{\mathbb{C}^{4p}} + \Big(B_a^*X, \widehat{F}_-(a)\Big)_{\mathbb{C}^{4q}} \\
= \ & \Big(X, A_a\widehat{F}_+(a)\Big)_{\mathbb{C}^{2(p+q)}} + \Big(X, B_a\widehat{F}_-(a)\Big)_{\mathbb{C}^{2(p+q)}} \\
= \ & \Big(X, A_a\widehat{F}_+(a) + B_a\widehat{F}_-(a)\Big)_{\mathbb{C}^{2(p+q)}} = 0.
\end{aligned}$$

This shows the nodal equilibrium conditions.

When $rank(A_a, B_a) = 2(p+q)$ and (4.2.11) hold for any $a \in V$, a direct verification shows that $\mathcal{L}$ under restriction (4.2.9) is a self-adjoint operator in $L^2(G)$. □

THEOREM 4.2.2 *Let the formal differential operator $\mathcal{L}$ be defined as (4.2.2), and let $\mathcal{Q}_{\pm}(a)$ be defined by (4.2.6) and (4.2.7) respectively. Suppose that for each $a \in V$ matrices $A_a = A_{2(p+q)\times 4p}$ and $B_a = B_{2(p+q)\times 4q}$ satisfy the conditions (4.2.11) and $rank(A_a, B_a) = 2(p+q)$. Define an operator $\mathcal{L}_0$ by $\mathcal{L}_0 = \mathcal{L}$ with domain*

$$\mathcal{D}(\mathcal{L}_0) = \{W \in H^4(E) \mid A_a \widehat{W}_+(a) + B_a \widehat{W}_-(a) = 0, \forall a \in V\}. \tag{4.2.14}$$

Then $\mathcal{L}_0$ is a positive operator if and only if A_a and B_a satisfy the condition

$$A_a \mathcal{Q}_+^{-1}(a)\mathcal{Q}_{+,1}(a)\mathcal{Q}_+^{-1}(a)A_a^* = B_a \mathcal{Q}_-^{-1}(a)\mathcal{Q}_{-,1}(a)\mathcal{Q}_-^{-1}(a)B_a^* \tag{4.2.15}$$

where

$$\mathcal{Q}_{\pm,1}(a) = \begin{bmatrix} O & -\mathbb{T}_{\pm}(a) & \mathbb{E}'_{\pm}(a) & \mathbb{E}_{\pm}(a) \\ O & O & -\mathbb{E}_{\pm}(a) & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix}$$

Proof For any $W \in H^4(E)$, we have

$$\begin{aligned}
(\mathcal{L}W, W)_{L^2} &= \sum_{j=1}^{n} \int_{e_j} (E_j(s) w_{j,ss}(s))_{ss} \overline{w_j(s)} ds - \sum_{j=1}^{n} \int_{e_j} (T_j(s) w_{j,s}(s))_s \overline{w_j(s)} ds \\
&+ \sum_{j=1}^{n} \int_{e_j} p_j(s) w_j(s) \overline{w_j(s)} ds \\
= \quad & \sum_{j=1}^{n} (E_j(s) w_{j,ss}(s))_s \overline{w_j(s)} \big|_0^1 - \sum_{j=1}^{n} E_j(s) w_{j,ss}(s)) \overline{w_{j,s}(s)}] \big|_0^1 - \sum_{j=1}^{n} T_j(s) w_{j,s}(s) \overline{w_j(s)} \big|_0^1 \\
&+ \sum_{j=1}^{n} \int_{e_j} (E_j(s)|w_{j,ss}(s)|^2 + T_j(s)|w_{j,s}(s)|^2 + p_j(s)|w_j(s)|^2) ds.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{j\in J^+(a_i)} \Big((E_j(s) w_{j,ss}(s))_s \overline{w_j(s)} - E_j(s) w_{j,ss}(s)) \overline{w_{j,s}(s)} - T_j(s) w_{j,s}(s) \overline{w_j(s)} \Big) \big|_{s=1} \\
&- \sum_{k\in J^-(a_i)} \Big((E_k(s) w_{k,ss}(s))_s \overline{w_k(s)} - E_k(s) w_{k,ss}(s) \overline{w_{k,s}(s)} - T_k(s) w_{k,s}(s) \overline{w_k(s)} \Big) \big|_{s=0} \\
= \quad & \left(\begin{bmatrix} O & -\mathbb{T}_+(a_i) & \mathbb{E}'_+(a_i) & \mathbb{E}_+(a_i) \\ O & O & -\mathbb{E}_+(a_i) & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix} \begin{bmatrix} W_+(a_i) \\ W'_+(a_i) \\ W''_+(a_i) \\ W'''_+(a_i) \end{bmatrix}, \begin{bmatrix} W_+(a_i) \\ W'_+(a_i) \\ W''_+(a_i) \\ W'''_+(a_i) \end{bmatrix} \right) \\
&- \left(\begin{bmatrix} O & -\mathbb{T}_-(a_i) & \mathbb{E}'_-(a_i) & \mathbb{E}_-(a_i) \\ O & O & -\mathbb{E}_-(a_i) & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix} \begin{bmatrix} W_-(a_i) \\ W'_-(a_i) \\ W''_-(a_i) \\ W'''_-(a_i) \end{bmatrix}, \begin{bmatrix} W_-(a_i) \\ W'_-(a_i) \\ W''_-(a_i) \\ W'''_-(a_i) \end{bmatrix} \right) \\
= \quad & \Big(\mathcal{Q}_{+,1}(a_i) \widehat{W}_+(a_i), \widehat{W}_+(a_i) \Big) - \Big(\mathcal{Q}_{-,1}(a_i) \widehat{W}_-(a_i), \widehat{W}_-(a_i) \Big),
\end{aligned}$$

we have

$$\begin{aligned}(\mathcal{L}W,W)_{L^2} &= \sum_{i=1}^{m}\Big(\mathcal{Q}_{+,1}(a_i)\widehat{W}_+(a_i),\widehat{W}_+(a_i)\Big)-\Big(\mathcal{Q}_{-,1}(a_i)\widehat{W}_-(a_i),\widehat{W}_-(a_i)\Big)\\ &\quad+\sum_{i=1}^{m}\int_{e_j}(E_j(s)|w_{j,ss}(s)|^2+T_j(s)|w_{j,s}(s)|^2+p_j(s)|w_j(s)|^2)ds.\end{aligned}$$

For any $W\in\mathcal{D}(\mathcal{L}_0)$ given, there exists corresponding an $X\in\mathbb{C}^{2(p+q)}$ such that

$$\widehat{W}_+(a_i)=\mathcal{Q}_+^{-1}(a_i)A_{a_i}^*X,\quad \widehat{W}_-(a_i)=-\mathcal{Q}_-^{-1}(a_i)B_{a_i}^*X$$

and hence

$$\begin{aligned}&\Big(\mathcal{Q}_{+,1}(a_i)\widehat{W}_+(a_i),\widehat{W}_+(a_i)\Big)_{\mathbb{C}^{4p}}-\Big(\mathcal{Q}_{-,1}(a_i)\widehat{W}_-(a_i),\widehat{W}_-(a_i)\Big)_{\mathbb{C}^{4q}}\\ =\;&\big(\mathcal{Q}_{+,1}(a_i)\mathcal{Q}_+^{-1}(a_i)A_{a_i}^*X,\mathcal{Q}_+^{-1}(a_i)A_{a_i}^*X\big)_{\mathbb{C}^{4p}}-\big(\mathcal{Q}_{-,1}(a_i)\mathcal{Q}_-^{-1}(a_i)B_{a_i}^*X,\mathcal{Q}_-^{-1}(a_i)B_{a_i}^*X\big)_{\mathbb{C}^{4q}}\\ =\;&\big(-A_{a_i}\mathcal{Q}_+^{-1}(a_i)\mathcal{Q}_{+,1}(a_i)\mathcal{Q}_+^{-1}(a_i)A_{a_i}^*X+B_{a_i}\mathcal{Q}_-^{-1}(a_i)\mathcal{Q}_{-,1}(a_i)\mathcal{Q}_-^{-1}(a_i)B_{a_i}^*X,X\big)_{\mathbb{C}^{2(p+q)}}\end{aligned}$$

where we have used equalities $\mathcal{Q}_\pm^{-1}(a)=-\mathcal{Q}_\pm^{-1}(a)$.

we only need to prove that the matrix

$$A_a\mathcal{Q}_+^{-1}(a)\mathcal{Q}_{+,1}(a)\mathcal{Q}_+^{-1}(a)A_a^*-B_a\mathcal{Q}_-^{-1}(a)\mathcal{Q}_{-,1}(a)\mathcal{Q}_-^{-1}(a)B_a^*$$

is Hermitian. In fact, noting that $\mathcal{Q}_\pm(a)=\mathcal{Q}_{\pm,1}(a)-\mathcal{Q}_{\pm,1}^*(a))$, we have

$$\begin{aligned}&\big(A_a\mathcal{Q}_+^{-1}(a)\mathcal{Q}_{+,1}(a)\mathcal{Q}_+^{-1}(a)A_a^*-B_a\mathcal{Q}_-^{-1}(a)\mathcal{Q}_{-,1}(a)\mathcal{Q}_-^{-1}(a)B_a^*\big)^*\\ =\;&A_a(\mathcal{Q}_+^{-1}(a))^*(\mathcal{Q}_{+,1}(a))^*(\mathcal{Q}_+^{-1}(a))^*A_a^*-B_a(\mathcal{Q}_-^{-1}(a))^*(\mathcal{Q}_{-,1}(a))^*(\mathcal{Q}_-^{-1}(a))^*B_a^*\\ =\;&A_a\mathcal{Q}_+^{-1}(a)\mathcal{Q}_{+,1}^*(a)\mathcal{Q}_+^{-1}(a)A_a^*-B_a\mathcal{Q}_-^{-1}(a)\mathcal{Q}_{-,1}^*(a)\mathcal{Q}_-^{-1}(a)B_a^*\\ =\;&A_a\mathcal{Q}_+^{-1}(a)[\mathcal{Q}_{+,1}(a)-\mathcal{Q}_+(a)]\mathcal{Q}_+^{-1}(a)A_a^*-B_a\mathcal{Q}_-^{-1}(a)[\mathcal{Q}_{-,1}(a)-\mathcal{Q}_-(a)]\mathcal{Q}_-^{-1}(a)B_a^*\\ =\;&A_a\mathcal{Q}_+^{-1}(a)\mathcal{Q}_{+,1}(a)\mathcal{Q}_+^{-1}(a)A_a^*-B_a\mathcal{Q}_-^{-1}(a)\mathcal{Q}_{-,1}(a)\mathcal{Q}_-^{-1}(a)B_a^*\\ &-[A_a\mathcal{Q}_+^{-1}(a)A_a^*-B_a\mathcal{Q}_-^{-1}(a)B_a^*]\\ =\;&A_a\mathcal{Q}_+^{-1}(a)\mathcal{Q}_{+,1}(a)\mathcal{Q}_+^{-1}(a)A_a^*-B_a\mathcal{Q}_-^{-1}(a)\mathcal{Q}_{-,1}(a)\mathcal{Q}_-^{-1}(a)B_a^*\end{aligned}$$

where we have used equality $A_a\mathcal{Q}_+^{-1}(a)A_a^*=B_a\mathcal{Q}_-^{-1}(a)B_a^*$. Therefore, $\mathcal{L}_0$ is positive if and only if

$$A_a\mathcal{Q}_+^{-1}(a)\mathcal{Q}_{+,1}(a)\mathcal{Q}_+^{-1}(a)A_a^*=B_a\mathcal{Q}_-^{-1}(a)\mathcal{Q}_{-,1}(a)\mathcal{Q}_-^{-1}(a)B_a^*.$$

The desired result follows. □

4.2.2 The structural equilibrium condition

In this subsection we discuss the whole structure equilibrium conditions. For each $W\in L^2(E)$, we introduce a vector-valued function $W(s)=[w_1(s),w_2(s),\cdots,w_n(s)]^T$.

Let $\mathcal{L}$ be defined by (4.2.2), we shall rewrite $\mathcal{L}$ into the matrix form. Setting

$$\mathbb{E}(s)=\mathrm{diag}[E_1(s),E_2(s),E_3(s),\cdots,E_n(s)];$$

$$\mathbb{T}(s)=\mathrm{diag}[T_1(s),T_2(s),T_3(s),\cdots,T_n(s)];$$

$$\mathbb{P}(s)=\mathrm{diag}[p_1(s),p_2(s),p_3(s),\cdots,p_n(s)];$$

and denote

$$\mathcal{L} = \frac{d^2}{ds^2}\left(\mathbb{E}(s)\frac{d^2}{ds^2}\right) - \frac{d}{ds}\left(\mathbb{T}(s)\frac{d}{ds}\right) + \mathbb{P}(s)$$
$$\mathcal{L}W(s) = [\mathcal{L}w_1(s), \mathcal{L}w_2(s), \cdots, \mathcal{L}w_n(s)]^T.$$

For any $W, F \in L^2(E)$, $W(s) = [w_1(s), w_2(s), \cdots, w_n(s)]^T$ and $F(s) = [f_1(s), f_2(s), \cdots, f_n(s)]^T$, we have

$$(W, F)_{L^2} = \int_G W(x)\overline{F(x)}dx = \sum_{j=1}^{n}\int_0^1 w_j(s)\overline{f_j(s)}ds = \int_0^1 (W(s), F(s))_{\mathbb{C}^n} ds.$$

For any $W, F \in H^4(E)$,

$$\begin{aligned}
& (\mathcal{L}W, F)_{L^2} - (W, \mathcal{L}F)_{L^2} \\
= \ & \int_0^1 (\mathcal{L}W(s), F(s))_{\mathbb{C}^n} ds - \int_0^1 (W(s), \mathcal{L}F(s))_{\mathbb{C}^n} ds \\
= \ & ((\mathbb{E}(x)W''(s))', F(s))_{\mathbb{C}^n}\Big|_0^1 - (\mathbb{E}(x)W''(s), F'(s))_{\mathbb{C}^n}\Big|_0^1 - (\mathbb{T}(s)W'(s), F(s))_{\mathbb{C}^n}\Big|_0^1 \\
& -(W(s), (\mathbb{E}(s)F''(s))')_{\mathbb{C}^n}\Big|_0^1 + (W'(s), \mathbb{E}(s)F''(s))_{\mathbb{C}^n}\Big|_0^1 + (W(s), \mathbb{T}(s)F'(s))_{\mathbb{C}^n}\Big|_0^1 \\
= \ & (-\mathbb{T}(s)W'(s) + \mathbb{E}'(s)W''(s) + \mathbb{E}(s)W'''(s), F(s))_{\mathbb{C}^n}\Big|_0^1 \\
& +(\mathbb{T}(s)W(s) - \mathbb{E}(s)W''(s), F'(s))_{\mathbb{C}^n}\Big|_0^1 \\
& (-\mathbb{E}'(s)W(s) + \mathbb{E}(x)W'(s), F''(s)))_{\mathbb{C}^n}\Big|_0^1 - (\mathbb{E}(s)W(xs), F'''(s))_{\mathbb{C}^n}\Big|_0^1 \\
= \ & \left(\begin{bmatrix} O_n & -\mathbb{T}(s) & \mathbb{E}'(s) & \mathbb{E}(s) \\ \mathbb{T}(s) & O_n & -\mathbb{E}(s) & O_n \\ -\mathbb{E}'(s) & \mathbb{E}(s) & O_n & O_n \\ -\mathbb{E}(s) & O_n & O_n & O_n \end{bmatrix}\begin{bmatrix} W(s) \\ W'(s) \\ W''(s) \\ W'''(s) \end{bmatrix}, \begin{bmatrix} F(s) \\ F'(s) \\ F''(s) \\ F'''(s) \end{bmatrix}\right)_{\mathbb{C}^{4n}} \Bigg|_0^1 \\
= \ & \left(\mathcal{Q}(1)\widehat{W}(1), \widehat{F}(1)\right)_{\mathbb{C}^{4n}} - \left(\mathcal{Q}(0)\widehat{W}(0), \widehat{F}(0)\right)_{\mathbb{C}^{4n}}
\end{aligned}$$

where

$$\mathcal{Q}(s) = \begin{bmatrix} O_n & -\mathbb{T}(s) & \mathbb{E}'(s) & \mathbb{E}(s) \\ \mathbb{T}(s) & O_n & -\mathbb{E}(s) & O_n \\ -\mathbb{E}'(s) & \mathbb{E}(s) & O_n & O_n \\ -\mathbb{E}(s) & O_n & O_n & O_n \end{bmatrix}, \quad \widehat{W}(s) = \begin{bmatrix} W(s) \\ W'(s) \\ W''(s) \\ W'''(s) \end{bmatrix}. \tag{4.2.16}$$

THEOREM 4.2.3 *Let G be a metric graph, and $\mathcal{L}$ be defined by (4.2.2). Suppose that the connective conditions at all vertices are given by*

$$A\widehat{W}(1) + B\widehat{W}(0) = 0, \quad A, B \in \mathbb{M}_{4n\times 4n}, \quad \operatorname{rank}(A, B) = 4n. \tag{4.2.17}$$

Then $\mathcal{L}$ is structural self-adjoint if only if A and B satisfy the condition

$$A\mathcal{Q}^{-1}(1)A^* = B\mathcal{Q}^{-1}(0)B^*. \tag{4.2.18}$$

Proof Let $W \in H^4(E)$ satisfy the connective conditions

$$A\widehat{W}(1) + B\widehat{W}(0) = 0, \quad A, B \in \mathbb{M}_{4n\times 4n}$$

where A and B satisfy the condition (4.2.18). Then there exists an $X \in \mathbb{C}^{4n}$ such that

$$\widehat{W}(1) = \mathcal{Q}^{-1}(1)A^*X, \quad \widehat{W}(0) = -\mathcal{Q}^{-1}(1)B^*X,$$

and hence for any $F \in H^4(E)$,

$$\begin{aligned}(\mathcal{L}W, F)_{L^2} - (W, \mathcal{L}F)_{L^2} &= \Big(\mathcal{Q}(1)\widehat{W}(1), \widehat{F}(1)\Big)_{\mathbb{C}^{4n}} - \Big(\mathcal{Q}(0)\widehat{W}(0), \widehat{F}(0)\Big)_{\mathbb{C}^{4n}} \\ &= \Big(X, A\widehat{F}(1) + B\widehat{F}(0)\Big)_{\mathbb{C}^{4n}}.\end{aligned}$$

Therefore, $\mathcal{L}$ is structural self-adjoint if and only if $A\widehat{F}(1) + B\widehat{F}(0) = 0$.

Conversely, if $\mathcal{L}$ is structure self adjoint, i.e.,

$$(\mathcal{L}W, F)_{L^2} - (W, \mathcal{L}F)_{L^2} = 0, \quad \forall W, F \in \mathcal{D}(\mathcal{L}) = \{W \in H^4(E) \mid A\widehat{W}(1) + B\widehat{W}(0) = 0\}.$$

Then $(\mathcal{L}W, F)_{L^2} = (W, \mathcal{L}F)_{L^2}$ implies

$$\Big(\mathcal{Q}(1)\widehat{W}(1), \widehat{F}(1)\Big)_{\mathbb{C}^{4n}} - \Big(\mathcal{Q}(0)\widehat{W}(0), \widehat{F}(0)\Big)_{\mathbb{C}^{4n}} = 0$$

and the connection conditions lead to

$$(A\widehat{W}(1) + B\widehat{W}(0), X)_{\mathbb{C}^{4n}} = (\widehat{W}(1), A^*X)_{\mathbb{C}^{4n}} + (\widehat{W}(0), B^*X)_{\mathbb{C}^{4n}} = 0, \forall X \in \mathbb{C}^{4n}$$

Comparing both equalities above we get that

$$\mathcal{Q}^*(1)\widehat{F}(1) = A^*X, \quad \mathcal{Q}^*(0)\widehat{F}(0) = -B^*X.$$

Note that $\mathcal{Q}^*(1) = -\mathcal{Q}(1)$ and $\mathcal{Q}^*(0) = -\mathcal{Q}(0)$. Thus

$$A\widehat{F}(1) + B\widehat{F}(0) = -A\mathcal{Q}^{-1}(1)A^*X + B\mathcal{Q}^{-1}(0)B^*X = 0, \quad \forall X \in \mathbb{C}^{4n}$$

The desired result (4.2.18) follows from above equality. □

THEOREM 4.2.4 *Let the formal differential operator $\mathcal{L}$ be defined as before, and let the connective condition be given by (4.2.17). Then $\mathcal{L}$ is a structural positive operator if only if A and B satisfy the condition*

$$A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^* = B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^* \tag{4.2.19}$$

where

$$\mathcal{Q}_1(s) = \begin{bmatrix} O_n & -\mathbb{T}(s) & \mathbb{E}'(s) & \mathbb{E}(s) \\ O_n & O_n & -\mathbb{E}(s) & O_n \\ O_n & O_n & O_n & O_n \\ O_n & O_n & O_n & O_n \end{bmatrix}_{4n\times 4n}.$$

Proof Firstly we prove that (4.2.19) implies (4.2.18), i.e., $A\mathcal{Q}^{-1}(1)A^* = B\mathcal{Q}^{-1}(0)B^*$. Suppose that A and B satisfy (4.2.19). Taking adjoint operation for (4.2.19) leads to

$$A\mathcal{Q}^{-1}(1)\mathcal{Q}_1^*(1)\mathcal{Q}^{-1}(1)A^* = B\mathcal{Q}^{-1}(0)\mathcal{Q}_1^*(0)\mathcal{Q}^{-1}(0)B^*$$

where we have used $(\mathcal{Q}^{-1}(s))^* = -\mathcal{Q}^{-1}(s)$. By the definition of $\mathcal{Q}_1(s)$, we have $\mathcal{Q}_1(s) - \mathcal{Q}_1^*(s) = \mathcal{Q}(s)$. Hence

$$\begin{aligned} & A\mathcal{Q}^{-1}(1)A^* - B\mathcal{Q}^{-1}(0)B^* \\ = \;& A\mathcal{Q}^{-1}(1)[\mathcal{Q}_1(1) - \mathcal{Q}_1^*(1)]\mathcal{Q}^{-1}(1)A^* - B\mathcal{Q}^{-1}(0)[\mathcal{Q}_1(0) - \mathcal{Q}_1^*(0)]\mathcal{Q}^{-1}(0)B^* \\ = \;& A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^* - B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^* \\ & -A\mathcal{Q}^{-1}(1)\mathcal{Q}_1^*(1)\mathcal{Q}^{-1}(1)A^* + B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^* = 0. \end{aligned}$$

Next, for any $W \in H^4(E)$, we have

$$\begin{aligned} & (\mathcal{L}W, W)_{L^2} - \int_0^1 [(\mathbb{E}(s)W''(s), W''(s))_{\mathbb{C}^n} + (\mathbb{T}(s)W'(s), W'(s))_{\mathbb{C}^n} + (\mathbb{P}(s)W(s), W(s))_{\mathbb{C}^n}]ds \\ = \;& ((E'(1)W''(1) + E(1)W'''(1), W(1))_{\mathbb{C}^n} - (E'(0)W''(0) + E(0)W'''(0), W(0))_{\mathbb{C}^n} \\ & -((E(1)W''(1), W'(1))_{\mathbb{C}^n} + (E(0)W''(0), W'(0))_{\mathbb{C}^n} \\ & -((T(1)W'(1), W(1))_{\mathbb{C}^n} + (T(0)W'(0), W(0))_{\mathbb{C}^n} \\ = \;& \left(\mathcal{Q}_1(1)\widehat{W}(1), \widehat{W}(1)\right)_{\mathbb{C}^{4n}} - \left(\mathcal{Q}_1(0)\widehat{W}(0), \widehat{W}(0)\right)_{\mathbb{C}^{4n}}. \end{aligned}$$

Therefore, $\mathcal{L}$ is structural positive if and only if

$$\left(\mathcal{Q}_1(1)\widehat{W}(1), \widehat{W}(1)\right)_{\mathbb{C}^{4n}} - \left(\mathcal{Q}_1(0)\widehat{W}(0), \widehat{W}(0)\right)_{\mathbb{C}^{4n}} = 0. \tag{4.2.20}$$

When (4.2.20) holds, we have

$$(\mathcal{L}W, W)_{L^2} = \int_0^1 [(\mathbb{E}(s)W''(s), W''(s))_{\mathbb{C}^n} + (\mathbb{T}(s)W'(s), W'(s))_{\mathbb{C}^n} + (\mathbb{P}(s)W(s), W(s))_{\mathbb{C}^n}]ds$$

Finally, we prove that $\mathcal{L}$ is structural positive if and only if the condition(4.2.19) holds. Let $\mathcal{L}$ be the structural positive operator, and let A and B satisfy the condition (4.2.18). Then for any $X \in \mathbb{C}^{4n}$, taking $W \in H^4(E)$ satisfying $\widehat{W}(1) = \mathcal{Q}^{-1}(1)A^*X$ and $\widehat{W}(0) = -\mathcal{Q}^{-1}(0)B^*X$, we have $A\widehat{W}(1) + B\widehat{W}(0) = 0$. Thus we have

$$\begin{aligned} 0 \;=\;& \left(\mathcal{Q}_1(1)\widehat{W}(1), \widehat{W}(1)\right)_{\mathbb{C}^{4n}} - \left(\mathcal{Q}_1(0)\widehat{W}(0), \widehat{W}(0)\right)_{\mathbb{C}^{4n}} \\ =\;& \left(\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^*X, \mathcal{Q}^{-1}(1)A^*X\right)_{\mathbb{C}^{4n}} - \left(\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*X, \mathcal{Q}^{-1}(0)B^*X\right)_{\mathbb{C}^{4n}} \\ =\;& -\left(A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^*X - B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*X, X\right)_{\mathbb{C}^{4n}}. \end{aligned}$$

Using the Hermitian property of the matrix $A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^* - B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*$, we get

$$A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^* = B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*.$$

Conversely, we suppose that A and B satisfy the condition (4.2.19), which implies (4.2.18), and the connection condition is given by (4.2.17). For any $W \in H^4(E)$ satisfying $AW(1) + BW(0) = 0$, according to Theorem 4.2.3, there exists an $X \in \mathbb{C}^{4n}$ such that

$$\widehat{W}(1) = \mathcal{Q}^{-1}(1)A^*X, \quad \widehat{W}(0) = -\mathcal{Q}^{-1}(0)B^*X.$$

Hence we have

$$\begin{aligned}
&\Big(\mathcal{Q}_1(1)\widehat{W}(1),\widehat{W}(1)\Big)_{\mathbb{C}^{4n}}-\Big(\mathcal{Q}_1(0)\widehat{W}(0),\widehat{W}(0)\Big)_{\mathbb{C}^{4n}}\\
=\ &\big(\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^*X,\mathcal{Q}^{-1}(1))A^*X\big)_{\mathbb{C}^{4n}}-\big(\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*X,\mathcal{Q}^{-1}(0)B^*X\big)_{\mathbb{C}^{4n}}\\
=\ &\big((-A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^*+B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*)X,X\big)_{\mathbb{C}^{4n}}=0
\end{aligned}$$

Therefore, $\mathcal{L}$ is structure positive. □

REMARK 4.2.1 *In the case of structure equilibrium, the connection conditions are not obviously dependent upon the structure of the graph.*

4.2.3 Some classical vertex conditions

In this subsection we discuss some classical local vertex conditions. Here we shall employ the notions of mechanics for a beam: $w(x)$ denotes the displacement of beam depart from the equilibrium state; $w_x(x)$ denotes the rotation angle of beam; $E(x)w_{xx}(x)$ denotes the bending moment and $(E(x)w_{xx})_x$ denotes the shearing force.

Let $G=(V,E)$ be a metric graph and $W\in H^4(E)$. At an interior vertex $a\in V$, we have equality

$$\begin{aligned}
&\Big(\mathcal{Q}_{+,1}(a)\widehat{W}_+(a),\widehat{W}_+(a)\Big)-\Big(\mathcal{Q}_{-,1}(a)\widehat{W}_-(a),\widehat{W}_-(a)\Big)\\
=\ &\sum_{j\in J^+(a)}\Big((E_j(s)w_{j,ss}(s))_s\overline{w_j(s)}-E_j(s)w_{j,ss}(s))\overline{w_{j,s}(s)}-T_j(s)w_{j,s}(s)\overline{w_j(s)}\Big)\big|_{s=1}\\
&-\sum_{k\in J^-(a)}\Big((E_k(s)w_{k,ss}(s))_s\overline{w_k(s)}-E_k(s)w_{k,ss}(s)\overline{w_{k,s}(s)}-T_k(s)w_{k,s}(s)\overline{w_k(s)}\Big)\big|_{s=0}\\
=\ &\sum_{j\in J^+(a)}((E_j(s)w_{j,ss}(s))_s-T_j(s)w_{j,s}(s))\,\overline{w_j(s)}\big|_{s=1}\\
&-\sum_{k\in J^-(a)}((E_k(s)w_{k,ss}(s))_s-T_k(s)w_{k,s}(s))\,\overline{w_k(s)}\big|_{s=0}\\
&-\sum_{j\in J^+(a)}[E_j(s)w_{j,ss}(s))\overline{w_{j,s}(s)}]\big|_{s=1}+\sum_{k\in J^-(a)}[E_k(s)w_{k,ss}(s)\overline{w_{k,s}(s)}]\big|_{s=0}.
\end{aligned}$$

§1. δ-type vertex conditions

At each vertex $a\in V$, we impose the geometry and the moment continuity conditions: the displacements of all edges jointed are continuous, and the bending moments also are continuous. Using (4.2.11), we deduce the local vertex conditions at $a\in V$:

$$\begin{cases}
w_j(1)=w_k(0)=w(a),\forall k\in J^-(a),j\in J^+(a)\\
E_j(1)w_{j,ss}(1)=E_k(0)w_{k,ss}(0)=U(a),\forall k\in J^-(a),j\in J^+(a)\\
\sum\limits_{j\in J^+(a)}w_{j,s}(1)-\sum\limits_{k\in J^-(a)}w_{k,s}(0)=0\\
\sum\limits_{j\in J^+(a)}[(E_j(s)w_{j,ss}(s))_s(s)-T_j(s)w_{j,s}(s)]_{s=1}\\
-\sum\limits_{k\in J^-(a)}(E_k(s)w_{k,ss}(s))_s(s)-T_k(s)w_{k,s}(s)]_{s=0}=0.
\end{cases}\tag{4.2.21}$$

The third condition is a geometrical condition and the last is the dynamic equilibrium condition.

§2. δ-type vertex conditions

At each $a \in V$ we impose the geometric constraints: the displacement of all edges jointed are continuous, and the rotation angles of the structure are continuous. Then the local vertex conditions are given by

$$\begin{cases} w_j(1) = w_k(0) = W(a), \forall k \in J^-(a), j \in J^+(a) \\ w_{j,s}(1) = w_{k,s}(0) = U(a), \forall k \in J^-(a), j \in J^+(a) \\ \sum\limits_{j \in J^+(a)} E_j(s) w_{j,ss}(1) - \sum\limits_{k \in J^-(a)} E_k(0) w_{k,ss}(0) = 0 \\ \sum\limits_{j \in J^+(a)} [(E_j(s) w_{j,ss}(s))_s(s) - T_j(s) w_{j,s}(s)]_{s=1} \\ - \sum\limits_{k \in J^-(a)} (E_k(s) w_{k,ss}(s))_s(s) - T_k(s) w_{k,s}(s)]_{s=0} = 0. \end{cases} \tag{4.2.22}$$

§3. δ'-type vertex conditions

At each $a \in V$ we assume that the dynamic continuity conditions: the bending moments and the shearing forces of the structure are continuous. Then the local vertex conditions are given by

$$\begin{cases} (E_j(s) w_{j,ss}(s))_s(1) - T_j(1) w_{j,s}(1) = -(E_k(s) w_{k,ss}(s))_s(0) + T_k(0) w_{j,s}(0) = F(a), \\ \forall k \in J^-(a), j \in J^+(a) \\ E_j(1) w_{j,ss}(1) = E_k(0) w_{k,ss}(0) = U(a), \forall k \in J^-(a), j \in J^+(a) \\ \sum\limits_{j \in J^+(a)} w_j(1) + \sum\limits_{k \in J^-(a)} w_k(0) = 0 \\ \sum\limits_{j \in J^+(a)} w_{j,s}(1) - \sum\limits_{k \in J^-(a)} w_{k,s}(0) = 0. \end{cases} \tag{4.2.23}$$

4.2.4 Euler-Bernoulli beam equation on graphs

Here we consider the Euler-Bernoulli beam equation on a metric graph G. Let $u(x,t)$ be a function defined on $G \times \mathbb{R}_+$, $u_j(s,t)$ be its normalized realization on $e_j \times \mathbb{R}_+$. Suppose that $u_j(s,t), j \in \{1, 2, \cdots, n\}$ satisfy the partial differential equation

$$m_j(s) \frac{\partial^2 u_j(s,t)}{\partial t^2} = -\frac{\partial^2}{\partial s^2} \left(E_j(s) \frac{\partial^2 u_j(s,t)}{\partial s^2} \right) + \frac{\partial}{\partial s} \left(T_j(s) \frac{\partial u_j(s,t)}{\partial s} \right) - p_j(s) u_j(s,t), \quad s \in (0,1), \tag{4.2.24}$$

where $m_j(s)$, $E_j(s)$ and $T_j(s)$ are positive continuous function, and $p_j(s)$ are nonnegative continuous functions.

We define a diagonal matrix

$$\mathbb{M}(s) = \mathrm{diag}[m_1(s), m_2(s), \cdots, m_n(s)]$$

then the equations (4.2.24) are equivalent to the vector-valued differential equations

$$\mathbb{M}(s)\frac{\partial^2 W(s,t)}{\partial t^2} = -\mathcal{L}W(s,t), \quad s \in (0,1) \tag{4.2.25}$$

where

$$\mathcal{L}W(s,t) = \frac{\partial^2}{\partial s^2}\left(\mathbb{E}(s)\frac{\partial^2 W(s,t)}{\partial s^2}\right) - \frac{\partial}{\partial s}\left(\mathbb{T}(s)\frac{\partial W(s,t)}{\partial s}\right) + \mathbb{P}(s)W(s,t).$$

Let $\mathcal{Q}(x)$ be defined by (4.2.16) and $A, B \in \mathbb{M}_{4n\times 4n}$ satisfy the conditions $rank(A,B) = 4n$ and (4.2.19), i.e.,

$$A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^* = B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*.$$

Then the differential operator $\mathcal{L}$ with domain

$$\mathcal{D}(\mathcal{L}) = \{W \in H^4(E) \mid A[W(1), W'(1), W''(1), W'''(1)]^T + B[W(0), W'(0), W''(0), W'''(0)]^T = 0\}$$

is a positive operator in $L^2(E)$ according to Theorem 4.2.4.

THEOREM 4.2.5 *Let G be a metric graph. Let $W(s,t)$ be a solution to (4.2.25) satisfying conditions*

$$A\widehat{W}(1,t) + B\widehat{W}(0,t) = 0, \quad \widehat{W}(s,t) = [W(s,t), W_s(s,t), W_{ss}(s,t), W_{sss}(s,t)]^T \tag{4.2.26}$$

The energy function of system (4.2.25) is defined as

$$\begin{aligned}\mathcal{E}(t) \quad &= \quad \frac{1}{2}\int_0^1 [(\mathbb{E}(s)W_{ss}(s,t), W_{ss}(s,t))_{\mathbb{R}^n} + (\mathbb{T}(s)W_s(s,t), W_s(s,t))_{\mathbb{R}^n}]ds \\ &\quad + \frac{1}{2}\int_0^1 [(\mathbb{P}(s)W(s,t), W(s,t))_{\mathbb{R}^n} + (\mathbb{M}(s)W_t(s,t), W_t(s,t))_{\mathbb{R}^n}]dx. \end{aligned} \tag{4.2.27}$$

Then the energy of the system is conservation.

Proof Let the energy function is defined by (4.2.27). Then we have

$$\begin{aligned}\frac{d\mathcal{E}(t)}{dt} &= \int_0^1 [(\mathbb{E}(s)W_{ss}(x,t), W_{sst}(x,t))_{\mathbb{R}^n} + (\mathbb{T}(s)W_s(s,t), W_{st}(s,t))_{\mathbb{R}^n}]ds \\ &\quad + \int_0^1 [(\mathbb{P}(s)W(s,t), W_t(s,t))_{\mathbb{R}^n} + (\mathbb{M}(s)W_{tt}(s,t), W_t(s,t))_{\mathbb{R}^n}]ds \\ &= \int_0^1 [(\mathbb{E}(s)W_{ss}(s,t), W_{sst}(s,t))_{\mathbb{R}^n} + (\mathbb{T}(s)W_s(s,t), W_{st}(s,t))_{\mathbb{R}^n}]ds \\ &\quad - \int_0^1 [((\mathbb{E}(s)W_{ss}(s,t))_{xx} - (\mathbb{T}(s)W_s(s,t))_x, W_t(s,t))_{\mathbb{R}^n} ds \\ &= -[((\mathbb{E}(s)W_{ss}(s,t))_s - \mathbb{T}(s)W_s(s,t), W_t(s,t))_{\mathbb{R}^n}\Big|_{s=0}^{s=1} \\ &\quad + (\mathbb{E}(s)W_{ss}(s,t), W_{st}(s,t))_{\mathbb{R}^n}\Big|_{s=0}^{s=1} \\ &= [-\mathbb{T}(s)W_s(s,t) + \mathbb{E}'(s)W_{ss}(x,t) + \mathbb{E}(s)W_{sss}(s,t), W_t(s,t))_{\mathbb{R}^n}\Big|_{s=0}^{s=1} \\ &\quad + (\mathbb{E}(s)W_{ss}(s,t), W_{st}(s,t))_{\mathbb{R}^n}\Big|_{s=0}^{s=1} \\ &= (\mathcal{Q}_1(1)\widehat{W}(1,t), \widehat{W}_t(1,t))_{R^{4n}} - (\mathcal{Q}_1(0)\widehat{W}(0,t), \widehat{W}_t(0,t))_{\mathbb{R}^{4n}}.\end{aligned}$$

Using condition (4.2.19), there exist $X(t) \in \mathbb{C}^{4n}$ such that

$$\widehat{W}(1,t) = \mathcal{Q}^{-1}(1)A^* X(t), \quad \widehat{W}(0,t) = -\mathcal{Q}^{-1}(0)B^* X(t),$$

and hence

$$\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} &= (\mathcal{Q}_1(1)\widehat{W}(1,t), \widehat{W}_t(1,t))_{\mathbb{R}^{4n}} - (\mathcal{Q}_1(0)\widehat{W}(0,t), \widehat{W}_t(0,t))_{\mathbb{R}^{4n}} \\
&= (\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^* X(t), \mathcal{Q}^{-1}(1)A^* X'(t))_{\mathbb{R}^{4n}} - (\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^* X(t), \mathcal{Q}^{-1}(0)B^* X'(t))_{\mathbb{R}^{4n}} \\
&= -((A\mathcal{Q}^{-1}(1)Q_1(1)\mathcal{Q}^{-1}(1)A^* - B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*)X(t), X'(t))_{\mathbb{R}^{4n}} = 0.
\end{aligned}$$

Therefore, the system (4.2.25) is the energy conservation. □

4.3 Timoshenko beam equations on graphs

Let G be a metric graph with the vertex set $V = \{a_1, a_2, \cdots, a_m\}$ and the edge set $E = \{e_1, e_2, \cdots, e_n\}$. Let $w(x,t)$ and $\varphi(x,t)$ be functions defined on $G \times \mathbb{R}_+$, $w_j(s,t)$ and $\varphi_j(s,t)$ be their normalized realization on $e_j \times \mathbb{R}_+$, respectively. If the pair $(w_j(s,t), \varphi_j(s,t)$ satisfy the partial differential equations

$$\begin{cases} \rho_j(s)\frac{\partial^2 w_j(s,t)}{\partial t^2} = \frac{\partial}{\partial s}\left(K_j(s)\frac{\partial w_j(s,t)}{\partial s} - \varphi_j(s,t)\right), \\ I_{\rho_j}(s)\frac{\partial^2 \varphi_j(s,t)}{\partial t^2} = \frac{\partial}{\partial s}\left(E_j(s)\frac{\partial \varphi_j(x,t)}{\partial x}\right) + \left(K_j(s)\frac{\partial w_j(s,t)}{\partial s} - \varphi_j(s,t)\right), \end{cases} \quad s \in (0,1), \tag{4.3.1}$$

where $\rho_j(s)$,$I_{\rho_j}(s)$, $E_j(s)$ and $K_j(s)$ are positive continuous differentiable function, then the pair $(w(x,t), \varphi(x,t))$ is called satisfying the Timoshenko beam equation on G.

The partial differential equations are defined on E, we need some connective and boundary conditions and the initial data to determine uniquely a pair functions.

4.3.1 Nodal condition for self-adjoint operator

We consider the self-adjoint property of the differential operator in $L^2(G) \times L^2(G)$, which is defined on each edge e_j by

$$\mathcal{L}\begin{bmatrix} w_j \\ \varphi_j \end{bmatrix} = \begin{bmatrix} (K_j(s)(w_{j,s}(s) - \varphi_j(s)))_s \\ (E_j(s)\varphi_{j,s}(s))_s + K_j(s)(w_{j,s}(s) - \varphi_j(s)) \end{bmatrix}, \quad s \in (0,1), \quad j = 1, 2, \cdots, n. \tag{4.3.2}$$

For any $F = [f,g] \in H^2(E) \times H^2(E)$, $W = [w, \varphi] \in H^2(E) \times H^2(E)$,

$$\begin{aligned}
&(\mathcal{L}W, F)_{L^2} - (W, \mathcal{L}F)_{L^2} \\
= &\sum_{j=1}^n \int_{e_j} (K_j(s)(w_{j,s}(s) - \varphi_j(s))_s \overline{f_j(s)} ds + \sum_{j=1}^n \int_{e_j} (E_j(s)(\varphi_{j,s}(s))_s \overline{g_j(s)} ds \\
&+ \sum_{j=1}^n \int_{e_j} (K_j(s)(w_{j,s}(s) - \varphi_j(s))\overline{g_j(s)} ds - \sum_{j=1}^n \int_{e_j} w_j(s)\overline{(K_j(s)(f_{j,s}(s) - g_j(s))_s} ds
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{j=1}^{n}\int_{e_j}\varphi_j(s)\overline{(E_j(s)(g_{j,s}(s))_s}ds-\sum_{j=1}^{n}\int_{e_j}\varphi_j(s)\overline{K_j(s)(f_{j,s}(s)-g_j(s))}ds\\
= & \sum_{j=1}^{n}K_j(s)(w_{j,s}(s)-\varphi_j(s))\overline{f_j(s)}\big|_0^1+\sum_{j=1}^{n}E_j(s)\varphi_{j,s}(s)\overline{g_j(s)}\big|_0^1\\
& -\sum_{j=1}^{n}w_j(s)\overline{(K_j(s)(f_{j,s}(s)-g_j(s)))}\big|_0^1-\sum_{j=1}^{n}\varphi_j(s)\overline{(E_j(s)(g_{j,s}(s))}\big|_0^1
\end{aligned}
$$

Set $J^+(a_i)=\{j_1,j_2,\cdots,j_p\}$, $J^-(a_i)=\{k_1,k_2,\cdots,k_q\}$. We define the local matrices at a_i by

$$
\begin{cases}
\mathbb{K}_+(a_i)=\mathrm{diag}(K_{j_1}(1),K_{j_2}(1),\cdots,K_{j_p}(1)),\\
\mathbb{K}_-(a_i)=\mathrm{diag}(K_{k_1}(0),K_{k_2}(0),\cdots,K_{k_q}(0)),\\
\mathbb{E}_+(a_i)=\mathrm{diag}(E_{j_1}(1),E_{j_2}(1),\cdots,E_{j_p}(1)),\\
\mathbb{E}_-(a_i)=\mathrm{diag}(E_{k_1}(0),E_{k_2}(0),\cdots,E_{k_q}(0))
\end{cases}
\tag{4.3.3}
$$

and the bond matrices

$$
\mathcal{Q}_+(a_i)=\begin{bmatrix}
O_p & -\mathbb{K}_+(a_i) & \mathbb{K}_+(a_i) & O_p\\
\mathbb{K}_+(a_i) & O_p & O_p & \mathbb{E}_+(a_i)\\
-\mathbb{K}_+(a_i) & O_p & O_p & O_p\\
O_p & -\mathbb{E}_+(a_i) & O_p & O_p
\end{bmatrix}_{4p\times 4p}
\tag{4.3.4}
$$

$$
\mathcal{Q}_-(a_i)=\begin{bmatrix}
O_q & -\mathbb{K}_-(a_i) & \mathbb{K}_-(a_i) & O_q\\
\mathbb{K}_-(a_i) & O_q & O_q & \mathbb{E}_-(a_i)\\
-\mathbb{K}_-(a_i) & O_q & O_q & O_q\\
O_q & -\mathbb{E}_-(a_i) & O_q & O_q
\end{bmatrix}_{4q\times 4q}.
\tag{4.3.5}
$$

For $W=[w,\varphi]\in H^1(E)\times H^1(E)$, set local column vectors

$$
W_+(a_i)=[w_{j_1}(1),w_{j_2}(1),\cdots,w_{j_p}(1)]^T,\quad W_-(a_i)=[w_{k_1}(0),w_{k_2}(0),\cdots,w_{k_q}(0)]^T,
$$

and

$$
\Phi_+(a_i)=[\varphi_{j_1}(1),\varphi_{j_2}(1),\cdots,\varphi_{j_p}(1)]^T,\quad \Phi_-(a_i)=[\varphi_{k_1}(0),\varphi_{k_2}(0),\cdots,\varphi_{k_q}(0)]^T.
$$

For $W=[w,\varphi],F=[f,g]\in H^2(E)\times H^2(E)$, we define vectors

$$
\widehat{W}_\pm(a_i)=\begin{bmatrix} W_\pm(a_i)\\ \Phi_\pm(a_i)\\ W'_\pm(a_i)\\ \Phi'_\pm(a_i)\end{bmatrix},\quad \widehat{F}_\pm(a_i)=\begin{bmatrix} F_\pm(a_i)\\ G_\pm(a_i)\\ F'_\pm(a_i)\\ G'_\pm(a_i)\end{bmatrix}.
$$

Then we have

$$\begin{aligned}
& \sum_{j\in J^+(a_i)} \Big(K_j(s)(w_{j,s}(s)-\varphi_j(s))\overline{f_j(s)} + E_j(s)\varphi_{j,s}(s)\overline{g_j(s)}\Big)_{s=1} \\
& - \sum_{j\in J^+(a_i)} \Big(K_j(s)w_j(s)\overline{(f_{j,s}(s)-g_j(s))} + E_j(s)\varphi_j(s)\overline{g_{j,s}(s)}\Big)_{s=1} \\
& - \sum_{k\in J^-(a_i)} \Big(K_j(s)(w_{j,s}(s)-\varphi_j(s))\overline{f_j(s)} + E_j(s)\varphi_{j,s}(s)\overline{g_j(s)}\Big)_{s=0} \\
& + \sum_{j\in J^-(a_i)} \Big(K_j(s)w_j(s)\overline{(f_{j,s}(s)-g_j(s))} + E_j(s)\varphi_j(s)\overline{g_{j,s}(s)}\Big)_{s=0} \\
= & \sum_{j\in J^+(a_i)} \Big(K_j(s)(w_{j,s}(s)-\varphi_j(s))\overline{f_j(s)} + [K_j(s)w_j(s)+E_j(s)\varphi_{j,s}(s)]\overline{g_j(s)}\Big)_{s=1} \\
& - \sum_{j\in J^+(a_i)} \Big(K_j(s)w_j(s)\overline{f_{j,s}(s)} + E_j(s)\varphi_j(s)\overline{g_{j,s}(s)}\Big)_{s=1} \\
& - \sum_{k\in J^-(a_i)} \Big(K_j(s)(w_{j,s}(s)-\varphi_j(s))\overline{f_j(s)} + [K_j(s)w_j(s)+E_j(s)\varphi_{j,s}(s)]\overline{g_j(s)}\Big)_{s=0} \\
& + \sum_{j\in J^-(a_i)} \Big(K_j(s)w_j(s)\overline{f_{j,s}(s)} + E_j(s)\varphi_j(s)\overline{g_{j,s}(s)}\Big)_{s=0} \\
= & \left(\begin{bmatrix} O_p & -\mathbb{K}_+(a_i) & \mathbb{K}_+(a_i) & O_p \\ \mathbb{K}_+(a_i) & O_p & O_p & \mathbb{E}_+(a_i) \\ -\mathbb{K}_+(a_i) & O_p & O_p & O_p \\ O_p & -\mathbb{E}_+(a) & O_p & O_p \end{bmatrix} \begin{bmatrix} W_+(a_i) \\ \Phi_+(a_i) \\ W'_+(a_i) \\ \Phi'_+(a_i) \end{bmatrix}, \begin{bmatrix} F_+(a_i) \\ G_+(a_i) \\ F'_+(a_i) \\ G'_+(a_i) \end{bmatrix}\right)_{\mathbb{C}^{4p}} \\
& - \left(\begin{bmatrix} O_q & -\mathbb{K}_-(a_i) & \mathbb{K}_-(a_i) & O_q \\ \mathbb{K}_-(a_i) & O_q & O_q & \mathbb{E}_-(a_i) \\ -\mathbb{K}_-(a_i) & O_q & O_q & O_q \\ O_q & -\mathbb{E}_-(a_i) & O_q & O_q \end{bmatrix} \begin{bmatrix} W_-(a_i) \\ \Phi_-(a_i) \\ W'_-(a_i) \\ \Phi'_-(a_i) \end{bmatrix}, \begin{bmatrix} F_-(a_i) \\ G_-(a_i) \\ F'_-(a_i) \\ G'_-(a_i) \end{bmatrix}\right)_{\mathbb{C}^{4q}} \\
= & \Big(\mathcal{Q}_+(a_i)\widehat{W}_+(a_i), \widehat{F}_+(a_i)\Big)_{\mathbb{C}^{4p}} - \Big(\mathcal{Q}_-(a_i)\widehat{W}_-(a_i), \widehat{F}_-(a_i)\Big)_{\mathbb{C}^{4q}}.
\end{aligned}$$

Therefore, we have

$$(\mathcal{L}W, F)_{L^2} - (W, \mathcal{L}F)_{L^2} = \sum_{i=1}^m \Big(\mathcal{Q}_+(a_i)\widehat{W}_+(a_i), \widehat{F}_+(a_i)\Big) - \Big(\mathcal{Q}_-(a_i)\widehat{W}_-(a_i), \widehat{F}_-(a_i)\Big) \quad (4.3.6)$$

THEOREM 4.3.1 *Let $\mathcal{L}$ be defined as (4.3.2), and let $\mathcal{Q}_\pm(a)$ be defined by (4.3.4) and (4.3.5) for each $a \in V$, respectively. Then the following statements are true*

1) At node $a \in V$, there are $2(p+q) = 2\#J(a)$ many linearly independent conditions, which have the form

$$A_a\widehat{W}_+(a) + B_a\widehat{W}_-(a) = 0, \quad \operatorname{rank}(A_a, B_a) = 2\#J(a) \quad (4.3.7)$$

where $A_a = A_{2(p+q)\times 4p}$ and $B_a = B_{2(p+q)\times 4q}$;

2) $\mathcal{L}$ with nodal conditions (4.3.7) is nodal equilibrium at a, i.e.,

$$\Big(\mathcal{Q}_+(a)\widehat{W}_+(a), \widehat{F}_+(a)\Big)_{\mathbb{C}^{4p}} - \Big(\mathcal{Q}_-(a)\widehat{W}_-(a), \widehat{F}_-(a)\Big)_{\mathbb{C}^{4q}} = 0 \quad (4.3.8)$$

if and only if A_a and B_a satisfy the condition

$$A_a \mathcal{Q}_+^{-1}(a) A_a^* = B_a \mathcal{Q}_-^{-1}(a) B_a^*. \tag{4.3.9}$$

In this case, $\mathcal{L}$ with nodal conditions (4.3.7) for all $a \in V$ is a self-adjoint operator.

Proof Since there are $2(p+q)$ many second-order differential equations in small neighborhood of vertex a, they have at most $2(p+q)$ number the connection conditions at a when we regard the network as a star shape graph in this neighborhood. So there are $2(p+q)$ many linearly independent conditions at a.

Suppose that the nodal conditions are given by (4.3.7). Then for any $X \in \mathbb{C}^{2(p+q)}$, we have

$$(A_a \widehat{W}_+(a), X)_{\mathbb{C}^{2(p+q)}} + (B_a \widehat{W}_-(a), X)_{\mathbb{C}^{2(p+q)}} = 0 \tag{4.3.10}$$

If $\mathcal{L}$ is nodal equilibrium at $a \in V$, i.e., for any $W, F \in H^2(E) \times H^2(E)$ satisfying (4.3.7), it holds that

$$\Big(\mathcal{Q}_+(a)\widehat{W}_+(a), \widehat{F}_+(a)\Big)_{\mathbb{C}^{4p}} - \Big(\mathcal{Q}_-(a)\widehat{W}_-(a), \widehat{F}_-(a)\Big)_{\mathbb{C}^{4q}} = 0, \tag{4.3.11}$$

comparing (4.3.10) to (4.3.11) we can get equalities

$$-\mathcal{Q}_+(a)\widehat{F}_+(a) = A_a^* X, \quad \mathcal{Q}_-(a)\widehat{F}_-(a) = B_a^* X$$

where we have used the equality $\mathcal{Q}_\pm^*(a) = -\mathcal{Q}_\pm(a)$, that leads to

$$\widehat{F}_+(a) = -\mathcal{Q}_+^{-1}(a) A_a^* X, \quad \widehat{F}_-(a) = \mathcal{Q}_-^{-1}(a) B_a^* X.$$

Thus,

$$0 = A_a \widehat{F}_+(a) + B_a \widehat{F}_-(a) = -A_a \mathcal{Q}_+^{-1}(a) A_a^* X + B_a \mathcal{Q}_-^{-1}(a) B_a^* X, \quad \forall X \in \mathbb{C}^{2(p+q)}.$$

So (4.3.9) holds.

Conversely, suppose that A_a and B_a in (4.3.7) satisfy the condition (4.3.9), $W \in H^2(E) \times H^2(E)$ is a functions satisfying the condition (4.3.7). By assumption (4.3.9), there exists an $X \in \mathbb{C}^{2(p+q)}$ such that

$$\widehat{W}_+(a) = \mathcal{Q}_+^{-1}(a) A_a^* X, \quad \widehat{W}_-(a) = -\mathcal{Q}_-^{-1}(a) B_a^* X.$$

Thus we have

$$\mathcal{Q}_+(a)\widehat{W}_+(a) = A_a^* X, \quad -\mathcal{Q}_-(a)\widehat{F}_-(a) = B_a^* X.$$

For any $F \in H^2(E) \times H^2(E)$ satisfying $A_a \widehat{F}_+(a) + B_a \widehat{F}_-(a) = 0$, we have

$$\begin{aligned}
& \Big(\mathcal{Q}_+(a)\widehat{W}_+(a), \widehat{F}_+(a)\Big)_{\mathbb{C}^{4p}} - \Big(\mathcal{Q}_-(a)\widehat{W}_-(a), \widehat{F}_-(a)\Big)_{\mathbb{C}^{4q}} \\
= & \Big(X, A_a \widehat{F}_+(a)\Big)_{\mathbb{C}^{2(p+q)}} + \Big(X, B_a \widehat{F}_-(a)\Big)_{\mathbb{C}^{2(p+q)}} \\
= & \Big(X, A_a \widehat{F}_+(a) + B_a \widehat{F}_-(a)\Big)_{\mathbb{C}^{2(p+q)}} = 0,
\end{aligned}$$

this means the nodal equilibrium at a.

If for all $a \in V$, the conditions (4.3.9) hold, a straightforward check shows that $\mathcal{L}$ under restriction (4.3.7) is a self-adjoint operator in $L^2(G)$. □

THEOREM 4.3.2 *Let $\mathcal{L}$ be defined as (4.3.2) and $\mathcal{Q}_{\pm}(a)$ be defined by (4.3.4) and (4.3.5) respectively. Suppose that $A_a = A_{2(p+q)\times 4p}$ and $B_a = B_{2(p+q)\times 4q}$ satisfy the conditions $rank(A_a, B_a) = 2(p+q)$ and (4.3.9). Define operator $\mathcal{L}_0$ by $\mathcal{L}_0 = -\mathcal{L}$ with domain*

$$\mathcal{D}(\mathcal{L}_0) = \{W \in H^2(E) \times H^2(E) \mid A_a\widehat{W}_+(a) + B_a\widehat{W}_-(a) = 0, \forall a \in V\} \tag{4.3.12}$$

where $\widehat{W}_+(a)$ and $\widehat{W}_-(a)$ for each $W = [w, \varphi] \in H^2(E) \times H^2(E)$ are defined as before. Then $\mathcal{L}_0$ is a nonnegative operator if and only if A_a and B_a satisfy the condition

$$A_a\mathcal{Q}_+^{-1}(a)\mathcal{Q}_{+,1}(a)\mathcal{Q}_+^{-1}(a)A_a^* = B_a\mathcal{Q}_-^{-1}(a)\mathcal{Q}_{-,1}(a)\mathcal{Q}_-^{-1}(a)B_a^* \tag{4.3.13}$$

where

$$\mathcal{Q}_{\pm,1}(a) = \begin{bmatrix} O & -\mathbb{K}_{\pm}(a) & \mathbb{K}_{\pm}(a) & O \\ O & O & O & \mathbb{E}_{\pm}(a) \\ O & O & O & O \\ O & O & O & O \end{bmatrix} \tag{4.3.14}$$

Proof Firstly we observe that the condition (4.3.13) implies (4.3.9). In fact, from (4.3.13) we can get

$$A_a(\mathcal{Q}_+^{-1}(a))^*\mathcal{Q}_{+,1}^*(a)(\mathcal{Q}_+^{-1}(a))^*A_a^* = B_a(\mathcal{Q}_-^{-1}(a))^*\mathcal{Q}_{-,1}^*(a)(\mathcal{Q}_-^{-1}(a))^*B_a^*$$

The relation $(\mathcal{Q}_{\pm}^{-1}(a))^* = -\mathcal{Q}_{\pm}^{-1}(a)$ leads to

$$A_a\mathcal{Q}_+^{-1}(a)\mathcal{Q}_{+,1}^*(a)\mathcal{Q}_+^{-1}(a)A_a^* = B_a\mathcal{Q}_-^{-1}(a)\mathcal{Q}_{-,1}^*(a)\mathcal{Q}_-^{-1}(a)B_a^*.$$

According definition of $\mathcal{Q}_{\pm,1}$ in (4.3.14), we have $\mathcal{Q}_{\pm}(a) = \mathcal{Q}_{\pm,1} - \mathcal{Q}_{\pm,1}^*$. This together with above and (4.3.13) yield

$$A_a\mathcal{Q}_+^{-1}(a)A_a^* = B_a\mathcal{Q}_-^{-1}(a)B_a^*.$$

Next, for any $w, \varphi \in H^2(E)$, we have

$$\begin{aligned}
(\mathcal{L}(w,\varphi),(w,\varphi))_{L^2} &= \sum_{j=1}^{n}\int_0^1 [((K_j(s)(w_{j,s}(s)-\varphi_j(s))_s)\overline{w_j(s)}ds \\
&\quad + \sum_{j=1}^{n}\int_0^1 ((E_j(s)\varphi_{j,s}(s))_s + K_j(s)(w_{j,s}(s)-\varphi_j(s)))\overline{\varphi_j(s)}]ds \\
&= \sum_{k=1}^{m}\sum_{j\in J^+(a_k)} K_j(s)(w_{j,s}(s)-\varphi_j(s))\overline{w_j(s)} + E_j(s)\varphi_{j,s}(s)\overline{\varphi_j(s)}\big|_{s=1} \\
&\quad - \sum_{k=1}^{m}\sum_{j\in J^-(a_k)} K_j(s)(w_{j,s}(s)-\varphi_j(s))\overline{w_j(s)} + E_j(s)\varphi_{j,s}(s)\overline{\varphi_j(s)}\big|_{s=0} \\
&\quad - \sum_{j=1}^{n}\int_0^1 [K_j(s)|(w_{j,s}(s)-\varphi_j(s)|^2 + E_j(s)|\varphi_{j,s}(s)|^2]ds.
\end{aligned}$$

Note that

$$\sum_{j\in J^+(a_k)} K_j(s)(w_{j,s}(s)-\varphi_j(s))\overline{w_j(s)} + E_j(s)\varphi_{j,s}(s)\overline{\varphi_j(s)}\big|_{s=1}$$

$$= \left(\begin{bmatrix} O_p & -\mathbb{K}_+(a_k) & \mathbb{K}_+(a_k) & O_p \\ O_p & O_p & O_p & \mathbb{E}_+(a_k) \\ O_p & O_p & O_p & O_p \\ O_p & O_p & O_p & O_p \end{bmatrix} \begin{bmatrix} W_+(a_i) \\ \Phi_+(a_i) \\ W'_+(a_k) \\ \Phi'_+(a_k) \end{bmatrix}, \begin{bmatrix} W_+(a_k) \\ \Phi_+(a_k) \\ W'_+(a_k) \\ \Phi'_+(a_k) \end{bmatrix} \right)_{4p\times 4p}$$

$$\sum_{j\in J^-(a_k)} K_j(s)(w_{j,s}(s)-\varphi_j(s))\overline{w_j(s)} + E_j(s)\varphi_{j,s}(s)\overline{\varphi_j(s)}\big|_{s=0}$$

$$= \left(\begin{bmatrix} O_q & -\mathbb{K}_-(a_k) & \mathbb{K}_-(a_k) & O_q \\ O_q & O_q & O_q & \mathbb{E}_-(a_k) \\ O_q & O_q & O_q & O_q \\ O_q & O_q & O_q & O_q \end{bmatrix} \begin{bmatrix} W_-(a_i) \\ \Phi_-(a_i) \\ W'_-(a_k) \\ \Phi'_-(a_k) \end{bmatrix}, \begin{bmatrix} W_-(a_k) \\ \Phi_-(a_k) \\ W'_-(a_k) \\ \Phi'_-(a_k) \end{bmatrix} \right)_{4q\times 4q}$$

Then we have

$$\begin{aligned} (\mathcal{L}(w,\varphi),(w,\varphi))_{L^2} &= \sum_{k=1}^m (\mathcal{Q}_{+,1}(a_k)\widehat{W}_+(a_k), \widehat{W}_+(a_k))_{\mathbb{C}^{4p}} - (\mathcal{Q}_{-,1}(a_k)\widehat{W}_-(a_k), \widehat{W}_-(a_k))_{\mathbb{C}^{4q}} \\ &\quad - \sum_{j=1}^n \int_0^1 [K_j(s)|(w_{j,s}(s)-\varphi_j(s)|^2 + E_j(s)|\varphi_{j,s}(s)|^2]ds. \end{aligned}$$

Therefore, $\mathcal{L}_0$ is nonnegative if and only if for each $a \in V$,

$$(\mathcal{Q}_{+,1}(a)\widehat{W}_+(a), \widehat{W}_+(a))_{\mathbb{C}^{4p}} - (\mathcal{Q}_{-,1}(a)\widehat{W}_-(a), \widehat{W}_-(a))_{\mathbb{C}^{4q}} = 0.$$

If $\mathcal{L}_0$ is nonnegative self-adjoint under the nodal equilibrium conditions (4.3.9), then for any $X \in \mathbb{C}^{2(p+q)}$, there exist $W = [w, \varphi]$ satisfy (4.3.7), i.e.,

$$A_a\widehat{W}_+(a) + B_a\widehat{W}_-(a) = 0,$$

such that

$$\widehat{W}_+(a) = \mathcal{Q}_+^{-1}(a)A_a^*X, \quad \widehat{W}_-(a) = -\mathcal{Q}_-^{-1}(a)B_a^*X$$

and hence

$$\begin{aligned} 0 &= \Big(\mathcal{Q}_{+,1}(a)\widehat{W}_+(a), \widehat{W}_+(a)\Big)_{\mathbb{C}^{4p}} - \Big(\mathcal{Q}_{-,1}(a)\widehat{W}_-(a), \widehat{W}_-(a)\Big)_{\mathbb{C}^{4q}} \\ &= \big(\mathcal{Q}_{+,1}(a)\mathcal{Q}_+^{-1}(a)A_a^*X, \mathcal{Q}_+^{-1}(a)A_a^*X\big)_{\mathbb{C}^{4p}} - \big(\mathcal{Q}_{-,1}(a)\mathcal{Q}_-^{-1}(a)B_a^*X, \mathcal{Q}_-^{-1}(a)B_a^*X\big)_{\mathbb{C}^{4q}} \\ &= \big(X, A_a(\mathcal{Q}_+^{-1}(a))^*\mathcal{Q}_{+,1}^*(a)\mathcal{Q}_+^{-1}(a)A_a^*X\big)_{\mathbb{C}^{2(p+q)}} - \big(X, B_a(\mathcal{Q}_-^{-1}(a))^*\mathcal{Q}_{-,1}^*(a)\mathcal{Q}_-^{-1}(a)B_a^*X\big)_{\mathbb{C}^{2(p+q)}}. \end{aligned}$$

From above we can get

$$A_a(\mathcal{Q}_+^{-1}(a))^*\mathcal{Q}_{+,1}^*(a)\mathcal{Q}_+^{-1}(a)A_a^* = B_a(\mathcal{Q}_-^{-1}(a))^*\mathcal{Q}_{-,1}^*(a)\mathcal{Q}_-^{-1}(a)B_a^*,$$

which is equivalent to

$$A_a(\mathcal{Q}_+^{-1}(a)\mathcal{Q}_{+,1}(a)\mathcal{Q}_+^{-1}(a)A_a^* = B_a(\mathcal{Q}_-^{-1}(a)\mathcal{Q}_{-,1}(a)\mathcal{Q}_-^{-1}(a)B_a^*.$$

The formula (4.3.13) follows.

Conversely, if A_a and B_a satisfy the conditions (4.3.13), as shown in the first step, we have

$$A_a\mathcal{Q}_+^{-1}(a))\mathcal{Q}_{+,1}^*(a)\mathcal{Q}_+^{-1}(a)A_a^* = B_a\mathcal{Q}_-^{-1}(a))\mathcal{Q}_{-,1}^*(a)\mathcal{Q}_-^{-1}(a)B_a^*$$

and

$$A_a\mathcal{Q}_+^{-1}(a)A_a^* = B_a\mathcal{Q}_-^{-1}(a)B_a^*.$$

For $W = [w,\varphi] \in H^2(E)\times H^2(E)$ satisfy $A_a\widehat{W}_+(a) + B_a\widehat{W}_-(a) = 0$, there exists an $X\in\mathbb{C}^{2(p+q)}$ such that

$$\widehat{W}_+(a) = \mathcal{Q}_+^{-1}(a))A_a^*X, \quad \widehat{W}_-(a) = \mathcal{Q}_-^{-1}(a))B_a^*X.$$

Thus,

$$\begin{aligned}
&\Big(\mathcal{Q}_{+,1}(a)\widehat{W}_+(a),\widehat{W}_+(a)\Big)_{\mathbb{C}^{4p}} - \Big(\mathcal{Q}_{-,1}(a)\widehat{W}_-(a),\widehat{W}_-(a)\Big)_{\mathbb{C}^{4q}}\\
=\;& \big(\mathcal{Q}_{+,1}(a)\mathcal{Q}_+^{-1}(a)A_a^*X, \mathcal{Q}_+^{-1}(a)A_a^*X\big)_{\mathbb{C}^{4p}}\\
&-\big(\mathcal{Q}_{-,1}(a)\mathcal{Q}_-^{-1}(a))B_a^*X, \mathcal{Q}_-^{-1}(a))B_a^*X\big)_{\mathbb{C}^{4q}}\\
=\;& \big(X, A_a(\mathcal{Q}_+^{-1}(a))^*\mathcal{Q}_{+,1}^*(a)\mathcal{Q}_+^{-1}(a))A_a^*X\big)_{\mathbb{C}^{2(p+q)}}\\
&-\big(X, B_a(\mathcal{Q}_-^{-1}(a))^*\mathcal{Q}_{-,1}^*(a)\mathcal{Q}_-^{-1}(a))B_a^*X\big)_{\mathbb{C}^{2(p+q)}} = 0.
\end{aligned}$$

Therefore, $\mathcal{L}_0$ is a nonnegative operator. □

4.3.2 The structural equilibrium condition

In this subsection we discuss the structure equilibrium condition. Let G be the graph with edge set $E=\{e_j\}$. For each $W=(w,\varphi)\in L^2(E)\times L^2(E)$, we always coincide with a vector-valued function $(W(s),\Phi(s))$

$$W(s) = [w_1(s), w_2(s),\cdots,w_n(s)]^T, \Phi(s) = [\varphi_1(s),\varphi_2(s),\cdots,\varphi_n(s)]^T.$$

In this way, the inner product in $L^2(G)\times L^2(G)$ becomes

$$((W,\Phi),(F,G))_{L^2} = \int_0^1 (W(s),F(s))_{\mathbb{C}^n}ds + \int_0^1(\Phi(s),G(s))_{\mathbb{C}^n}ds.$$

Let $\mathcal{L}$ be defined by (4.3.2). Setting

$$\mathbb{K}(s) = \mathrm{diag}[K_1(s),K_2(s),\cdots,K_n(s)]$$

and

$$\mathbb{E}(s) = \mathrm{diag}[E_1(s),E_2(s),\cdots,E_n(s)],$$

we rewrite $\mathcal{L}$ into the matrix form

$$\mathcal{L}\begin{pmatrix} W(s)\\ \Phi(s)\end{pmatrix} = \begin{pmatrix} (\mathbb{K}(s)(W'(s)-\Phi(s)))'\\ (\mathbb{E}(s)\Phi'(s))' + \mathbb{K}(s)(W'(s)-\Phi(s))\end{pmatrix}, \quad s\in(0,1).$$

Thus we have

$$\begin{aligned}
& (\mathcal{L}(W,\Phi),(F,G))_{L^2} - ((W,\Phi),\mathcal{L}(F,G))_{L^2} \\
= \; & (\mathbb{K}(s)(W'(s)-\Phi(s)),F)_{\mathbb{C}^n}\Big|_0^1 + (\mathbb{E}(s)\Phi'(s),G(s))_{\mathbb{C}^n}\Big|_0^1 \\
& -(W(s),\mathbb{K}(s)(F'(s)-G(s)))_{\mathbb{C}^n}\Big|_0^1 - (\Phi(s),\mathbb{E}(s)G'(s))_{\mathbb{C}^n}\Big|_0^1 \\
= \; & \left(\begin{bmatrix} 0 & -\mathbb{K}(s) & \mathbb{K}(s) & O_n \\ \mathbb{K}(s) & O_n & O_n & \mathbb{E}(s) \\ -\mathbb{K}(s) & O_n & O_n & O_n \\ O_n & -\mathbb{E}(s) & O_n & O_n \end{bmatrix}\begin{bmatrix} W(s) \\ \Phi(s) \\ W'(s) \\ \Phi'(s)\end{bmatrix}, \begin{bmatrix} F(s) \\ G(s) \\ F'(s) \\ G'(s)\end{bmatrix}\right)_{\mathbb{C}^{4n}}\Bigg|_0^1 \\
= \; & \left(\mathcal{Q}(1)\widehat{W}(1),\widehat{F}(1)\right)_{\mathbb{C}^{4n}} - \left(\mathcal{Q}(0)\widehat{W}(0),\widehat{F}(0)\right)_{\mathbb{C}^{4n}}
\end{aligned}$$

where

$$\mathcal{Q}(s) = \begin{bmatrix} 0 & -\mathbb{K}(s) & \mathbb{K}(s) & O_n \\ \mathbb{K}(s) & O_n & O_n & \mathbb{E}(s) \\ -\mathbb{K}(s) & O_n & O_n & O_n \\ O_n & -\mathbb{E}(s) & O_n & O_n \end{bmatrix}, \quad \widehat{W}(s) = \begin{bmatrix} W(s) \\ \Phi(s) \\ W'(s) \\ \Phi'(s)\end{bmatrix} \tag{4.3.15}$$

Now let the connection condition of $\mathcal{L}$ defined on graph G be given by

$$A\widehat{W}(1) + B\widehat{W}(0) = 0, \quad A, B \in \mathbb{M}_{4n\times 4n}, \quad rank(A,B) = 4n \tag{4.3.16}$$

where $\widehat{W}(s)$ is defined as (4.3.15).

THEOREM 4.3.3 *Let the formal differential operator $\mathcal{L}$ on $L^2(E)\times L^2(E)$ be defined as (4.3.2). Let A and B be elements in $\mathbb{M}_{4n\times 4n}$. Then $\mathcal{L}$ with the connective condition (4.3.16) is structural self-adjoint if only if A and B satisfy the condition*

$$A\mathcal{Q}^{-1}(1)A^* = B\mathcal{Q}^{-1}(0)B^*. \tag{4.3.17}$$

Proof It is clear that $\mathcal{L}$ is self adjoint if and only if (W,Φ) and (F,G) satisfy (4.3.16) such that

$$\left(\mathcal{Q}(1)\widehat{W}(1),\widehat{F}(1)\right)_{\mathbb{C}^{4n}} - \left(\mathcal{Q}(0)\widehat{W}(0),\widehat{F}(0)\right)_{\mathbb{C}^{4n}} = 0$$

where

$$\widehat{W}(s) = [W(s),\Phi(s),W'(s),\Phi'(s)]^T, \quad \widehat{F}(s) = [F(s),G(s),F'(s),G'(s)]^T.$$

Since (W,Φ), (F,G) satisfy (4.3.16), i.e.,

$$A\widehat{W}(1) + B\widehat{W}(0) = 0, \quad A\widehat{F}(1) + B\widehat{F}(0) = 0,$$

for any $X \in \mathbb{C}^{4n}$, we have

$$(\widehat{W}(1), A^*X)_{\mathbb{C}^{4n}} + (\widehat{W}(0), B^*X)_{\mathbb{C}^{4n}} = 0.$$

Comparing above equality with

$$\left(\mathcal{Q}(1)\widehat{W}(1),\widehat{F}(1)\right)_{\mathbb{C}^{4n}} - \left(\mathcal{Q}(0)\widehat{W}(0),\widehat{F}(0)\right)_{\mathbb{C}^{4n}} = 0$$

we get

$$\mathcal{Q}^*(1)\widehat{F}(1) = A^*X, \quad \mathcal{Q}^*(0)\widehat{F}(0) = -B^*X.$$

Note that $\mathcal{Q}^*(s) = -\mathcal{Q}(s)$ and $\mathcal{Q}^{-1}(s)$ exist, and hence

$$\widehat{F}(1) = -\mathcal{Q}^{-1}(1)A^*X, \quad \widehat{F}(0) = \mathcal{Q}^{-1}(0)B^*X.$$

Thus we have

$$0 = A\widehat{F}(1) + B\widehat{F}(0) = -A\mathcal{Q}^{-1}(1)A^*X + B\mathcal{Q}^{-1}(0)B^*X, \quad \forall X \in \mathbb{C}^{4n}$$

Therefore, $A\mathcal{Q}^{-1}(1)A^*X = B\mathcal{Q}^{-1}(0)B^*$.

Conversely, if A and B satisfy condition (4.3.17), then for any W and F satisfy

$$\widehat{W}(1) = \mathcal{Q}^{-1}(1)A^*X, \quad \widehat{W}(0) = -\mathcal{Q}^{-1}(0)B^*X,$$

and

$$\widehat{F}(1) = \mathcal{Q}^{-1}(1)A^*Y, \quad \widehat{F}(0) = -\mathcal{Q}^{-1}(0)B^*Y.$$

we have $A\widehat{W}(1) + B\widehat{W}(0) = 0$ and $A\widehat{F}(1) + B\widehat{F}(0) = 0$, and

$$\begin{aligned}
&\left(\mathcal{Q}(1)\widehat{W}(1), \widehat{F}(1)\right)_{\mathbb{C}^{4n}} - \left(\mathcal{Q}(0)\widehat{W}(0), \widehat{F}(0)\right)_{\mathbb{C}^{4n}} \\
= \;& \left(A^*X, \mathcal{Q}^{-1}(1)A^*Y\right)_{\mathbb{C}^{4n}} - \left(B^*X, \mathcal{Q}^{-1}(0)B^*Y\right)_{\mathbb{C}^{4n}} \\
= \;& \left(X, (A\mathcal{Q}^{-1}(1)A^* - B\mathcal{Q}^{-1}(0)B^*)Y\right)_{\mathbb{C}^{4n}} = 0.
\end{aligned}$$

The desired result follows. □

THEOREM 4.3.4 *Let $\mathcal{L}$ on $L^2(G)\times L^2(G)$ be defined as (4.3.2). Let A and B be the elements in $\mathbb{M}_{4n\times 4n}$. Then $-\mathcal{L}$ with connection condition (4.3.16) is a structural positive operator if only if A and B satisfy the condition*

$$A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^* = B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^* \tag{4.3.18}$$

where

$$\mathcal{Q}_1(s) = \begin{bmatrix} O_n & -\mathbb{K}(s) & \mathbb{K}(s) & O_n \\ O_n & O_n & O_n & \mathbb{E}(s) \\ O_n & O_n & O_n & O_n \\ O_n & O_n & O_n & O_n \end{bmatrix} \tag{4.3.19}$$

Proof For any $W, \Phi \in H^2(E)$, we have

$$\begin{aligned}
(\mathcal{L}(W,\Phi),(W,\Phi))_{L^2} &= \sum_{j=1}^n \int_0^1 [((K_j(s)(w_{j,s}(s) - \varphi_j(s))_s)\overline{w_j(s)}ds \\
&\quad + \sum_{j=1}^n \int_0^1 ((E_j(s)\varphi_{j,s}(s))_s + K_j(s)(w_{j,s}(s) - \varphi_j(s)))\overline{\varphi_j(s)}]ds \\
&= \sum_{j=1}^n K_j(s)(w_{j,s}(s) - \varphi_j(s))\overline{w_j(s)}\big|_0^1 + E_j(s)\varphi_{j,s}(s)\overline{\varphi_j(s)}\big|_0^1 \\
&\quad - \sum_{j=1}^n \int_0^1 [K_j(s)|(w_{j,s}(s) - \varphi_j(s)|^2 + E_j(s)|\varphi_{j,s}(s)|^2]ds.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{j=1}^{n} K_j(s)(w_{j,s}(s)-\varphi_j(s))\overline{w_j(s)}+E_j(s)\varphi_{j,s}(s)\overline{\varphi_j(s)} \\
= &\left(\begin{bmatrix} O_n & -\mathbb{K}(s) & \mathbb{K}(s) & O_n \\ O_n & O_n & O_n & \mathbb{E}(s) \\ O_n & O_n & O_n & O_n \\ O_n & O_n & O_n & O_n \end{bmatrix} \begin{bmatrix} W(s) \\ \Phi(s) \\ W'(s) \\ \Phi'(s) \end{bmatrix}, \begin{bmatrix} W(s) \\ \Phi(s) \\ W'(s) \\ \Phi'(s) \end{bmatrix} \right)_{4n\times 4n}
\end{aligned}$$

Then we have

$$\begin{aligned}
(\mathcal{L}(W,\Phi),(W,\Phi))_{L^2} &= (\mathcal{Q}_1(1)\widehat{W}(1),\widehat{W}(1))_{\mathbb{C}^n}-(\mathcal{Q}_1(0)\widehat{W}(0),\widehat{W}(0))_{\mathbb{C}^{4n}} \\
&\quad -\sum_{j=1}^{n}\int_0^1 [K_j(s)|(w_{j,s}(s)-\varphi_j(s)|^2+E_j(s)|\varphi_{j,s}(s)|^2]ds.
\end{aligned}$$

Obviously, $-\mathcal{L}$ is structure positive if and only if

$$(\mathcal{Q}_1(1)\widehat{W}(1),\widehat{W}(1))_{\mathbb{C}^n}-(\mathcal{Q}_1(0)\widehat{W}(0),\widehat{W}(0))_{\mathbb{C}^{4n}}=0.$$

If $-\mathcal{L}$ is structure positive, then it is self adjoint and hence

$$\widehat{W}(1)=\mathcal{Q}^{-1}(1)A^*X,\quad \widehat{W}(0)=-\mathcal{Q}^{-1}(0)B^*X,\quad X\in\mathbb{C}^{4n}.$$

Thus we get

$$\begin{aligned}
0 &= (\mathcal{Q}_1(1)\widehat{W}(1),\widehat{W}(1))_{\mathbb{C}^n}-(\mathcal{Q}_1(0)\widehat{W}(0),\widehat{W}(0))_{\mathbb{C}^{4n}} \\
&= (\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^*X,\mathcal{Q}^{-1}(1)A^*X)_{\mathbb{C}^n}-(\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*X,\mathcal{Q}^{-1}(0)B^*X)_{\mathbb{C}^{4n}} \\
&= -(A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^*X-B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*X,X)_{\mathbb{C}^{4n}},\quad \forall X
\end{aligned}$$

where we have used the equality $(\mathcal{Q}^{-1}(s))^*=-\mathcal{Q}^{-1}(s)$. Using relations $A\mathcal{Q}^{-1}(1))A^*=B\mathcal{Q}^{-1}(0)B^*$ and $\mathcal{Q}(s)=\mathcal{Q}_1(s)-\mathcal{Q}_1^*(s)$, we can prove that

$$A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^*-B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*$$

is a Hermitian matrix. Therefore, we have

$$A\mathcal{Q}^{-1}(1)\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^*=B\mathcal{Q}^{-1}(0)\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*.$$

The formula (4.3.18) is proven.

Conversely, we assume that A and B satisfy (4.3.18). Since the equality (4.3.18) implies

$$A\mathcal{Q}^{-1}(1)\mathcal{Q}_1^*(1)\mathcal{Q}^{-1}(1)A^*=B\mathcal{Q}^{-1}(0)\mathcal{Q}_1^*(0)\mathcal{Q}^{-1}(0)B^*$$

Since $\mathcal{Q}(s)=\mathcal{Q}_1(s)-\mathcal{Q}_1^*(s)$, so we have $A\mathcal{Q}^{-1}(1)A^*=B\mathcal{Q}^{-1}(0)B^*$. Therefore, under the condition (4.3.18), $-\mathcal{L}$ is a self adjoint operator. Then for any $W,\Phi\in H^2(E)$ satisfy

$$A\widehat{W}(1)+B\widehat{W}(0)=0,$$

there exists an $X \in \mathbb{C}^{4n}$ such that

$$\widehat{W}(1) = \mathcal{Q}^{-1}(1)A^*X, \quad \widehat{W}(0) = -\mathcal{Q}^{-1}(0)B^*X,$$

and hence

$$\begin{aligned}
& (\mathcal{Q}_1(1)\widehat{W}(1), \widehat{W}(1))_{\mathbb{C}^n} - (\mathcal{Q}_1(0)\widehat{W}(0), \widehat{W}(0))_{\mathbb{C}^{4n}} \\
= \ & (\mathcal{Q}_1(1)\mathcal{Q}^{-1}(1)A^*X, \mathcal{Q}^{-1}(1)A^*X)_{\mathbb{C}^n} - (\mathcal{Q}_1(0)\mathcal{Q}^{-1}(0)B^*X, \mathcal{Q}^{-1}(0)B^*X)_{\mathbb{C}^{4n}} \\
= \ & (X, A\mathcal{Q}^{-1}(1)\mathcal{Q}_1^*(1)\mathcal{Q}^{-1}(1)A^*X)_{\mathbb{C}^n} - (X, B\mathcal{Q}^{-1}(0)\mathcal{Q}_1^*(0)\mathcal{Q}^{-1}(0)B^*X)_{\mathbb{C}^{4n}} = 0.
\end{aligned}$$

So $-\mathcal{L}$ is a structure positive operator. □

In what follows, we shall calculate the condition (4.3.18). Firstly we calculate

$$\begin{aligned}
& \mathcal{Q}^{-1}(s)\mathcal{Q}_1(s)\mathcal{Q}^{-1}(s) \\
= \ & \mathcal{Q}^{-1}(s)\begin{bmatrix} O_n & -\mathbb{K}(s) & \mathbb{K}(s) & O_n \\ O_n & O_n & O_n & \mathbb{E}(s) \\ O_n & O_n & O_n & O_n \\ O_n & O_n & O_n & O_n \end{bmatrix}\begin{bmatrix} O_n & O_n & -\mathbb{K}^{-1}(s) & O_n \\ O_n & O_n & O_n & -\mathbb{E}^{-1}(s) \\ \mathbb{K}^{-1}(s) & O_n & O_n & -\mathbb{E}^{-1}(s) \\ O_n & \mathbb{E}^{-1}(s) & \mathbb{E}^{-1}(s) & O_n \end{bmatrix} \\
= \ & \begin{bmatrix} O_n & O_n & -\mathbb{K}^{-1}(s) & O_n \\ O_n & O_n & O_n & -\mathbb{E}^{-1}(s) \\ \mathbb{K}^{-1}(s) & O_n & O_n & -\mathbb{E}^{-1}(s) \\ O_n & \mathbb{E}^{-1}(s) & \mathbb{E}^{-1}(s) & O_n \end{bmatrix}\begin{bmatrix} I_n & O_n & O_n & O_n \\ O_n & I_n & I_n & O_n \\ O_n & O_n & O_n & O_n \\ O_n & O_n & O_n & O_n \end{bmatrix} \\
= \ & \begin{bmatrix} O_n & O_n & O_n & O_n \\ O_n & O_n & O_n & O_n \\ \mathbb{K}^{-1}(s) & O_n & O_n & O_n \\ O_n & \mathbb{E}^{-1}(s) & \mathbb{E}^{-1}(s) & O_n \end{bmatrix}
\end{aligned}$$

Denote by

$$\mathcal{Q}_\dagger(s) = \begin{bmatrix} O_n & O_n & O_n & O_n \\ O_n & O_n & O_n & O_n \\ \mathbb{K}^{-1}(s) & O_n & O_n & O_n \\ O_n & \mathbb{E}^{-1}(s) & \mathbb{E}^{-1}(s) & O_n \end{bmatrix}$$

Then we have

$$\mathcal{Q}^{-1}(s) = \mathcal{Q}_\dagger(s) - \mathcal{Q}_\dagger^*(s)$$

Therefore we have the following result.

COROLLARY 4.3.1 *Let $\mathcal{L}$ on $L^2(E) \times L^2(E)$ be defined as (4.3.2). Let A and B be the elements in $\mathbb{M}_{4n\times 4n}$. Then $-\mathcal{L}$ with connection condition (4.3.16) is a structural positive operator if only if A and B satisfy the condition*

$$A\mathcal{Q}_\dagger(1)A^* = B\mathcal{Q}_\dagger(0)B^* \tag{4.3.20}$$

4.3.3 Timoshenko beam equations on metric graphs

Now let us return to Timoshenko beam equation on metric graph G. Let $w(x,t)$ and $\varphi(x,t)$ be functions defined on $G\times\mathbb{R}_+$, $w_j(s,t)$ and $\varphi_j(s,t)$ be their normalized realization on $e_j\times\mathbb{R}_+$. Suppose that the pair $(w_j(s,t),\varphi_j(s,t))$ satisfy the partial differential equations

$$\begin{cases} \rho_j(s)\frac{\partial^2 w_j(s,t)}{\partial t^2}=\frac{\partial}{\partial s}\left(K_j(s)\frac{\partial w_j(s,t)}{\partial s}-\varphi_j(s,t)\right), \\ I_{\rho_j}(s)\frac{\partial^2 \varphi_j(s,t)}{\partial t^2}=\frac{\partial}{\partial s}\left(E_j(s)\frac{\partial \varphi_j(x,t)}{\partial x}\right)+\left(K_j(s)\frac{\partial w_j(s,t)}{\partial s}-\varphi_j(s,t)\right), \end{cases} \quad s\in(0,1),$$

where $\rho_j(s)$,$I_{\rho_j}(s)$, $E_j(s)$ and $K_j(s)$ are positive continuous functions.

For the sake of simplicity, we introduce vector-valued function. Define matrices

$$\mathbb{M}_\rho(s)=\mathrm{diag}[\rho_1(s),\rho_2(s),\cdots,\rho_n(s)]$$

and

$$\mathbb{I}_\rho(s)=\mathrm{diag}[I_{\rho_1}(s),I_{\rho_2}(s),\cdots,I_{\rho_n}(s)]$$

Then the vector form of Timoshenko beam is

$$\begin{cases} \mathbb{M}_\rho(s)\frac{\partial^2 W(s,t)}{\partial t^2}=\frac{\partial}{\partial s}\left(\mathbb{K}(s)\frac{\partial W(s,t)}{\partial s}-\Phi(s,t)\right), \\ \mathbb{I}_\rho(s)\frac{\partial^2 \Phi(s,t)}{\partial t^2}=\frac{\partial}{\partial s}\left(\mathbb{E}(s)\frac{\partial \Phi(x,t)}{\partial x}\right)+\left(\mathbb{K}(s)\frac{\partial W(s,t)}{\partial s}-\Phi(s,t)\right), \end{cases} \quad s\in(0,1), \tag{4.3.21}$$

The energy function of (4.3.21) is defined by

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\int_0^1(\mathbb{K}(s)(W_s(s,t)-\Phi(s,t)),(W_s(s,t)-\Phi(s,t)))_{\mathbb{R}^n}+(\mathbb{M}_\rho(s)W_t(s,t),W_t(s,t))_{\mathbb{R}^n}ds\\ &\quad+\frac{1}{2}\int_0^1(\mathbb{E}(s)\Phi_s(s,t),\Phi_s(s,t))_{\mathbb{R}^n}+(\mathbb{I}_\rho(s)\Phi_t(s,t),\Phi_t(s,t))_{\mathbb{R}^n}ds. \end{aligned}\tag{4.3.22}$$

THEOREM 4.3.5 *Let partial differential equations on $L^2(E)\times L^2(E)$ be defined as (4.3.21). Let A and B be the elements in $\mathbb{M}_{4n\times 4n}$ satisfy the condition (4.3.20). Then (4.3.21) with the connective condition*

$$A[W(1,t),\Phi(1,t),W_s(1,t),\Phi_s(1,t)]^T+B[W(0,t),\Phi(0,t),W_s(0,t),\Phi_s(0,t)]^T=0$$

is well-posed and energy conservation under appropriate initial data.

The proof is a straightforward verification, the detail is omitted.

4.3.4 Some classical vertex conditions

In this subsection we give some classical local vertex conditions. Let $w(x,t)$ and $\varphi(x,t)$ be functions defined on $G\times\mathbb{R}_+$, $w_j(s,t)$ and $\varphi_j(s,t)$ be their normalized realization on $e_j\times\mathbb{R}_+$. Suppose that the pair $(w_j(s,t),\varphi_j(s,t)$ satisfy the differential equations

$$\begin{cases} \rho_j(s)\frac{\partial^2 w_j(s,t)}{\partial t^2}=\frac{\partial}{\partial s}\left(K_j(s)\frac{\partial w_j(s,t)}{\partial s}-\varphi_j(s,t)\right), \\ I_{\rho_j}(s)\frac{\partial^2 \varphi_j(s,t)}{\partial t^2}=\frac{\partial}{\partial s}\left(E_j(s)\frac{\partial \varphi_j(x,t)}{\partial x}\right)+\left(K_j(s)\frac{\partial w_j(s,t)}{\partial s}-\varphi_j(s,t)\right), \end{cases} \quad s\in(0,1),$$

where $\rho_j(s)$,$I_{\rho_j}(s)$, $E_j(s)$ and $K_j(s)$ are positive continuous functions.

§1. δ-type vertex conditions

At vertex $a \in V$ we have the geometric constraints: the displacements and rotation angle of all jointing edges are continuous. Then the local vertex conditions at $a \in V$ are given by

$$\begin{cases} w_j(1) = w_k(0) = w(a), \forall k \in J^-(a), j \in J^+(a) \\ \varphi_j(1) = \varphi_k(0) = \varphi(a), \forall k \in J^-(a), j \in J^+(a) \\ \sum\limits_{j \in J^+(a)} K_j(w_{j,s}(1) - \varphi_j(1)) - \sum\limits_{k \in J^-(a)} K_k(w_{k,s}(0) - \varphi_k(0)) = 0 \\ \sum\limits_{j \in J^+(a)} E_j(1)\varphi_{j,s}(1) - \sum\limits_{k \in J^-(a)} E_k(0)\varphi_{k,s}(0) = 0. \end{cases} \tag{4.3.23}$$

The last two conditions are the dynamic equilibrium conditions.

§2. δ'-type vertex conditions

At node $a \in V$ we assume that forces and moments of the structure are continuous. Then the local vertex conditions at a are given by

$$\begin{cases} K_j(1)(w_{j,s}(1) - \varphi_j(1)) = K_k(0)(w_{k,s}(0) - \varphi_k(0)) = F(a), \\ \forall k \in J^-(a), j \in J^+(a) \\ E_j(1)\varphi_{j,s}(1) = E_k(0)\varphi_{j,s}(0) = M(a), \forall k \in J^-(a), j \in J^+(a) \\ \sum\limits_{j \in J^+(a)} w_j(1) - \sum\limits_{k \in J^-(a)} w_k(0) = 0 \\ \sum\limits_{j \in J^+(a)} \varphi_j(1) - \sum\limits_{k \in J^-(a)} \varphi_j(0) = 0. \end{cases} \tag{4.3.24}$$

§3. δ-type elastic support vertex conditions

At node $a \in V$ we impose the geometric constraints: the displacements and ration angles of all jointing edges are continuous, and there is an elastic support at a with hooke's constants $T(a)$ and $M(a)$. Then the local vertex conditions at a are given by

$$\begin{cases} w_j(1) = w_k(0) = w(a), \forall k \in J^-(a), j \in J^+(a) \\ \varphi_j(1) = \varphi_k(0) = \varphi(a), \forall k \in J^-(a), j \in J^+(a) \\ \sum\limits_{j \in J^+(a)} K_j(1)(w_{j,s}(1) - \varphi_j(1)) - \sum\limits_{k \in J^-(a)} K_k(0)(w_{k,s}(0) - \varphi_k(0)) - T(a)w(a) = 0, \\ \sum\limits_{j \in J^+(a)} E_j(1)\varphi_{j,s}(1) - \sum\limits_{k \in J^-(a)} E_k(0)\varphi_{k,s}(0) - M(a)\varphi(a) = 0. \end{cases} \tag{4.3.25}$$

Chapter 5

Networks of Strings and Design of Controllers

5.1 Networks of strings with elastic supports

Let $G = (V, E)$ be a metric graph without isolated vertex, $u(x,t)$ be a function defined on G. Suppose that $u(x,t)$ satisfies the wave equation on each $e_j \in E$, i.e.,

$$m_j(s)u_{j,tt}(s,t) = (T_j(s)u_{j,s}(s,t))_s - q_j(s)u_j(s), \quad s \in (0,1) \tag{5.1.1}$$

where $T_j(s)$, $m_j(s)$ are positive continuous functions and $q_j(s)$ are nonnegative functions (or called potentials).

Assume that the structure is continuous at each vertex and there is an elastic support at each vertex $a \in V$. The energy function of the system is defined as

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\sum_{j=1}^{n}\int_{e_j}[T_j(s)|u_{j,s}(s,t)|^2 + q_j(s)|u_j(s,t)|^2]ds \\ &\quad + \frac{1}{2}\sum_{j=1}^{n}\int_{e_j} m_j(s)|u_t(s,t)|^2 ds + \frac{1}{2}\sum_{i=1}^{m} k(a_i)|u(a_i,t)|^2,\end{aligned}$$

then we have

$$\begin{aligned}\frac{d\mathcal{E}(t)}{dt} &= \sum_{j=1}^{n}\int_{e_j} T_j(s)u_{j,s}(s,t)u_{j,st}(s,t)ds + q_j(s)u_j(s,t)u_{j,t}(s,t)ds \\ &\quad + \sum_{j=1}^{n}\int_{e_j} m_j(s)u_{j,tt}(s,t)u_{j,t}(s,t)ds + \sum_{i=1}^{m} k(a_i)u(a_i,t)u_t(a_i,t) \\ &= \sum_{j=1}^{n} T_j(s)u_{j,s}(s,t)u_{j,t}(s,t)\big|_0^1 + \sum_{i=1}^{m} k(a_i)u(a_i,t)u_t(a_i,t)\end{aligned}$$

$$= \sum_{i=1}^{m}\left(\sum_{j\in J^+(a_i)} T_j(1)u_{j,s}(1,t)u_{j,t}(1,t) - \sum_{k\in J^-(a_i)} T_k(0)u_{k,s}(0,t)u_{k,t}(0,t)\right) + \sum_{i=1}^{m} k(a_i)u(a_i,t)u_t(a_i,t).$$

Using the continuity conditions at the vertices

$$u(a_i,t) = u_k(0,t) = u_j(1,t), \quad \forall k \in J^-(a_i), j \in J^+(a_i), \quad a_i \in V_{int}$$

and

$$u(a_i,t) = u_k(0,t), \quad \forall k \in J^-(a_i), a_i \in \partial G; \quad u(a_i,t) = u_j(1,t), \quad j \in J^+(a_i), \quad a_i \in \partial G$$

we get that

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} &= \sum_{i=1}^{m}\left(\sum_{j\in J^+(a_i)} T_j(1)u_{j,s}(1,t)u_{j,t}(1,t) - \sum_{k\in J^-(a_i)} T_k(0)u_{k,s}(0,t)u_{k,t}(0,t)\right) \\ &\quad + \sum_{i=1}^{m} k(a_i)u(a_i,t)u_t(a_i,t) \\ &= \sum_{i=1}^{m}\left(\sum_{j\in J^+(a_i)} T_j(1)u_{j,s}(1,t) - \sum_{k\in J^-(a_i)} T_k(0)u_{k,s}(0,t) + k(a_i)u(a_i,t)\right) u_t(a_i,t) \end{aligned}$$

Therefore, we derive the dynamic conditions of the elastic system at all vertices from the geometric conditions:

1) Continuity conditions

$$u(a_j,t) = u_k(0,t) = u_i(1,t), \quad \forall k \in J^-(a_j), i \in J^+(a_j), \tag{5.1.2}$$

2) Dynamical conditions

$$\sum_{j\in J^+(a)} T_j(1)u_{j,s}(1,t) - \sum_{k\in J^-(a)} T_k(0)u_{k,s}(0,t) + k(a)u(a,t) = f(a,t), \quad a \in V \tag{5.1.3}$$

where $f(a,t)$ is an exterior force acting on a. Such a network is said to be a continuous network. If all $k(a) = 0, \forall a \in V$, then there is no elastic support on the network.

5.1.1 Vectorization form of networks of strings

Let G be a directed graph without isolated vertex. Let function $u(x,t)$ defined on G be normalized and satisfy the wave equation on E.

Let

$$U(s,t) = (u_1(s,t), u_2(s,t), \cdots, u_n(s,t)), \quad s \in (0,1)$$

$$U(v,t) = (u(a_1,t), u(a_2,t), \cdots, u(a_m,t)),$$

and define matrices

$$\begin{cases} \mathbb{T}(s) = \mathrm{diag}(T_1(s), T_2(s), \cdots, T_n(s)), \\ \mathbb{M}(s) = \mathrm{diag}(m_1(s), m_2(s), \cdots, m_n(s)), \\ \mathbb{Q}(s) = \mathrm{diag}(q_1(s), q_2(x), \cdots, q_n(s)), \\ \mathbb{K}(v) = \mathrm{diag}(k(a_1), k(a_2), \cdots, k(a_m)) \end{cases} \tag{5.1.4}$$

Then the differential equations (5.1.1) on E can be rewritten into

$$\mathbb{M}(s)U_{tt}(s,t) = (\mathbb{T}(s)U_s(s,t))_s - \mathbb{Q}(s)U(s,t), \quad s \in (0,1)$$

and the connective and boundary conditions (5.1.2) are rewritten as

$$\exists U(v,t) \in \mathbb{C}^m, s.t. \begin{array}{l} U(1,t) = (\Phi^+)^T U(v,t), \\ U(0,t) = (\Phi^-)^T U(v,t). \end{array} \tag{5.1.5}$$

The vector form of the dynamic condition (5.1.3) is

$$\Phi^+ \mathbb{T}(1)U_s(1,t) - \Phi^- \mathbb{T}(0)U_s(0,t)) + \mathbb{K}(v)U(v,t) = F(v,t) \in \mathbb{C}^m. \tag{5.1.6}$$

Thus the network of strings can be rewritten into

$$\begin{cases} \mathbb{M}(s)U_{tt}(s,t) = (\mathbb{T}(s)U_s(s,t))_s - \mathbb{Q}(s)U(s,t), \quad s \in (0,1) \\ \exists U(v,t) \in \mathbb{C}^m, s.t. U(1,t) = (\Phi^+)^T U(v,t), \\ U(0,t) = (\Phi^-)^T U(v,t); \\ \Phi^+ \mathbb{T}(1)U_s(1,t) - \Phi^- \mathbb{T}(0)U_s(0,t) + \mathbb{K}(v)U(v,t) = F(v,t) \in \mathbb{C}^m, \\ U(s,0) = U_0(s), U_t(s,0) = U_1(s). \end{cases} \tag{5.1.7}$$

where $U_0(s)$ and $U_1(s)$ are the appropriate initial data.

5.1.2 Design of observers and feedback controllers

Since the energy function of the system (5.1.7) is given by

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2}\int_0^1 [(\mathbb{T}(s)U_s(s,t), U_s(s,t))_{\mathbb{C}^n} + (\mathbb{Q}(s)U(s,t), U(s,t))_{\mathbb{C}^n}]ds \\ &\quad + \frac{1}{2}\int_0^1 (\mathbb{M}(s)U_t(s,t), U_t(s,t))_{\mathbb{C}^n} ds + \frac{1}{2}(\mathbb{K}(v)U(v,t), U(v,t))_{\mathbb{C}^m} \end{aligned}$$

and

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} &= \int_0^1 (\mathbb{T}(s)U_s(x,t), U_{st}(x,t))_{\mathbb{C}^n} ds + \int_0^1 (\mathbb{Q}(s)U(s,t), U_t(s,t))_{\mathbb{C}^n} ds \\ &\quad + \int_0^1 (\mathbb{M}(s)U_{tt}(s,t), U_t(s,t))_{\mathbb{C}^n} ds + (\mathbb{K}(v)U(v,t), U_t(v,t))_{\mathbb{C}^m} \\ &= (\mathbb{T}(1)U_s(1,t), U_s(1,t))_{\mathbb{C}^n} - (\mathbb{T}(0)U_s(0,t), U_t(0,t))_{\mathbb{C}^n} + (\mathbb{K}(v)U(v,t), U_t(v,t))_{\mathbb{C}^m}. \end{aligned}$$

Using the continuity condition,$U_t(0,t) = (\Phi^-)^T U_t(v,t)$, $U_t(1,t) = (\Phi^+)^T U_t(v,t)$, we get

$$\begin{aligned}\frac{d\mathcal{E}(t)}{dt} &= (\mathbb{T}(1)U_s(1,t), U_t(1,t))_{\mathbb{C}^n} - (\mathbb{T}(0)U_s(0,t), U_t(0,t))_{\mathbb{C}^n} + (\mathbb{K}(v)U(v,t), U_t(v,t))_{\mathbb{C}^m}\\ &= (\Phi^+\mathbb{T}(1)U_s(1,t) - \Phi^-\mathbb{T}(0)U_s(0,t) + \mathbb{K}(v)U(v,t), U_t(v,t))_{\mathbb{C}^m}\\ &= (F(v,t), U_t(v,t))_{\mathbb{C}^m}\end{aligned}$$

where $F(v,t)$ is control input. Thus we have

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t (F(v,t), U_t(v,t))_{\mathbb{C}^m} dt.$$

By the duality principle of the system, we choose the observation (dual to its controllers) of the system

$$Y(t) = \mathcal{S}U_t(v,t) \in \mathbb{C}^m$$

where $\mathcal{S}$ is a vertex selection matrix of observation, which is a 0-1 diagonal matrix. Take the feedback control law as

$$F(v,t) = -\Gamma Y(t) = -\Gamma\mathcal{S}U_t(v,t) \in \mathbb{C}^m$$

where Γ is a positive gain matrix from $\mathbb{C}^m$ to $\mathbb{C}^m$. Thus the closed loop system is

$$\begin{cases} \mathbb{M}(s)U_{tt}(s,t) = (\mathbb{T}(s)U_s(s,t))_s - \mathbb{Q}(s)U(s,t), \quad s\in(0,1)\\ \exists U(v,t)\in\mathbb{V}, s.t. U(1,t) = (\Phi^+)^T U(v,t),\\ U(0,t) = (\Phi^-)^T U(v,t),\\ \Phi^+\mathbb{T}(1)U_x(1,t) - \Phi^-\mathbb{T}(0)U_x(0,t) + \mathbb{K}(v)U(v,t) = -\Gamma\mathcal{S}U_t(v,t) \in \mathbb{C}^m\\ U(s,0) = U_0(s), U_t(s,0) = U_1(s), \quad s\in(0,1). \end{cases} \tag{5.1.8}$$

Note that the vertex selection matrix of observation $\mathcal{S}$ is always a diagonal matrix, whose diagonal entries are 0 or 1. Usually the feedback gain matrix Γ is also a diagonal matrix. Therefore $\Gamma\mathcal{S}$ denotes the design of feedback controllers. With these feedback controllers, the energy of the closed loop system satisfies equality

$$\mathcal{E}(t) + \int_0^t (\Gamma\mathcal{S}U_t(v,t), U_t(v,t))_{\mathbb{C}^m} dt = \mathcal{E}(0).$$

Obviously, $\mathcal{E}(t) \le \mathcal{E}(0)$. This means that the energy of the closed loop system is dissipative.

EXAMPLE 5.1.1 *We consider a continuous network of strings with nodal supports, whose structure is shown as Fig.5.1.1.*

The directed edges are defined by

$$\begin{array}{llll} e_1=(a_1,a_5) & e_2=(a_2,a_6) & e_3=(a_3,a_7) & e_4=(a_{10},a_4)\\ e_5=(a_5,a_6) & e_6=(a_6,a_8) & e_7=(a_5,a_7) & e_8=(a_7,a_8)\\ e_9=(a_8,a_9) & e_{10}=(a_9,a_{10}) & e_{11}=(a_8,a_{10}) & \end{array}$$

The boundary $\partial G = \{a_1,a_2,a_3,a_4\}$.

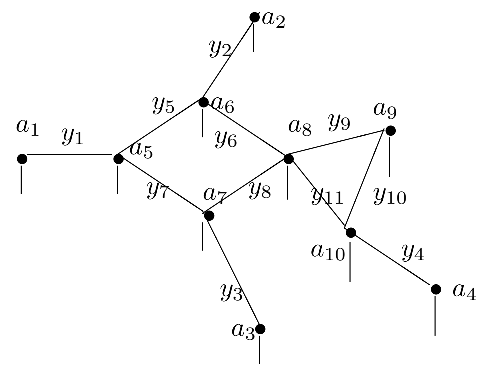

Fig. 5.1.1. A continuous network of strings with circuits

Assume that the parameter increasing coincides with the direction of G. Let $y_j(s,t)$ be the displacement of the string on e_j and satisfy equation

$$m_j(s)y_{j,tt}(s,t) = (T_j(s)y_{j,s}(s,t))_s - q_j(s)y_j(s,t), \quad s \in (0,1)$$

The boundaries and all interior nodes have the velocity feedback controllers with gain $\alpha_j = \alpha(a_j) \geq 0, j \in \{1, 2, \cdots, 10\}$.

The motion of the closed loop system is governed by the partial differential equations

$$\begin{cases} m_j(s)y_{j,tt}(s,t) = (T_j(s)y_{j,s}(s,t))_s - q_i(s)y_j(s,t), \quad s \in (0,1), j = 1,2,\cdots,11 \\ -T_1(0)y_{1,s}(0,t) + k(a_1)y(a_1,t) = -\alpha_1 y_t(a_1,t), \quad y_1(0,t) = y(a_1,t); \\ -T_2(0)y_{2,s}(0,t) + k(a_2)y(a_2,t) = -\alpha_2 y_t(a_2,t), \quad y_2(0,t) = y(a_2,t) \\ -T_3(0)y_{3,s}(0,t) + k(a_3)y(a_3,t) = -\alpha_3 y_t(a_3,t), \quad y_3(0,t) = y(a_3,t) \\ T_4(1)y_{4,s}(1,t) + k(a_4)y(a_4,t) = -\alpha_4 y_t(a_4,t), \quad y_4(1,t) = y(a_4,t) \\ y_1(1,t) = y_5(0,t) = y_7(0,t) = y(a_5,t), \\ T_1(1)y_{1,s}(1,t) - T_5(0)y_{5,s}(0,t) - T_7(0)y_{7,s}(0,t) + k(a_5)y(a_5,t) = -\alpha_5 y_t(a_5,t), \\ y_2(1,t) = y_5(1,t) = y_6(0,t) = y(a_6,t), \\ T_2(1)y_{2,s}(1,t) + T_5(1)y_{5,s}(1,t) - T_6(0)y_{6,s}(0,t) + k(a_6)y(a_6,t) = -\alpha_6 y_t(a_6,t), \\ y_3(1,t) = y_7(1,t) = y_8(0,t) = y(a_7,t), \\ T_3(1)y_{1,s}(1,t) + T_7(1)y_{7,s}(1,t) - T_8(0)y_{8,s}(0,t) + k(a_7)y(a_7,t) = -\alpha_7 y_t(a_7,t), \\ y_6(1,t) = y_8(1,t) = y_9(0,t) = y_{11}(0,t) = y(a_8,t), \\ T_6(1)y_{6,s}(1,t) + T_8(1)y_{8,s}(1,t) - T_9(0)y_{9,s}(0,t) - T_{11}(0)y_{11,s}(0,t) + k(a_8)y(a_8,t) = -\alpha_8 y_t(a_8,t), \\ y_9(1,t) = y_{10}(0,t) = y(a_9,t), \\ T_9(1)y_{9,1}(1,t) - T_{10}(0)y_{10,s}(0,t) + k(a_9)y(a-9,t) = -\alpha_9 y_t(a_9,t), \\ y_{10}(1,t) = y_{11}(1,t) = y_4(0,t) = y(a_{10},t), \\ T_{10}(1)y_{10,s}(1,t) + T_{11}(1)y_{11,s}(1,t) - T_4(0)y_{4,x}(0,t) + k(a_{10})y(a_{10},t) = -\alpha_{10} y_t(a_{10},t) \end{cases} \tag{5.1.9}$$

with appropriate initial data.

The incidence matrix Φ is given by

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \end{array} \\
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\end{array}$$

The vertices-valued space is $\mathbb{C}^{10}$. The product of selection matrix of vertex observation and the feedback gain matrix is given by

$$\Gamma\mathcal{S} = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_{10}), \quad \alpha_j \geq 0.$$

and the elastic constant matrix is

$$\mathbb{K}(v) = \mathrm{diag}(k(a_1), k(a-2), \cdot, k(a_{10})).$$

The state vector is

$$Y(x,t) = (y_1(x,t), y_2(x,t), \cdots, y_{11}(x,t))$$

and the coefficients matrices

$$\mathbb{T}(s) = \mathrm{diag}(T_1(s), T_2(s), \cdots, T_{11}(s)),$$

$$\mathbb{M}(s) = \mathrm{diag}(m_1(s), m_2(s), \cdots, m_{11}(s))$$

and

$$\mathbb{Q}(s) = \mathrm{diag}(q_1(s), q_2(s), \cdots, q_{11}(s)).$$

Thus we can rewrite (5.1.9) into a vector-valued differential equations in $\mathbb{C}^n$

$$\begin{cases} \mathbb{M}(s)Y_{tt}(s,t) = (\mathbb{T}(s)Y_s(s,t))_s - \mathbb{Q}(s)Y(s,t), x \in (0,1) \\ \exists Y(v,t) \in \mathbb{C}^{10}, s.t.\ Y(1,t) = (\Phi^+)^T Y(v,t), \\ Y(0,t) = (\Phi^-)^T Y(v,t), \\ \Phi^+\mathbb{T}(1)Y_s(1,t) - \Phi^-\mathbb{T}(0)Y_s(0,t) + \mathbb{K}(v)Y(v,t) = -\Gamma\mathcal{S}Y_t(v,t) \in \mathbb{C}^{10} \\ Y(s,0) = Y_0(s), \quad Y_t(s,0) = Y_1(s) \end{cases} \tag{5.1.10}$$

where $Y_0(s)$ and $Y_1(s)$ are appropriate initial data. □

In the previous example, we have

$$\Phi^+\mathbb{T}(1)Y_s(1,t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ T_4(1)y_{4,s}(1,t) \\ T_1(1)y_{1,s}(1,t) \\ T_2(1)y_{2,s}(1,t) + T_5(1)y_{5,s}(1,t) \\ T_3(1)y_{3,s}(1,t) + T_7(1)y_{7,s}(1,t) \\ T_6(1)y_{6,s}(1,t) + T_8(1)y_{8,s}(1,t) \\ T_9(1)y_{9,s}(1,t) \\ T_{10}(1)y_{10,s}(1,t) + T_{11}(1)y_{11,s}(1,t) \end{pmatrix}$$

$$\Phi^-\mathbb{T}(0)Y_s(0,t) = \begin{pmatrix} T_1(0)y_{1,s}(0,t) \\ T_2(0)y_{2,s}(0,t) \\ T_3(0)y_{3,s}(0,t) \\ 0 \\ T_5(0)y_{5,s}(0,t) + T_7(0)y_{7,s}(0,t) \\ T_6(0)y_{6,s}(0,t) \\ T_8(0)y_{8,s}(0,t) \\ T_9(0)y_{9,s}(0,t) + T_{11}(0)y_{11,s}(0,t) \\ T_{10}y_{10,s}(0,t) \\ T_4y_{4,s}(0,t) \end{pmatrix}$$

Obviously,

$$\begin{aligned} &\Phi^+\mathbb{T}(0)Y_s(1,t) - \Phi^-\mathbb{T}(0)Y_s(0,t) \\ = &\begin{pmatrix} -T_1(0)y_{1,s}(0,t) \\ -T_2(0)y_{2,s}(0,t) \\ -T_3(0)y_{3,s}(0,t) \\ T_4(1)y_{4,s}(1,t) \\ T_1(1)y_{1,s}(1,t) - T_5(0)y_{5,s}(0,t) - T_7(0)y_{7,s}(0,t) \\ T_2(1)y_{2,s}(1,t) + T_5(1)y_{5,s}(1,t) - T_6(0)y_{6,s}(0,t) \\ T_3(1)y_{3,s}(1,t) + T_7(1)y_{7,s}(1,t) - T_8(0)y_{8,s}(0,t) \\ T_6(1)y_{6,s}(1,t) + T_8(1)y_{8,s}(1,t) - T_9(0)y_{9,s}(0,t) - T_{11}(0)y_{11,s}(0,t) \\ T_9(1)y_{9,x}(1,t) - T_{10}(0)y_{10,x}(0,t) \\ T_{10}(1)y_{10,x}(1,t) + T_{11}(1)y_{11,x}(1,t) - T_4(0)y_{4,x}(0,t) \end{pmatrix} \end{aligned}$$

is an element in $\mathbb{C}^{10}$. This shows that the number of the dynamic conditions is not larger than the number of vertices provided that the network is continuous.

EXAMPLE 5.1.2 *Let G be a directed graph without boundary, whose structure is shown in Fig. 5.1.2. We consider a continuous network of strings defined on the graph G with elastic supports at all nodes.*

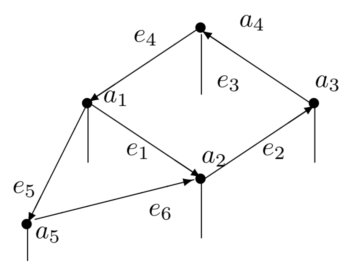

Fig. 5.1.2 A continuous network without boundary

The displacement of string on e_j is $y_j(x,t)$, the direction of parameter x coincides with that of the edges, and satisfies

$$m_j y_{j,tt}(x,t) = T_j y_{j,xx}(x,t), \quad x \in (0,1)$$

The connective conditions are

$$\begin{aligned}
&y(a_1,t) = y_1(0,t) = y_4(1,t) = y_5(0,t) \quad y(a_3,t) = y_3(0,t) = y_2(1,t) \\
&y(a_2,t) = y_2(0,t) = y_1(1,t) = y_6(1,t) \quad y(a_4,t) = y_4(0,t) = y_3(1,t) \\
&y(a_5,t) = y_5(1,t) = y_6(0,t)
\end{aligned}$$

Take $\Gamma = \mathrm{diag}(\alpha(a_1), \alpha(a_2), \cdots \alpha(a_5))$ and $\mathbb{K}(v) = \mathrm{diag}(k(a_1), k(a_2), \cdots k(a_5))$. The dynamic conditions are

$$\begin{aligned}
&T_4 y_{4,x}(1,t) - T_1 y_{1,x}(0,t) - T_5 y_{5,x}(0,t) + k(a_1) y(a_1,t) = -\alpha(a_1) y_t(a_1,t) \\
&T_5 y_{5,x}(1,t) - T_6 y_{6,x}(0,t) + k(a_5) y(a_5,t) = -\alpha(a_5) y_t(a_5,t) \\
&T_1 y_{1,x}(1,t) + T_6 y_{6,x}(1,t) - T_2 y_{2,x}(0,t) + k(a_2) y(a_2,t) = -\alpha(a_2) y_t(a_2,t) \\
&T_2 y_{2,x}(1,t) - T_3 y_{3,x}(0,t) + k(a_3) y(a_3,t) = -\alpha(a_3) y_t(a_3,t) \\
&T_3 y_{3,x}(1,t) - T_4 y_{4,x}(0,t) + k(a_4) y(a_4,t) = -\alpha(a_4) y_t(a_4,t)
\end{aligned}$$

The continuity connection conditions can be written into

$$\begin{pmatrix} y_1(0,t) \\ y_2(0,t) \\ y_3(0,t) \\ y_4(0,t) \\ y_5(0,t) \\ y_6(0,t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y(a_1,t) \\ y(a_2,t) \\ y(a_3,t) \\ y(a_4,t) \\ y(a_5,t) \end{pmatrix}$$

$$\begin{pmatrix} y_1(1,t) \\ y_2(1,t) \\ y_3(1,t) \\ y_4(1,t) \\ y_5(1,t) \\ y_6(1,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y(a_1,t) \\ y(a_2,t) \\ y(a_3,t) \\ y(a_4,t) \\ y(a_5,t) \end{pmatrix}$$

The dynamic conditions are rewritten into

$$\begin{aligned} \Phi^{+}\mathbb{T}Y_x(1,t) &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} T_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_6 \end{pmatrix} \begin{pmatrix} y_{1,x}(1,t) \\ y_{2,x}(1,t) \\ y_{3,x}(1,t) \\ y_{4,x}(1,t) \\ y_{5,x}(1,t) \\ y_{6,x}(1,t) \end{pmatrix} \\ &= \begin{pmatrix} T_4 y_{4,x}(1,t) \\ T_1 y_{1,x}(1,t) + T_6 y_{6,x}(1,t) \\ T_2 y_{2,x}(1,t) \\ T_3 y_{3,x}(1,t) \\ T_5 y_{5,x}(1,t) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \Phi^{-}\mathbb{T}Y_x(0,t) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_5 & \\ 0 & 0 & 0 & 0 & 0 & T_6 \end{pmatrix} \begin{pmatrix} y_{1,x}(0,t) \\ y_{2,x}(0,t) \\ y_{3,x}(0,t) \\ y_{4,x}(0,t) \\ y_{5,x}(0,t) \\ y_{6,x}(0,t) \end{pmatrix} \\ &= \begin{pmatrix} T_1 y_{1,x}(0,t) + T_5 y_{5,x}(0,t) \\ T_2 y_{2,x}(0,t) \\ T_3 y_{3,x}(0,t) \\ T_4 y_{4,x}(0,t) \\ T_6 y_{6,x}(0,t) \end{pmatrix} \end{aligned}$$

The controllers and elastic supports are

$$\Gamma\mathcal{S}Y_t(v,t) = \begin{pmatrix} \alpha(a_1) & 0 & 0 & 0 & 0 \\ 0 & \alpha(a_2) & 0 & 0 & 0 \\ 0 & 0 & \alpha(a_3) & 0 & 0 \\ 0 & 0 & 0 & \alpha(a_4) & 0 \\ 0 & 0 & 0 & 0 & \alpha(a_5) \end{pmatrix} \begin{pmatrix} y_t(a_1,t) \\ y_t(a_2,t) \\ y_t(a_3,t) \\ y_t(a_4,t) \\ y_t(a_5,t) \end{pmatrix}$$

$$\mathbb{K}(v)Y(v,t) = \begin{pmatrix} k(a_1) & 0 & 0 & 0 & 0 \\ 0 & k(a_2) & 0 & 0 & 0 \\ 0 & 0 & k(a_3) & 0 & 0 \\ 0 & 0 & 0 & k(a_4) & 0 \\ 0 & 0 & 0 & 0 & k(a_5) \end{pmatrix} \begin{pmatrix} y(a_1,t) \\ y(a_2,t) \\ y(a_3,t) \\ y(a_4,t) \\ y(a_5,t) \end{pmatrix}.$$

5.2 Boundary and internal controllers

Here we distinguish the boundary controllers and interior controllers. Let the selection matrix of vertex observation S be a 0-1 diagonal matrix of the form

$$\mathcal{S} = \text{diag}(s(a_1), s(a_2), \cdots, s(a_m)).$$

If $s(a_j) = 0, \forall \alpha_j \in \partial G$ and $s(a) = 1, \forall a \in V_{int}$, then $\mathcal{S}$ is said to be the interior observation and $\Gamma\mathcal{S}$ to be interior controllers; if $s(a) = 0, \forall a \in V_{int}$, then $\mathcal{S}$ is said to be the boundary observation and $\Gamma\mathcal{S}$ is called the boundary controllers.

Define the interior subspace and the boundary subspace respectively by

$$\mathbb{V}_1 = \{(x_1, x_2, \cdots, x_m) \mid x_j = 0, a_j \in \partial G\}, \quad \mathbb{V}_2 = \{(x_1, x_2, \cdots, x_m) \mid x_j = 0, a_j \in V_{int}\}$$

Obviously, $\mathbb{V}_1 \oplus \mathbb{V}_2 = \mathbb{C}^m$. Define a projection matrix $\mathcal{P}$ from $\mathbb{C}^m$ to $\mathbb{V}_1$.

When the system has boundary controllers, we have

$$\begin{cases} \mathbb{M}(s)U_{tt}(s,t) = (\mathbb{T}(s)U_s(s,t))_s - \mathbb{Q}(s)U(s,t), \quad s \in (0,1) \\ \exists U(v,t) \in \mathbb{V}, s.t. U(1,t) = (\Phi^+)^T U(v,t), \\ U(0,t) = (\Phi^-)^T U(v,t) \\ \mathcal{P}\Phi^+\mathbb{T}(1)U_s(1,t) - \mathcal{P}\Phi^-\mathbb{T}(0)U_s(0,t) + \mathcal{P}\mathbb{K}(v)U(v,t) = 0 \in \mathbb{V}_1 \\ (I-\mathcal{P})\Phi^+\mathbb{T}(1)U_s(1,t) - (I-\mathcal{P})\Phi^-\mathbb{T}(0)U_s(0,t) + (I-\mathcal{P})\mathbb{K}(v)U(v,t) = -(I-\mathcal{P})\Gamma\mathcal{S}U_t(v,t) \\ U(s,0) = U_0(s), \quad U_t(s,0) = U_1(s). \end{cases} \tag{5.2.1}$$

If the system has only internal controllers, then we have

$$\begin{cases} \mathbb{M}(s)U_{tt}(s,t) = (\mathbb{T}(s)U_s(s,t))_s - \mathbb{Q}(s)U(s,t), \quad s \in (0,1) \\ \exists U(v,t) \in \mathbb{V}, s.t. U(1,t) = (\Phi^+)^T U(v,t), \\ U(0,t) = (\Phi^-)^T U(v,t) \\ \mathcal{P}\Phi^+\mathbb{T}(1)U_s(1,t) - \mathcal{P}\Phi^-\mathbb{T}(0)U_s(0,t) + \mathcal{P}\mathbb{K}(v)U(v,t) = -\mathcal{P}\Gamma\mathcal{S}U_t(v,t) \in \mathbb{V}_1 \\ (I-\mathcal{P})\Phi^+\mathbb{T}(1)U_s(1,t) - (I-\mathcal{P})\Phi^- T(0)U_s(0,t) + (I-\mathcal{P})\mathbb{K}(v)U(v,t) = 0 \\ U(s,0) = U_0(s),\ U_t(s,0) = U_1(s). \end{cases} \tag{5.2.2}$$

EXAMPLE 5.2.1 *Let G be a directed graph whose structure is shown as Fig. 5.2.1. We set up the boundary controllers at a_5, a_6, a_7 and a_8, without interior controller.*

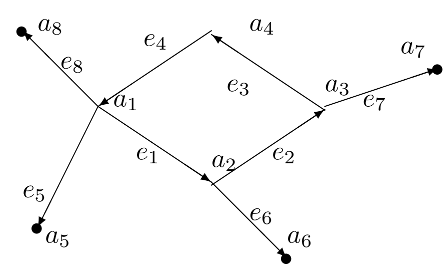

Fig. 5.2.1. Boundary controllers at a_5, a_6, a_7 and a_8, without interior controller.

In this example, the vertex selection matrix $\mathcal{S}$ and space $\mathbb{V}_1$ are respectively

$$\mathcal{S} = \mathrm{diag}(0,0,0,0,1,1,1,1), \quad \mathbb{V}_1 = \{(x_1,x_2,x_3,x_4,0,0,0,0) \mid x_j \in \mathbb{R}\}.$$

EXAMPLE 5.2.2 *Let G be a planar directed graph whose structure is shown as Fig. 5.2.2. We impose the interior controllers, without boundary controller.*

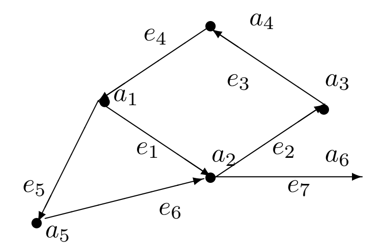

Fig. 5.2.2. All are internal controllers without boundary controller

In this example, the vertex selection matrix $\mathcal{S}$ and space $\mathbb{V}_1$ are respectively

$$\mathcal{S} = \mathrm{diag}(1,1,1,1,1,0), \quad \mathbb{V}_1 = \{(x_1,x_2,x_3,x_4,x_5,0) \mid x_j \in \mathbb{R}\}.$$

5.2.1 Mixed boundary conditions if $\partial G \neq \emptyset$

Let $\partial G \neq \emptyset$ and $\mathbb{K}(v) = 0$. Let $u(x,t)$ defined on G satisfy the wave equation on E. Assume that $u(x,t) \in C(G)$. If there is at least one $a \in \partial G$, $u(a,t) = 0$, which is called the Dirichlet boundary; and there is at least one $a \in \partial G$ such that it is of Neumann boundary. Then the network is said to be with mixed boundary.

Define a subspace $\mathbb{V}$ by

$$\mathbb{V} = \{(w(a_1), w(a_2), \cdots, w(a_m) \mid w(a) = 0, a \in \partial G_D\} \subset \mathbb{C}^m$$

where ∂G_D denotes the Dirichlet boundary. Let $\mathcal{P}$ be the orthogonal projection from $\mathbb{C}^m$ to $\mathbb{V}$. We design the feedback controllers for the networks with mixed boundary.

Take the feedback control law as

$$F(v,t) = -\Gamma \mathcal{S} U_t(v,t) \in \mathbb{V}$$

where $\Gamma\mathcal{S}$ is a nonnegative matrix. Thus the closed loop system is

$$\begin{cases} \mathbb{M}(s)U_{tt}(s,t) = (\mathbb{T}(s)U_s(s,t))_s - \mathbb{Q}(s)U(s,t), \quad s \in (0,1) \\ \exists U(v,t) \in \mathbb{V}, s.t. U(1,t) = (\Phi^+)^T U(v,t), \\ U(0,t) = (\Phi^-)^T U(v,t), \\ \mathcal{P}\Phi^+ T(1)U_s(1,t) - \mathcal{P}\Phi^- T(0)U_s(0,t) = -\Gamma\mathcal{S}U_t(v,t) \in \mathbb{V} \\ U(s,0) = U_0(s), \quad U_t(s,0) = U_1(s). \end{cases} \tag{5.2.3}$$

Here we again distinguish the boundary controllers and interior controllers.

EXAMPLE 5.2.3 *Let G be a planar directed graph, and the structure be shown in Fig. 5.2.3. In this example we impose the interior controllers without boundary controller.*

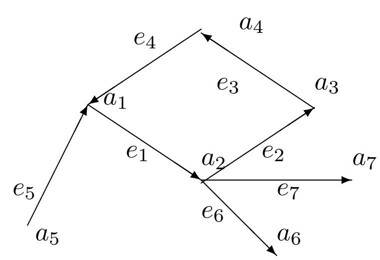

Fig. 5.2.3. Mixed boundary conditions: a_5 is a Dirichlet boundary, a_6 and a_7 are Neumann boundary, others are interior controllers

5.3 Discontinuous networks of strings

5.3.1 Semi-continuous networks

Let G be a planar metric graph. We parameterize the graph satisfying condition that for each $a \in V_{int}$, there are at least one incoming edge and one outgoing edge. That is, $J(a) = J^+(a) \cup J^-(a)$ satisfy the condition that $\#J^+(a) \geq 1$ and $\#J^-(a) \geq 1$.

Assume that an elastic structure coincides with the graph when it is in equilibrium position. The elastic structure undergoes the small vibration in a plane, whose motion on each edge e_j

is governed by the partial differential equation

$$m_j(s)w_{j,tt}(s,t) = (T_j(s)w_{j,s}(s,t))_s - q_j(s)w_j(s,t), \quad s \in (0,1) \tag{5.3.1}$$

We impose the network with the following the geometric and dynamic conditions:

1) if $a \in \partial G$ such that $J(a) = J^+(a)$, we have

$$\lim_{s\to 1} w_j(s,t) = w_j(1,t) = w(a,t), \quad j \in J^+(a), \tag{5.3.2}$$

and the dynamic boundary conditions

$$T_j(1)w_{j,s}(1,t) = -\alpha(a)w_t(a,t), \quad j \in J^+(a). \tag{5.3.3}$$

2) if $a \in \partial G$ with $J(a) = J^-(a)$, then

$$\lim_{s\to 0} w_j(s,t) = w_j(0,t) = w(a,t) = 0, \quad j \in J^-(a) \tag{5.3.4}$$

which means that the components of the elastic structure are clamped at every $a \in \partial G_D$.

At the interior node $a \in V_{int}$

$$\lim_{s\to 1} w_j(s,t) = w_j(1,t), \quad j \in J^+(a), \quad \lim_{s\to 0} w_j(s,t) = w_j(0,t), \quad j \in J^-(a)$$

satisfy the connective conditions:

If $\#J^+(a) > 1$, one imposes the continuity condition of the moment in the incoming direction. Accordingly, in the incoming direction of at this vertex, there exists a single-valued moment such that

$$T_r(1)w_{r,s}(1,t) = T_j(1)w_{j,s}(1,t), \quad \forall j,r \in J^+(a). \tag{5.3.5}$$

Corresponding geometric condition is

$$\sum_{i\in J^+(a)} w_i(1,t) = w(a,t); \tag{5.3.6}$$

For $J^-(a)$, we impose the rigid connection condition in the outgoing direction, which transfers the force. Then the geometric conditions are

$$w(a,t) = w_j(0,t), \quad \forall j \in J^-(a) \tag{5.3.7}$$

and the dynamic condition is

$$T_r(1)w_{r,s}(1,t) - \sum_{j\in J^-(a)} T_j(0)w_{j,s}(0,t) = -\alpha(a)w_t(a,t) \tag{5.3.8}$$

where $r \in J^+(a)$.

If $\#J^+(a) = 1$, $r \in J^+(a)$, then the above condition becomes the rigid connection. Hence we have the geometric conditions

$$w(a,t) = w_j(0,t) = w_r(1,t), \quad \forall j \in J^-(a) \tag{5.3.9}$$

and the corresponding dynamic conditions (including damping) become

$$T_r(1)w_{r,s}(1,t) - \sum_{j\in J^-(a)} T_j(0)w_{j,s}(0,t) = -\alpha(a)w_t(a,t). \tag{5.3.10}$$

Here $\alpha(a), a \in V$ are the positive feedback gain constants. Thus the complete description of the network is given by

$$\begin{cases} m_j(s)w_{j,tt}(s,t) = (T_j(s)w_{j,s}(s,t))_s - q_j(s)w_j(s,t), s \in (0,1), j = 1,2\cdots,n \\ w_j(0,t) = 0, \quad a \in \partial G_D, \quad j \in J(a) = J^-(a) \\ \sum_{j\in J^+(a)} w_j(1,t) = w(a,t) = w_i(0,t), i \in J^-(a), a \in V_{int} \\ T_r(1)w_{r,s}(1,t) - \sum_{j\in J^-(a)} T_j(0)w_{j,s}(0,t) = -\alpha(a)w_t(a,t), r \in J^+(a), \\ T_j(1)w_{j,s}(1,t) = -\alpha(a)w_t(a,t), a \in \partial G, J(a) = J^+(a). \end{cases} \tag{5.3.11}$$

Set $W(x,t) = (w_1(s,t), w_2(s,t), \cdots, w_n(s,t))^T$ and

$$\mathbb{M}(s) = \operatorname{diag}(m_1(s), m_2(s), \cdots, m_n(s)), \quad \mathbb{T}(s) = \operatorname{diag}(T_1(s), T_2(s), \cdots, T_n(s))$$

$$W(v,t) = (w(a_1,t), w(a_2,t), \cdots, w(a_m,t))^T$$

$$\mathbb{V} = \{(w(a_1), w(a_2), \cdots, w(a_m)) \mid w(a_j) = 0, a_j \in \partial G_D\}.$$

Let $\mathbb{U}$ be the isomorphism from $\mathbb{V}$ to $\mathbb{C}^q$, where $q = m - \#\partial G_D = \dim \mathbb{V}$.

The conditions (5.3.2) and (5.3.6) say that the network is of incoming flow continuity, i.e.,

$$W(v,t) = (\Phi^+)W(1,t).$$

The conditions (5.3.4) and (5.3.7) indicate that the network is of outgoing continuity, i.e.,

$$W(0,t) = (\Phi^-)^T W(v,t).$$

Since the incoming is flow continuous, the outgoing is continuous in usual, so the network is said to be semi-continuous. In the case of semi-continuity, we have

$$W(0,t) = (\Phi^-)^T(\Phi^+)W(1,t).$$

Thus the conditions (5.3.3) and (5.3.8) can be rewritten into

$$\mathcal{P}(\Phi^+)^T\mathbb{T}(1)W_s(1,t) - \mathcal{P}(\Phi^-)^T\mathbb{T}(0)W_s(0,t) = -\Gamma\mathcal{S}W_t(v,t) \in \mathbb{V}$$

where $\mathcal{P} : \mathbb{C}^m \to \mathbb{V}$. Therefore, we have

$$\begin{cases} \mathbb{M}(s)W_{tt}(s,t) = (\mathbb{T}(s)W_s(s,t))_s - \mathbb{Q}(s)W(s,t), \quad s \in (0,1) \\ W(0,t) = \mathbf{C}W(1,t), \\ \mathcal{P}(\Phi^+)^T\mathbb{T}(1)W_s(1,t) - \mathcal{P}(\Phi^-)^T\mathbb{T}(0)W_s(0,t) = -\Gamma\mathcal{S}W_t(v,t) \in \mathbb{V} \\ W(s,0) = W_0(s), \quad W_t(s,0) = W_1(s) \end{cases} \tag{5.3.12}$$

where $\mathbf{C} = (\Phi^-)^T(\Phi^+)$, $W_0(s)$ and $W_1(s)$ are the initial state of the system.

The system (5.3.12) comes from design of feedback controllers for the semi-continuous network. The energy function of the network is defined by

$$\mathcal{E}(t) = \frac{1}{2}\sum_{j=1}^{n}\int_0^1 (T_j(s)|w_{j,s}(s,t)|^2 + m_j(s)|w_{j,t}(s,t)|^2 + q_j(s)|w_j(s,t)|^2)ds.$$

Thus

$$\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} &= \sum_{j=1}^{n}\int_0^1 (T_j(s)w_{j,s}(s,t)w_{j,st}(s,t)+m_j(s)w_{j,t}(s,t)w_{j,tt}(s,t))+q_j(s)w_j(s,t)w_{j,t}(s,t))ds\\
&= \sum_{j=1}^{n}[T_j(1)w_{j,s}(1,t)w_{j,t}(1,t)-\sum_{j=1}^{n}T_j(0)w_{j,s}(0,t)w_{j,t}(0,t)\\
&= \sum_{k=1}^{m}\left[\sum_{j\in J^+(a_k)}T_j(1)w_j(1,t)w_{j,t}(1,t)-\sum_{j\in J^-(a_k)}T_j(0)w_{j,s}(0,t)w_{j,t}(0,t)\right]\\
&= \sum_{k=1}^{m}\left[T_k(1)w_{k,s}(1,t)\sum_{j\in J^+(a_k)}w_{j,t}(1,t)-w_t(a_k,t)\sum_{j\in J^-(a_k)}T_j(0)w_{j,s}(0,t)\right]\\
&= \sum_{k=1}^{m}w_t(a_k,t)\left[T_k(1)w_{k,s}(1,t)-\sum_{j\in J^-(a_k)}T_j(0)w_{j,s}(0,t)\right]\\
&= \sum_{k=1}^{m}w_t(a_k,t)f(a_k,t).
\end{aligned}$$

The feedback control law is

$$f(a_k,t)=-\alpha(a_k)w_t(a_k,t),\quad k=1,2,\cdots,m.$$

where $\alpha(a_k)\geq 0$. Hence we get the dynamic conditions

$$T_k(1)w_{k,s}(1,t)-\sum_{j\in J^-(a_k)}T_j(0)w_{j,s}(0,t)=-\alpha(a_k)w(a_k,t).$$

Thus we have

$$\frac{d\mathcal{E}(t)}{dt}=-\sum_{k=1}^{m}\alpha(a_k)|w_t(a_k,t)|^2.$$

In the design of controllers, we can take different form. Since $w(a,t)=w_i(0,t)=\sum_{j\in J^+(a)}w_j(1,t)$, we have

$$\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} &= \sum_{k=1}^{m}\left[\sum_{j\in J^+(a_k)}T_j(1)w_{j,s}(1,t)w_{j,t}(1,t)-\sum_{j\in J^-(a_k)}T_j(0)w_{j,s}(0,t)w_{j,t}(0,t)\right]\\
&= \sum_{k=1}^{m}\left[\sum_{j\in J^+(a_k)}T_j(1)w_{j,s}(1,t)w_{j,t}(1,t)-\sum_{j\in J^+(a)}w_{j,t}(1,t)\sum_{i\in J^-(a_k)}T_i(0)w_{i,s}(0,t)\right]\\
&= \sum_{k=1}^{m}\sum_{j\in J^+(a)}w_{j,t}(1,t)\left[T_j(1)w_{j,s}(1,t)-\sum_{i\in J^-(a_k)}T_i(0)w_{i,s}(0,t)\right].
\end{aligned}$$

If all $w_{j,t}(1,t)$ are observable (measurable), we take

$$\left[T_jw_{j,s}(1,t)-\sum_{i\in J^-(a_k)}T_iw_{i,s}(0,t)\right]=-\alpha_jw_{j,t}(1,t),$$

this leads to

$$\frac{d\mathcal{E}(t)}{dt} = -\sum_{k=1}^{m} \sum_{j \in J^+(a)} \alpha_j |w_{j,t}(1,t)|^2.$$

In this case, the corresponding closed loop system is

$$\begin{cases} \mathbb{M}(s)W_{tt}(s,t) = (\mathbb{T}(s)W_s(s,t))_s - \mathbb{Q}(s)W(s,t), \quad s \in (0,1) \\ W(0,t) = \mathbf{C}W(1,t), \\ \mathbb{T}(1)W_s(1,t) - \mathbf{C}^T\mathbb{T}(0)W_s(0,t) = -\Gamma\mathcal{S}W_t(1,t) \\ W(s,0) = W_0(s), W_t(s,0) = W_1(s). \end{cases}$$

The difference of both is that the above is the endpoints measurement of each string, while (5.3.12) is the node measurement.

EXAMPLE 5.3.1 *In this example we consider a discontinuous network of strings with multiple circuits. Let G be a planar graph, whose structure is shown as Fig.5.3.1. The directed edges are defined by*

$$\begin{array}{llll} \gamma_1 = (a_1, a_5) & \gamma_2 = (a_2, a_6) & \gamma_3 = (a_3, a_7) & \gamma_4 = (a_{10}, a_4) \\ \gamma_5 = (a_5, a_6) & \gamma_6 = (a_6, a_8) & \gamma_7 = (a_5, a_7) & \gamma_8 = (a_7, a_8) \\ \gamma_9 = (a_8, a_9) & \gamma_{10} = (a_9, a_{10}) & \gamma_{11} = (a_8, a_{10}) & \end{array}$$

And let $y(x,t)$ be a function defined on G and satisfy the wave equation on each edge γ_j:

$$m_j y_{j,tt}(x,t) = T_j y_{j,xx}(x,t)$$

Moreover, we assume that $y(x,t)$ satisfies the semi-continuous condition at all interior nodes.

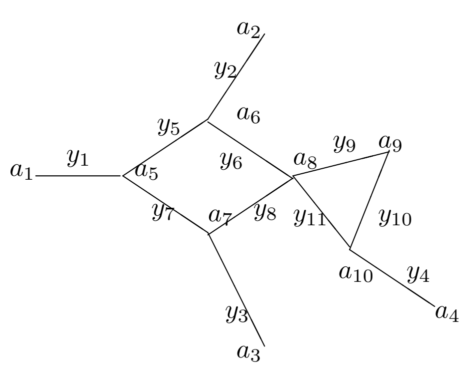

Fig. 5.3.1. A discontinuous network of strings with circuits

According to (5.3.11), the motion of the network is governed by partial differential equations

$$\begin{cases} m_j y_{j,tt}(x,t) = T_j y_{j,xx}(x,t),\ x \in (0,1), \\ y_1(0,t) = y_2(0,t) = y_3(0,t) = 0; \quad y_4(0,t) = y_{10}(1,t) + y_{11}(1,t); \\ y_5(0,t) = y_7(0,t) = y_1(1,t); \quad y_6(0,t) = y_2(1,t) + y_5(1,t); \\ y_8(0,t) = y_7(1,t) + y_3(1,t); \quad y_9(0,t) = y_6(1,t) + y_8(1,t); \\ y_{10}(0,t) = y_9(1,t); \quad y_{11}(0,t) = y_6(1,t) + y_8(1,t); \\ T_1 y_{1,x}(1,t) - T_5 y_{5,x}(0,t) - T_7 y_{7,x}(0,\ t) = -\alpha(\alpha_5) y_t(a_5,t), \\ T_2 y_{2,x}(1,t) + T_5 y_{5,x}(1,t) - T_6 y_{6,x}(0,t) = -\alpha(a_6) y_t(a_6,t), \\ T_3 y_{3,x}(1,t) + T_7 y_{7,x}(1,t) - T_8 y_{8,x}(0,t) = -\alpha(a_7) y_t(a_7,t), \\ T_4 y_{4,x}(1,t) = -\alpha(a_4) y_t(a_4,t), \\ T_6 y_{6,x}(1,t) + T_8 y_{8,x}(1,t) - T_9 y_{9,x}(0,t) - T_{11} y_{11,x}(0,t) = -\alpha(a_8) y_t(a_8,t), \\ T_9 y_{9,x}(1,t) - T_{10} y_{10,x}(0,t) = -\alpha(a_9) y_t(a_9,t), \\ T_{10} y_{10,x}(1,t) + T_{11} y_{11,x}(1,t) - T_4 y_{4,x}(0,t) = -\alpha(a_{10}) y_t(a_{10},t), \\ y_k(x,0) = y_{k,0}(x),\ y_k(x,0) = y_{k,1}(x),\ k = 1,2,\cdots,11. \end{cases} \tag{5.3.13}$$

The connection matrix $\mathbf{C} = (\Phi^-)^T(\Phi^+)$ is given by

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the connective matrix gives the relation of the network, from which we can reconstruct partly the graph.

The relation between vertices and edges for the forces are given by

$$(\Phi^+)^T\mathbb{T}(1)Y_x(1,t) = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&0\\ 1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&1&0&0&0 \end{bmatrix} \begin{bmatrix} T_1(1)y_{1,x}(1,t)\\ T_2(1)y_{2,x}(1,t)\\ T_3(1)y_{3,x}(1,t)\\ T_4(1)y_{4,x}(1,t)\\ T_5(1)y_{5,x}(1,t)\\ T_6(1)y_{6,x}(1,t)\\ T_7(1)y_{7,x}(1,t)\\ T_8(1)y_{8,x}(1,t)\\ T_9(1)y_{9,x}(1,t)\\ T_{10}(1)y_{10,x}(1,t)\\ T_{11}(1)y_{11,x}(1,t) \end{bmatrix}.$$

$$(\Phi^-)^T\mathbb{T}(0)Y_x(0,t) = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&0&1&0&0 \end{bmatrix} \begin{bmatrix} T_1(0)y_{1,x}(0,t)\\ T_2(0)y_{2,x}(0,t)\\ T_3(0)y_{3,x}(0,t)\\ T_4(0)y_{4,x}(0,t)\\ T_5(0)y_{5,x}(0,t)\\ T_6(0)y_{6,x}(0,t)\\ T_7(0)y_{7,x}(0,t)\\ T_8(0)y_{8,x}(0,t)\\ T_9(0)y_{9,x}(0,t)\\ T_{10}(0)y_{10,x}(0,t)\\ T_{11}(0)y_{11,x}(0,t) \end{bmatrix}.$$

Thus, the forces at all vertices are

$$(\Phi^+)^T\mathbb{T}(1)Y_x(1,t) - (\Phi^-)^T\mathbb{T}(0)Y_x(0,t)$$

$$= \begin{bmatrix} -T_1(0)y_{1,x}(0,t) \\ -T_2(0)y_{2,x}(0,t) \\ -T_3(0)y_{3,x}(0,t) \\ T_4(1)y_{4,x}(1,t) \\ T_1(1)y_{1,x}(1,t) - T_5(0)y_{5,x}(0,t) - T_7(0)y_{7,x}(0,t) \\ T_2(1)y_{2,x}(1,t) + T_5(1)y_{5,x}(1,t) - T_6(0)y_{6,x}(0,t) \\ T_3(1)y_{3,x}(1,t) + T_7(1)y_{7,x}(0,t) - T_8(0)y_{8,x}(0,t) \\ T_6(1)y_{6,x}(1) + T_8(1)y_{8,x}(1,t) - T_9(0)y_{9,x}(0,t) - T_{11}(1)y_{11,x}(0,t) \\ T_9(1)y_{9,x}(1,t) - T_{10}(0)y_{10,x}(0,t) \\ T_{10}(1)y_{10,x}(1,t) + T_{11}(1)y_{11,x}(1,t) - T_4(0)y_{4,x}(0,t) \end{bmatrix}.$$

At all the controlled vertices, the dynamic conditions are

$$\begin{aligned} & \mathcal{P}(\Phi^+)^T\mathbb{T}(1)Y_x(1,t) - \mathcal{P}(\Phi^-)^T\mathbb{T}(0)Y_x(0,t) \\ = & \begin{bmatrix} 0 \\ 0 \\ 0 \\ T_4(1)y_{4,x}(1,t) \\ T_1(1)y_{1,x}(1,t) - T_5(0)y_{5,x}(0,t) - T_7(0)y_{7,x}(0,t) \\ T_2(1)y_{2,x}(1,t) + T_5(1)y_{5,x}(1,t) - T_6(0)y_{6,x}(0,t) \\ T_3(1)y_{3,x}(1,t) + T_7(1)y_{7,x}(0,t) - T_8(0)y_{8,x}(0,t) \\ T_6(1)y_{6,x}(1) + T_8(1)y_{8,x}(1,t) - T_9(0)y_{9,x}(0,t) - T_{11}(1)y_{11,x}(0,t) \\ T_9(1)y_{9,x}(1,t) - T_{10}(0)y_{10,x}(0,t) \\ T_{10}(1)y_{10,x}(1,t) + T_{11}(1)y_{11,x}(1,t) - T_4(0)y_{4,x}(0,t) \end{bmatrix} \\ = & -\begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha(a_4)y_t(a_4,t) \\ \alpha(a_5)y_t(a_5,t) \\ \alpha(a_6)y_t(a_6,t) \\ \alpha(a_7)y_t(a_7,t) \\ \alpha(a_8)y_t(a_8,t) \\ \alpha(a_9)y_t(a_9,t) \\ \alpha(a_{10})y_t(a_{10},t) \end{bmatrix} = -\Gamma\mathcal{S}Y_t(v,t) \in \mathbb{V}. \end{aligned}$$

The projection in the control space is

$$
\begin{aligned}
&\mathbb{U}P(\Phi^+)^T\mathbb{T}(1)Y_x(1,t) - \mathbb{U}P(\Phi^-)^T\mathbb{T}(0)Y_x(0,t)\\
= &\begin{bmatrix}
T_4(1)y_{4,x}(1,t)\\
T_1(1)y_{1,x}(1,t) - T_5(0)y_{5,x}(0,t) - T_7(0)y_{7,x}(0,t)\\
T_2(1)y_{2,x}(1,t) + T_5(1)y_{5,x}(1,t) - T_6(0)y_{6,x}(0,t)\\
T_3(1)y_{3,x}(1,t) + T_7(0)y_{7,x}(0,t) - T_8(0)y_{8,x}(0,t)\\
T_6(1)y_{6,x}(1) + T_8(1)y_{8,x}(1,t) - T_9(0)y_{9,x}(0,t) - T_{11}(1)y_{11,x}(0,t)\\
T_9(1)y_{9,x}(1,t) - T_{10}(0)y_{10,x}(0,t)\\
T_{10}(1)y_{10,x}(1,t) + T_{11}(1)y_{11,x}(1,t) - T_4(0)y_{4,x}(0,t)
\end{bmatrix}\\
= &-\begin{bmatrix}
\alpha(a_4)y_t(a_4,t)\\
\alpha(a_5)y_t(a_5,t)\\
\alpha(a_6)y_t(a_6,t)\\
\alpha(a_7)y_t(a_7,t)\\
\alpha(a_8)y_t(a_8,t)\\
\alpha(a_9)y_t(a_9,t)\\
\alpha(a_{10})w_t(a_{10},t)
\end{bmatrix} = -\mathbb{U}\Gamma\mathcal{S}Y_t(v,t) \in \mathbb{C}^q.
\end{aligned}
$$

Similarly we calculate $\mathbb{T}(1)Y_x(1,t) - \mathbf{C}^T\mathbb{T}(0)Y_x(0,t)$

$$
\begin{aligned}
&\mathbb{T}(1)Y_x(1,t) - \mathbf{C}^T\mathbb{T}(0)Y_x(0,t)\\
= &\begin{bmatrix}
T_1(1)y_{1,x}(1,t)\\
T_2(1)y_{2,x}(1,t)\\
T_3(1)y_{3,x}(1,t)\\
T_4(1)y_{4,x}(1,t)\\
T_5(1)y_{5,x}(1,t)\\
T_6(1)y_{6,x}(1,t)\\
T_7(1)y_{7,x}(1,t)\\
T_8(1)y_{8,x}(1,t)\\
T_9(1)y_{9,x}(1,t)\\
T_{10}(1)y_{10,x}(1,t)\\
T_{11}(1)y_{11,x}(1,t)
\end{bmatrix} - \begin{bmatrix}
T_5(0)y_{5,x}(0,t) + T_7(0)y_{7,x}(0,t)\\
T_6(0)y_{6,x}(0,t)\\
T_8(0)y_{8,x}(0,t)\\
0\\
T_6(0)y_{6,x}(0,t)\\
T_9(0)y_{9,x}(0,t) + T_{11}(0)y_{11,x}(0,t)\\
T_8(0)y_{8,x}(0,t)\\
T_9(0)y_{9,x}(0,t) + T_{11}(0)y_{11,x}(0,t)\\
T_{10}(0)y_{10,x}(0,t)\\
T_4(0)y_{4,x}(0,t)\\
T_4(0)y_{4,x}(0,t)
\end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} T_1(1)y_{1,x}(1,t) - T_5(0)y_{5,x}(0,t) - T_7(0)y_{7,x}(0,t) \\ T_2(1)y_{2,x}(1,t) - T_6(0)y_{6,x}(0,t) \\ T_3(1)y_{3,x}(1,t) - T_8(0)y_{8,x}(0,t) \\ T_4(1)y_{4,x}(1,t) \\ T_5(1)y_{5,x}(1,t) - T_6(0)y_{6,x}(0,t) \\ T_6(1)y_{6,x}(1,t) - T_9(0)y_{9,x}(0,t) - T_{11}(0)y_{11,x}(0,t) \\ T_7(1)y_{7,x}(1,t) - T_8(0)y_{8,x}(0,t) \\ T_8(1)y_{8,x}(1,t) - T_9(0)y_{9,x}(0,t) - T_{11}(0)y_{11,x}(0,t) \\ T_9(1)y_{9,x}(1,t) - T_{10}(0)y_{10,x}(0,t) \\ T_{10}(1)y_{10,x}(1,t) - T_4(0)y_{4,x}(0,t) \\ T_{11}(1)y_{11,x}(1,t) - T_4(0)y_{4,x}(0,t) \end{bmatrix}.$$

EXAMPLE 5.3.2 *In this example we consider a semi-continuous network of strings with multiple circuits. Let G be a planar directed graph, whose structure be shown in Fig.5.3.2.*

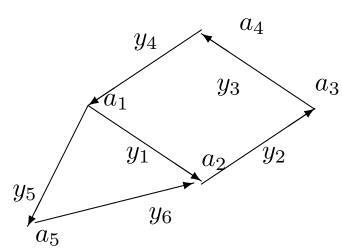

Fig. 5.3.2. A complex network of strings without boundary

Let $y_j(x,t)$ be the displacement of the strings network on the edge γ_j. They satisfy the wave equations

$$m_j y_{j,tt}(x,t) = T_j y_{j,xx}(x,t), \quad x \in (0,1), j = 1,2,3,4.$$

Suppose that the displacements of the network of strings satisfy the Kirchhoff law at the nodes a_2 and continuity conditions at the other nodes. The connective conditions of displacement $y_j(x,t)$ at the nodes are given by

$$\begin{aligned} &y_1(0,t) = y_4(1,t) \quad y_2(0,t) = y_1(1,t) + y_6(1,t) \\ &y_3(0,t) = y_2(1,t) \quad y_4(0,t) = y_3(1,t) \\ &y_5(0,t) = y_4(1,t) \quad y_6(0,t) = y_5(1,t) \end{aligned}$$

Assume that all $y_{j,t}(1,t)$ are measurable. The dynamic conditions are

$$\begin{aligned}
&T_1 y_{1,x}(1,t) - T_2 y_{2,x}(0,t) = -\alpha_1 y_{1,t}(1,t),\\
&T_2 y_{2,x}(1,t) - T_3 y_{3,x}(0,t) = -\alpha_2 y_{2,t}(1,t),\\
&T_3 y_{3,x}(1,t) - T_4 y_{4,x}(0,t) = -\alpha_3 y_{3,t}(1,t),\\
&T_4 y_{4,x}(1,t) - T_1 y_{1,x}(0,t) - T_5 y_{5,x}(0,t) = -\alpha_4 y_{4,t}(1,t),\\
&T_5 y_{5,x}(1,t) - T_6 y_{6,x}(0,t) = -\alpha_5 y_{5,t}(1,t),\\
&T_6 y_{6,x}(1,t) - T_2 y_{2,x}(0,t) = -\alpha_6 y_{6,t}(1,t).
\end{aligned}$$

Setting $Y(x,t) = [y_1(x,t), y_2(x,t), \cdots, y_6(x,t)]^\tau$,

$$\mathbb{M} = \mathrm{diag}(m_1, m_2, \cdots, m_6), \quad \mathbb{T} = \mathrm{diag}(T_1, T_2, \cdots, T_6),$$

$$\Gamma\mathcal{S} = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_6)$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus we have

$$\begin{cases} \mathbb{M}Y_{tt}(x,t) = \mathbb{T}Y_{xx}(x,t), x \in (0,1) \\ Y(0,t) = CY(1,t) \\ \mathbb{T}Y_x(1,t) - C^T\mathbb{T}Y_x(0,t) = -\Gamma\mathcal{S}Y_t(1,t). \end{cases}$$

5.3.2 Discontinuous networks

In previous subsection, we discuss the semi-continuous networks of strings, in which the matrix $C = (\Phi^-)^T\Phi^+$ gives the connective condition between edge and edge in the network. In this subsection we assume that the relation of edges C is given. However, for given C, the structural of the graph may be very complicated, that may include the loop and parallel edges. Whatever, our discussion is based on the connection matrix.

Let $Y(x,t)$ be a vector-valued function defined on the interval $(0,1)$ and satisfy the wave equation

$$\mathbb{M}(x)Y_{tt}(x,t) = (\mathbb{T}(x)Y_x(x,t))_x - \mathbb{Q}(x)Y(x,t), \quad x \in (0,1).$$

Assume that the connective condition is given by

$$Y(0,1) = CY(1,t), \quad \det(I - C) \neq 0.$$

Here we seek for the dynamic condition for this system.

The energy function of the network is defined by

$$\mathcal{E}(t)=\frac{1}{2}\int_0^1(\mathbb{T}(x)Y_x(x,t),Y_x(x,t))_{\mathbb{C}^n}+(\mathbb{M}(x)Y_t(x,t),Y_t(x,t))_{\mathbb{C}^n}+(\mathbb{Q}(x)Y(x,t),Y(x,t))_{\mathbb{C}^n}dx.$$

Thus

$$\begin{aligned}\frac{d\mathcal{E}(t)}{dt} &= \int_0^1(\mathbb{T}(x)Y_{xt}(x,t),Y_x(x,t))_{\mathbb{C}^n}+(\mathbb{M}(x)Y_{tt}(x,t),Y_t(x,t))_{\mathbb{C}^n}+(\mathbb{Q}(x)Y(x,t),Y_t(x,t))_{\mathbb{C}^n}dx\\ &= (\mathbb{T}(1)Y_x(1,t),Y_t(1,t))_{c^n}-(\mathbb{T}(0)Y_x(0,t),Y_t(0,t))_{\mathbb{C}^n}\\ &= (\mathbb{T}(1)Y_x(1,t),Y_t(1,t))_{\mathbb{C}^n}-(\mathbb{T}(0)Y_x(0,t),CY_t(1,t))_{\mathbb{C}^n}\\ &= (\mathbb{T}(1)Y_x(1,t)-C^T\mathbb{T}(0)Y_x(0,t),Y_t(1,t))_{\mathbb{C}^n}\end{aligned}$$

Let Γ be a non-negative matrix. We take control law as

$$\mathbb{T}(1)Y_x(1,t)-C^T\mathbb{T}(0)Y_x(0,t)=F(t)=-\Gamma\mathcal{S}Y_t(1,t).$$

Then the closed loop system is

$$\begin{cases}\mathbb{M}(x)Y_{tt}(x,t)=(\mathbb{T}(x)Y_x(x,t))_x-\mathbb{Q}(x)Y(x,t), \quad x\in(0,1)\\ Y(0,t)=CY(1,t),\\ \mathbb{T}(1)Y_x(1,t)-C^T\mathbb{T}(0)Y_x(0,t)=-\Gamma\mathcal{S}Y_t(1,t)\\ Y(x,0)=Y_0(x), \quad Y_t(x,0)=Y_1(x)\end{cases} \tag{5.3.14}$$

where $\det(I-C)\neq 0$.

EXAMPLE 5.3.3 *Let G be a planar directed graph, and the structure be shown as Fig.5.3.3.*

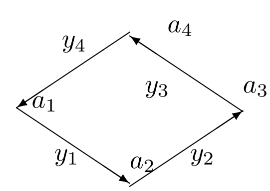

Fig. 5.3.3. A discontinuous network without boundary

Let $y_j(x,t)$ be the displacement of the strings network on the edge γ_j. They satisfy the wave equations

$$m_jy_{j,tt}(x,t)=T_jy_{j,xx}(x,t), \quad x\in(0,1), j=1,2,3,4.$$

Suppose that the displacements of the network of strings are continuous at the nodes a_2, a_3 and a_4, but at a_1 there is a jump at rate β, $\beta\neq 1$. The connective conditions of displacement $y_j(x,t)$ at the nodes are given by

$$\begin{aligned} &y_1(0,t)=\beta y_4(1,t) \quad && y_2(0,t)=y_1(1,t)\\ &y_3(0,t)=y_2(1,t) && y_4(0,t)=y_3(1,t)\end{aligned}$$

The dynamic conditions are

$$\begin{aligned}
T_1y_{1,x}(1,t) - T_2y_{2,x}(0,t) &= -\alpha_1 y_{1,t}(1,t)\\
T_2y_{2,x}(1,t) - T_3y_{3,x}(0,t) &= -\alpha_2 y_{2,t}(1,t)\\
T_3y_{3,x}(1,t) - T_4y_{4,x}(0,t) &= -\alpha_3 y_{3,t}(1,t)\\
T_4y_{4,x}(1,t) - \beta T_1y_{1,x}(0,t) &= -\alpha_4 y_{4,t}(1,t).
\end{aligned}$$

Setting $Y(x,t) = [y_1(x,t), y_2(x,t), y_3(x,t), y_4(x,t)]^T$,

$$\mathbb{M} = \mathrm{diag}(m_1, m_2, m_3, m_4), \quad \mathbb{T} = \mathrm{diag}(T_1, T_2, T_3, T_4),$$

$$\Gamma = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus we have

$$\begin{cases} \mathbb{M}Y_{tt}(x,t) = \mathbb{T}Y_{xx}(x,t), x \in (0,1) \\ Y(0,t) = CY(1,t) \\ \mathbb{T}Y_x(1,t) - C^T\mathbb{T}Y_x(0,t) = -\Gamma Y_t(1,t). \end{cases}$$

EXAMPLE 5.3.4 *Let G be a planar directed graph with the parallel edges, whose structure is shown as Fig.5.3.4.*

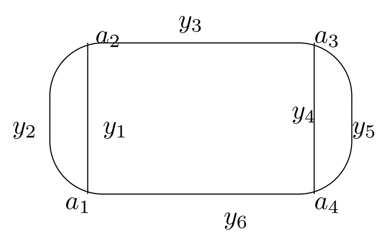

Fig. 5.3.4. A discontinuous network with parallel edges

Let $y_j(x,t)$ be the displacement of the strings network on the edge γ_j. They satisfy the wave equations

$$m_jy_{j,tt}(x,t) = T_jy_{j,xx}(x,t), \quad x \in (0,1), j = 1,2,3,4.$$

Suppose that the displacements of the network of strings satisfy the Kirchhoff law at nodes a_1 a_2 and a_4. y_1 and y_2, y_4 and y_5 are the parallel edges. The connective conditions of displacement $y_j(x,t)$ at the nodes are given by

$$\begin{array}{ll}
y_1(0,t) = \beta y_6(1,t) & y_2(0,t) = (1-\beta)y_6(1,t)\\
y_3(0,t) = y_2(1,t) + y_1(1,t) & y_4(0,t) = \alpha_1 y_3(1,t)\\
y_5(0,t) = \alpha_2 y_3(1,t) & y_6(0,t) = y_4(1,t) + y_5(1,t)
\end{array}$$

where $\alpha_1 + \alpha_2 \neq 1$.

Obviously, the system have not a constant solution. One define the energy function

$$E(t) = \frac{1}{2}\sum_{j=1}^{6}\int_0^1 [T_j y_{j.x}^2(x,t) + m_j y_{j,t}^2(x,t)]dx$$

Using the energy function, we have

$$\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} &= \sum_{j=1}^{6}[T_j y_{j,x}(1,t)y_{j,t}(1,t) - T_j y_{j,x}(0,t)y_{j,t}(0,t)] \\
&= [T_6 y_{6,x}(1,t)y_{6,t}(1,t) - T_1 y_{1,x}(0,t)y_{1,t}(0,t) - T_2 y_{2,x}(0,t)y_{2,t}(0,t)] \\
&\quad +[T_1 y_{1,x}(1,t)y_{1,t}(1,t) + T_2 y_{2,x}(1,t)y_{2,t}(1,t) - T_3 y_{3,x}(0,t)y_{3,t}(0,t)] \\
&\quad +[T_3 y_{3,x}(1,t)y_{3,t}(1,t) - T_4 y_{4,x}(0,t)y_{4,t}(0,t) - T_5 y_{5,x}(0,t)y_{5,t}(0,t)] \\
&\quad +[T_4 y_{4,x}(1,t)y_{4,t}(1,t) + T_5 y_{5,x}(1,t)y_{5,t}(1,t) - T_6 y_{6,x}(0,t)y_{6,t}(0,t)] \\
&= [T_6 y_{6,x}(1,t) - \beta T_1 y_{1,x}(0,t) - (1-\beta)T_2 y_{2,x}(0,t)]y_{6,t}(1,t) \\
&\quad +[T_1 y_{1,x}(1,t) - T_3 y_{3,x}(0,t)]y_{1,t}(1,t) + [T_2 y_{2,x}(1,t) - T_3 y_{3,x}(0,t)]y_{2,t}(1,t) \\
&\quad +[T_3 y_{3,x}(1,t) - \alpha T_4 y_{4,x}(0,t) - (1-\alpha)T_5 y_{5,x}(0,t)]y_{3,t}(1,t) \\
&\quad +[T_4 y_{4,x}(1,t) - T_6 y_{6,x}(0,t)]y_{4,t}(1,t) + [T_5 y_{5,x}(1,t) - T_6 y_{6,x}(0,t)]y_{5,t}(1,t)
\end{aligned}$$

Assume that all $y_{j,t}(1,t)$ are measurable, then the dynamic conditions are

$$\begin{aligned}
&T_6 y_{6,x}(1,t) - \beta T_1 y_{1,x}(0,t) - (1-\beta)T_2 y_{2,x}(0,t) = -\gamma_6 y_{6,t}(1,t) \\
&T_1 y_{1,x}(1,t) - T_3 y_{3,x}(0,t)] = \gamma_1 y_{1,t}(1,t) \\
&T_2 y_{2,x}(1,t) - T_3 y_{3,x}(0,t) = -\gamma_2 y_{2,t}(1,t) \\
&T_3 y_{3,x}(1,t) - \alpha_1 T_4 y_{4,x}(0,t) - \alpha_2 T_5 y_{5,x}(0,t) = -\gamma_3 y_{3,t}(1,t) \\
&T_4 y_{4,x}(1,t) - T_6 y_{6,x}(0,t) = -\alpha_4 y_{4,t}(1,t) \\
&T_5 y_{5,x}(1,t) - T_6 y_{6,x}(0,t) = -\alpha 5 y_{5,t}(1,t).
\end{aligned}$$

Setting $Y(x,t) = [y_1(x,t), y_2(x,t), y_3(x,t), y_4(x,t), y_5(x,t), y_6(x,t)]^T$,

$$\mathbb{M} = \mathrm{diag}(m_1, m_2, m_3, m_4, m_5, m_6), \quad \mathbb{T} = \mathrm{diag}(T_1, T_2, T_3, T_4, T_5, T_6),$$

$$\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6)$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 & 0 & (1-\beta) \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Thus we have

$$\begin{cases} \mathbb{M}Y_{tt}(x,t) = \mathbb{T}Y_{xx}(x,t), x \in (0,1) \\ Y(0,t) = CY(1,t) \\ \mathbb{T}Y_x(1,t) - C^T\mathbb{T}Y_x(0,t) = -\Gamma\mathcal{S}Y_t(1,t). \end{cases}$$

5.3.3 Twist curve

The discontinuous model given in (5.3.14) has more extensive applications. In this subsection, we discuss a mathematical problem. Assume that there exist four curves twisting at one common node, each curve at time t satisfies the wave equation

$$m_j w_{j,tt}(x,t) = T_j w_{j,xx}(x,t), \quad x \in (0,1).$$

Herein we give an example for this type curves.

EXAMPLE 5.3.5 *Let G be a planar directed graph with parallel edges, whose structure is shown as Fig.5.3.5.*

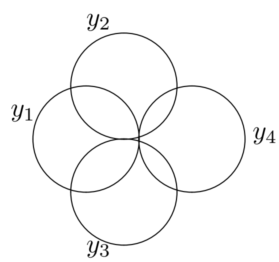

Fig. 5.3.5. A twisting curve

Suppose that the network of strings satisfy the following connective conditions:

$$y_1(0,t) = y_3(1,t) + y_4(1,t) \quad y_2(0,t) = y_1(1,t) + y_4(1,t)$$
$$y_3(0,t) = y_1(1,t) + y_2(1,t) \quad y_4(0,t) = y_2(1,t) + y_3(1,t).$$

The connective matrix is given by

$$C = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

which satisfies $\det(I - C) \neq 0$. The motion of twisting curves system with nodal damping is governed by the partial equations

$$\begin{cases} \mathbb{M}Y_{tt}(x,t) = \mathbb{T}Y_{xx}(x,t), \quad x \in (0,1) \\ Y(0,t) = CY(1,t), \quad t > 0, \\ \mathbb{T}Y_x(1,t) - C^T\mathbb{T}Y_x(0,t) = -\Gamma Y_t(1,t), \\ Y(x,0) = Y_0(x), \quad Y_t(x,0) = Y_1(x), x \in (0,1). \end{cases} \tag{5.3.15}$$

5.4 Nodal dynamic

In this section one will discuss the nodal dynamics. One only considers the interior node $a \in V_{int}$. Here one assumes that for each $a \in V_{int}$, there is at least one incoming edge and one outgoing edge. For each $j \in J^+(a)$, one regards the value $w_j(1,t)$ as the input of the node system, and for $i \in J^-(a)$, $w_i(0,t)$ as output of the node system.

$w(a,t)$ denotes the behavior of the node system, the output of the system is defined as

$$w_i(0,t) = \omega_i(a) f(w(a,t))$$

where f is the output function, which is possible linear function. The nodal dynamic is described as

$$\frac{dw(a,t)}{dt} = -\frac{1}{R(a)} w(a,t) + \sum_{j \in J^+(a)} \beta_j(a) w_j(1,t) - \sum_{i \in J^-(a)} \gamma_i(a) w_i(0,t) + z(a,t) \tag{5.4.1}$$

where $\beta_j(a)$ denotes the input rate and $\gamma_i(a)$ denotes the output feedback gain constants, $z(a,t)$ denotes the potential.

Chapter 6

Abstract Second Order Hyperbolic Systems

In this chapter we study an abstract second order hyperbolic system valued in $\mathbb{C}^n$ with appropriate boundary conditions. We prove that the system is well-posed and associates with a C_0 semigroup in a Hilbert state space. Further we show under certain conditions that the spectra of the system operator are located in the vertical strip, and there is a sequence of the eigenvectors and generalized eigenvectors that forms a Riesz basis with parenthesis for the Hilbert state space, and hence the system satisfies the spectrum determined growth assumption.

6.1 Introduction

Many mechanical systems, such as cable, spacecraft with flexible attachments or robots with flexible links, contain certain parts whose dynamic behavior can be rigorous described by partial differential equations

$$\mathbb{M}\frac{\partial^2 Y(x,t)}{\partial t^2} = \mathbb{T}\frac{\partial^2 Y(x,t)}{\partial x^2} + \mathbb{P}\frac{\partial Y(x,t)}{\partial x} + \mathbb{Q}Y(x,t) \quad x \in (0,1),\ t > 0, \tag{6.1.1}$$

with appropriate boundary conditions, where $Y(x,t)$ is a function valued in $\mathbb{C}^n$, $\mathbb{M}, \mathbb{T}, \mathbb{P}$ and $\mathbb{Q}$ are $n \times n$ matrices, $\mathbb{M}$ and $\mathbb{T}$ are positive definite matrices.

For such systems, we study not only their dynamic behavior but also control problem in actual problems. So many scientists have made effort in this aspect and have designed a lot of passive and active controllers to achieve control aim. In recent years, boundary control of the system governed by partial differential equations have become an important research area. For many concrete systems, a lot of boundary feedback controllers are used to stabilize the system, for example, [69], [16], [71], [22], [111] for one-dimensional wave system, [5][6] [34] and the references therein for string network, [61], [81], [110], [113] for Timoshenko system, [93] and [94] for one dimensional selling porous solid system. However, stability analysis of corresponding closed loop system is a difficult work. It becomes much more difficult when one uses the spectral analysis method, this is because one asserts stability of system from its spectral distribution only when the system satisfies the spectrum determined growth assumption, which means that decay rate of the system is determined via spectrum of the system operator. For a distributed parameter system, to prove that the system satisfies the spectrum determined growth assumption itself is a tough problem. Note that if the system is a Riesz one, that is, the multiplicities of eigenvalues are uniformly bounded, and there is a sequence of eigenvectors and generalized eigenvectors that forms a Riesz basis for Hilbert state space, then the spectrum determined growth assumption holds. So verification of Riesz basis property in many literatures becomes an important component, (see, [22], [110], [113]). Recent, we find a method to verify the Riesz basis property (see, [117],[119]). In the present chapter we shall use this method to discuss Riesz basis property of the system (6.1.1) attached appropriate boundary conditions. More precisely saying, we shall study the following system valued in $\mathbb{C}^n$

$$\begin{cases} \mathbb{M}Y_{tt}(x,t) = \mathbb{T}Y_{xx}(x,t), \quad x \in (0,1),\ t>0, \\ Y(0,t) = CY(1,t), \quad t>0, \\ \mathbb{T}Y_x(1,t) - C^T\mathbb{T}Y_x(0,t) = -\Gamma Y_t(1,t), \quad t>0, \\ Y(x,0) = Y_0(x), \quad Y_t(x,0) = Y_1(x), \quad x \in (0,1). \end{cases} \tag{6.1.2}$$

where $\mathbb{M}$ and $\mathbb{T}$ are positive definite $n\times n$ matrices, Γ is a nonnegative matrix, and C is a real $n \times n$ matrix satisfying $\det(I-C) \neq 0$, and C^T denotes the transpose of matrix C. We shall prove that the system is well-posed, under certain conditions, the eigenvectors and generalized eigenvectors of the system generate a Riesz basis for the Hilbert state space.

It is worth mentioned that system (6.1.2) is different from those systems in literature mentioned above; it has coupled equations and non-separable boundary conditions. Those properties cause some difficulty in mathematical treat. However, such a system has many important applications.

6.2 Well-posed-ness of abstract differential equations

In this section we shall formulate system (6.1.2) into a Hilbert space, and then discuss the well-posed-ness of the system.

Let abstract hyperbolic system valued in $\mathbb{C}^n$ be given by

$$\begin{cases} \mathbb{M}Y_{tt}(x,t) = \mathbb{T}Y_{xx}(x,t), \quad x \in (0,1),\ t>0, \\ Y(0,t) = CY(1,t), \quad t>0, \\ \mathbb{T}Y_x(1,t) - C^T\mathbb{T}Y_x(0,t) = -\Gamma Y_t(1,t), \quad t>0, \\ Y(x,0) = Y_0(x), \quad Y_t(x,0) = Y_1(x), \quad x \in (0,1) \end{cases} \tag{6.2.1}$$

where $\mathbb{M}$ and $\mathbb{T}$ are positive definite matrices, Γ is a nonnegative matrix, and C is a real matrix satisfying $\det(I-C) \neq 0$.

Set

$$V_E^k(0,1) = \left\{ f \in H^k\left([0,1], \mathbb{C}^n\right) \,\middle|\, f(0) = Cf(1) \right\}$$

where $H^k((0,1),\mathbb{C}^n)$ is the Sobolev space of order k.

Let

$$\mathcal{H} = V_E^1(0,1) \times L^2([0,1],\mathbb{C}^n)$$

equipped inner product

$$\langle (f_1,f_2),(g_1,g_2)\rangle_{\mathcal{H}} = \int_0^1 (\mathbb{T}f_1'(x), g_1'(x))dx + \int_0^1 (\mathbb{M}f_2(x), g_2(x))dx,$$

here and hereafter we always denote by $(\cdot,\cdot)$ the inner product in $\mathbb{C}^n$ and by $\langle \cdot,\cdot\rangle_{\mathcal{H}}$ the inner product in $\mathcal{H}$.

It is easy to see that

$$\|(f_1,f_2)\|_{\mathcal{H}} = \left(\int_0^1 (\mathbb{T}f_1'(x), f_1'(x))dx + \int_0^1 (\mathbb{M}f_2(x), f_2(x))dx\right)^{1/2}$$

is a norm on $\mathcal{H}$, and $\mathcal{H}$ is a Hilbert space.

Define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{A}) = \left\{(f,g) \in V_E^2(0,1) \times V_E^1(0,1) \;\middle|\; \mathbb{T}f'(1) - C^T\mathbb{T}f'(0) = -\Gamma g(1)\right\} \tag{6.2.2}$$

$$\mathcal{A}(f,g) = (g(x), \mathbb{M}^{-1}\mathbb{T}f''(x)), \quad \forall (f,g) \in \mathcal{D}(\mathcal{A}). \tag{6.2.3}$$

With help of above notation we can rewrite (6.2.1) into an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \frac{d}{dt}Z(t) = \mathcal{A}Z(t), \quad t > 0, \\ Z(t) = (Y(x,t), Y_t(x,t)), \\ Z(0) = (Y_0(x), Y_1(x)). \end{cases} \tag{6.2.4}$$

THEOREM 6.2.1 *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, then $\mathcal{A}$ is a dissipative operator, $\mathcal{A}^{-1}$ exists and is compact on $\mathcal{H}$, and hence $\mathcal{A}$ generates a C_0 semigroup of contraction on $\mathcal{H}$.*

Proof Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. For any $(f,g) \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned} \langle \mathcal{A}(f,g),(f,g)\rangle_{\mathcal{H}} &= \int_0^1 (\mathbb{T}g'(x), f'(x))dx + \int_0^1 (\mathbb{M}(\mathbb{M}^{-1}\mathbb{T})f''(x), g(x))dx \\ &= (\mathbb{T}g(x), f'(x))\Big|_0^1 - \int_0^1 (\mathbb{T}g(x), f''(x))dx + \int_0^1 (\mathbb{T}f''(x), g(x))dx, \end{aligned}$$

$$\begin{aligned} \langle (f,g),\mathcal{A}(f,g)\rangle_{\mathcal{H}} &= \int_0^1 (\mathbb{T}f'(x), g'(x))dx + \int_0^1 (\mathbb{M}g(x), (\mathbb{M}^{-1}\mathbb{T}f''(x))dx \\ &= (\mathbb{T}f'(x), g(x))\Big|_0^1 - \int_0^1 (\mathbb{T}f''(x), g(x))dx + \int_0^1 (\mathbb{T}g(x), f''(x))dx, \end{aligned}$$

and hence

$$\begin{aligned} \Re\langle \mathcal{A}(f,g),(f,g)\rangle_{\mathcal{H}} &= \Re(\mathbb{T}g(x), f'(x))\Big|_0^1 \\ &= \Re(\mathbb{T}f'(1), g(1)) - \Re(Tf'(0), g(0)) \\ &= \Re(\mathbb{T}f'(1), g(1)) - \Re(\mathbb{T}f'(0), Cg(1)) \end{aligned}$$

$$
\begin{aligned}
&= \Re(\mathbb{T}f'(1) - C^T\mathbb{T}f'(0), g(1)) \\
&= -(\Gamma g(1), g(1)) \leq 0
\end{aligned} \tag{6.2.5}
$$

where we have used the conditions that $g(0) = Cg(1)$, $\mathbb{T}f'(1) - C^T\mathbb{T}f'(0) = -\Gamma g(1)$ and Γ is a nonnegative matrix. So $\mathcal{A}$ is a dissipative operator.

Now we show that $0 \in \rho(\mathcal{A})$. For any given $(u, v) \in \mathcal{H}$, we consider solvability of equation

$$\mathcal{A}(f, g) = (u, v), \quad (f, g) \in \mathcal{D}(\mathcal{A}),$$

i.e.,

$$
\begin{cases}
g(x) = u(x), & x \in [0, 1], \\
\mathbb{M}^{-1}\mathbb{T}f''(x) = v(x), & x \in (0, 1).
\end{cases} \tag{6.2.6}
$$

For the second equation in (6.2.6), integrating from x to 1 leads to

$$\mathbb{T}f'(1) - \mathbb{T}f'(x) = \int_x^1 \mathbb{M}v(s)ds \quad x \in (0, 1), \tag{6.2.7}$$

and

$$(1-x)\mathbb{T}f'(1) - \mathbb{T}f(1) + \mathbb{T}f(x) = \int_x^1 dr \int_r^1 \mathbb{M}v(s)ds, \quad x \in (0, 1). \tag{6.2.8}$$

From (6.2.7) and (6.2.8) we get

$$\mathbb{T}f'(1) - \mathbb{T}f'(0) = \int_0^1 \mathbb{M}v(s)ds, \tag{6.2.9}$$

$$\mathbb{T}f'(1) - \mathbb{T}f(1) + \mathbb{T}f(0) = \int_0^1 dr \int_r^1 \mathbb{M}v(s)ds. \tag{6.2.10}$$

Acting C^T on both sides of (6.2.9), combining condition $\mathbb{T}f'(1) - C^T\mathbb{T}f'(0) = -\Gamma g(1) = -\Gamma u(1)$, yields

$$(I - C^T)\mathbb{T}f'(1) = -\Gamma u(1) - \int_0^1 C^T\mathbb{M}v(s)ds. \tag{6.2.11}$$

Since $\det(I - C^T) \neq 0$, we have

$$\mathbb{T}f'(1) = -[I - C^T]^{-1}\left[\Gamma u(1) + \int_0^1 C^T\mathbb{M}v(s)ds\right]. \tag{6.2.12}$$

Substituting $f(0) = Cf(1)$ into (6.2.10) yields

$$\mathbb{T}f'(1) - \mathbb{T}f(1) + \mathbb{T}Cf(1) = \int_0^1 dr \int_r^1 \mathbb{M}v(s)ds. \tag{6.2.13}$$

Thus we get from (6.2.12) and (6.2.13) that

$$
\begin{aligned}
f(1) &= -[I - C]^{-1}\mathbb{T}^{-1}[I - C^T]^{-1}\left[\Gamma u(1) + \textstyle\int_0^1 C^T\mathbb{M}v(s)ds\right] \\
&\quad -[I - C]^{-1}\mathbb{T}^{-1}\textstyle\int_0^1 dr \int_r^1 \mathbb{M}v(s)ds.
\end{aligned} \tag{6.2.14}
$$

Therefore,

$$f(x) = f(1) - (1-x)f'(1) + \mathbb{T}^{-1}\int_x^1 dr \int_r^1 \mathbb{M}v(s)ds$$

$$\begin{aligned} &= -\left[xI+(I-C)^{-1}C\right]\mathbb{T}^{-1}(I-C^T)^{-1}\left[\Gamma u(1)+\int_0^1 C^T\mathbb{M}v(s)ds\right] \\ &\quad -(I-C)^{-1}C\mathbb{T}^{-1}\int_0^1 dr\int_r^1 \mathbb{M}v(s)ds-\mathbb{T}^{-1}\int_0^x dr\int_r^1 \mathbb{M}v(s)ds. \end{aligned} \tag{6.2.15}$$

Let f be given by (6.2.15) and $g(x)=u(x)$, then $(f,g)\in\mathcal{D}(\mathcal{A})$, and $\mathcal{A}(f,g)=(u,v)$. So the inverse operator theorem reads that $0\in\rho(\mathcal{A})$. Note that $u\in V_E^1(0,1)$ and f has an integral representation. The Sobolev's Embedding Theorem asserts that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. Therefore, the Lumer-Philips theorem (cf. [92]) reads that $\mathcal{A}$ generates a C_0 semigroup of contraction on $\mathcal{H}$. □

COROLLARY 6.2.1 *Let $\mathcal{A}$ be defined by (6.2.2) and (6.2.3) and $S(t)$ be the semigroup generated by $\mathcal{A}$. Then the following statements are true.*

1) $\sigma(\mathcal{A})$ consists of all isolated eigenvalues of finite multiplicity;

2)If $\Gamma>0$ and $-1\notin\sigma(C)$, then $\sigma(\mathcal{A})\subset\{\lambda\in\mathbb{C}\mid \Re\lambda<0\}$, and hence $S(t)$ is asymptotically stable.

Proof The first assertion follows from $\mathcal{A}^{-1}$ being a compact operator on $\mathcal{H}$. Here we mainly prove the second assertion.

Since we have assumed that $\det(I-C)\neq 0$, which implies that $1\notin\sigma(C)$, we prove that, when $-1\notin\sigma(C)$, it holds that $\Re\lambda<0$ for any $\lambda\in\sigma(\mathcal{A})$.

By the contradictory method, if it is not true then there is at least one $\lambda\in\sigma(\mathcal{A})$ with $\Re\lambda=0$. Clearly, $\lambda\neq 0$. Let $(f,g)\in\mathcal{D}(\mathcal{A})$ be corresponding an eigenvector. Then we have $g(x)=\lambda f(x)$ and

$$0=\Re\lambda||(f,g)||_{\mathcal{H}}^2=\Re\lambda\langle(f,g),(f,g)\rangle_{\mathcal{H}}=\Re\langle\mathcal{A}(f,g),(f,g)\rangle_{\mathcal{H}}=-(\Gamma g(1),g(1))\leq 0.$$

Since Γ is a positive definite matrix, it must be $g(1)=0$, and hence $f(1)=0$. So the vector-valued function $f(x)$ satisfies the following differential equation

$$\begin{cases} \lambda^2\mathbb{M}f(x)=\mathbb{T}f''(x), & x\in(0,1) \\ f(0)=0=f(1), & \mathbb{T}f'(1)-C^T\mathbb{T}f'(0)=0. \end{cases} \tag{6.2.16}$$

Since $\mathbb{M}$ and $\mathbb{T}$ are positive definite matrices, so is matrix $\mathbb{T}^{-1/2}\mathbb{M}\mathbb{T}^{-1/2}$. Set $B^2=\mathbb{T}^{-1/2}\mathbb{M}\mathbb{T}^{-1/2}$, where B also is a positive definite matrix. Then the general solution of (6.2.16) has the form

$$f(x)=\mathbb{T}^{-1/2}\sinh(x\lambda B)v, \quad v\in\mathbb{C}^n.$$

Substituting above into the boundary conditions in (6.2.16) lead to

$$\begin{cases} \mathbb{T}^{1/2}\sinh(\lambda B)v=0, \\ \lambda\left(\mathbb{T}^{1/2}B\cosh(\lambda B)-C^T\mathbb{T}^{1/2}B\right)v=0 \end{cases}$$

From $(\sinh\lambda B)v=0$ we get that $\sinh\lambda BBv=0$ or equivalently $e^{\lambda B}Bv=e^{-\lambda B}Bv$, this leads to

$$[e^{\lambda B}-I][e^{\lambda B}+I]Bv=0.$$

So $Bv\in\mathcal{N}(e^{\lambda B}+I)\cup\mathcal{N}(e^{\lambda B}-I)$.

On the other hands, since $\lambda \neq 0$, multiply matrix $\mathbb{T}^{-\frac{1}{2}}$ both sides of the second equation and adding the both equations yield

$$[e^{\lambda B} - \mathbb{T}^{-1/2}C^T\mathbb{T}^{1/2}]Bv = 0.$$

We rewrite above into the following forms

$$[e^{\lambda B} - I]Bv = [T^{-1/2}C^T\mathbb{T}^{1/2} - I]Bv$$

and

$$[e^{\lambda B} + I]Bv = [T^{-1/2}C^T\mathbb{T}^{1/2} + I]Bv.$$

Notice that $\det(\mathbb{T}^{-1/2}C^T\mathbb{T}^{1/2}-I) = \det(C^T-I) \neq 0$. If $-1 \notin \sigma(C)$, then $\det(T^{-1/2}C^T\mathbb{T}^{1/2}+I) = \det(C^T+I) \neq 0$. If $Bv \in \mathcal{N}(e^{\lambda B} - I)$, then the first equality leads to $Bv = 0$; if $Bv \in \mathcal{N}(e^{\lambda B} + I)$, then the second equation leads to $Bv = 0$. So $v = 0$, i.e., $f(x) = 0$, and hence $(f, g) = (0, 0)$. This contradicts that (f, g) is an eigenvector of $\mathcal{A}$. Therefore, it holds that $\Re\lambda < 0$ for any $\lambda \in \sigma(\mathcal{A})$. The second assertion is proved. The asymptotical stability of $S(t)$ follows from Lyubich and Phóng's theorem [70]. $\square$

REMARK 6.2.1 *In Corollary 6.2.1, $\Gamma > 0$ is merely a sufficient condition for asymptotic stability. If for $\Gamma \geq 0$ one can deduce $f(1) = 0$ from $\Gamma f(1) = 0$, then the result is still true. However, $-1 \notin \sigma(C)$ is a necessary condition for stability of the system. The condition $\det(I - C) \neq 0$ does not ensure that there is no eigenvalue on the imaginary axis. Here we correct an error in [125].*

EXAMPLE 6.2.1 *In this example, we show that $\det(I - C) \neq 0$ is insufficient for there being no eigenvalue on the imaginary axis. Let us consider a network of strings.*

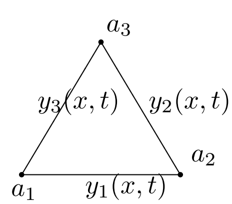

Fig 6.2.1. A triangle circuit network

The string equations are

$$y_{j,tt}(x,t) = y_{j,xx}(x,t), \quad x \in (0,1), j = 1,2,3$$

and the connective conditions

$$\begin{aligned}
&y_1(0,t) = -y_3(1,t), \quad y_2(0,t) = y_1(1,t), \quad y_3(0,t) = y_2(1,t)\\
&y_{1,x}(1,t) - y_{2,x}(0,t) = -\alpha_1 y_{1,t}(1,t)\\
&y_{2,x}(1,t) - y_{3,x}(0,t) = -\alpha_2 y_{2,t}(1,t)\\
&y_{3,x}(1,t) + y_{1,x}(0,t) = -\alpha_3 y_{3,t}(1,t).
\end{aligned}$$

Here

$$C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

satisfies $\det(I - C) \neq 0$. *Direct calculation shows that* $\lambda = i(2k+1)\pi, k \in \mathbb{N}$ *are eigenvalues of the system.*

6.3 Spectral analysis of $\mathcal{A}$

In order to study property of the semigroup $S(t)$ generated by $\mathcal{A}$, we need to learn some spectral properties of $\mathcal{A}$. In this section, we shall study distribution of $\sigma(\mathcal{A})$, the completeness and Riesz basis property of the eigenvector and generalized eigenvector of $\mathcal{A}$.

We begin with the eigenvalue problem. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}$ and (f, g) be corresponding an eigenvector. Then we have

$$\begin{cases} g(x) = \lambda f(x), \quad x \in [0,1], \\ \lambda^2 \mathbb{M} f(x) = \mathbb{T} f''(x), \quad x \in (0,1), \\ f(0) = C f(1), \\ \mathbb{T} f'(1) - C^T \mathbb{T} f'(0) = -\lambda \Gamma f(1). \end{cases} \tag{6.3.1}$$

Set

$$\widehat{f}(x) = \mathbb{T}^{1/2} f(x), \quad \widehat{g}(x) = \mathbb{T}^{1/2} g(x), \quad B^2 = \mathbb{T}^{-1/2} \mathbb{M} \mathbb{T}^{-1/2},$$

where B is a positive definite matrix. Then Eqs.(6.3.1) is equivalent to the following equation

$$\begin{cases} \widehat{g}(x) = \lambda \widehat{f}(x), \quad x \in [0,1], \\ \lambda^2 B^2 \widehat{f}(x) = \widehat{f}''(x), \quad x \in (0,1), \\ \widehat{f}(0) = \mathbb{T}^{1/2} C \mathbb{T}^{-1/2} \widehat{f}(1), \\ \widehat{f}'(1) - \mathbb{T}^{-1/2} C^T \mathbb{T}^{1/2} \widehat{f}'(0) = -\lambda \mathbb{T}^{-1/2} \Gamma \mathbb{T}^{-1/2} \widehat{f}(1). \end{cases} \tag{6.3.2}$$

Clearly, the general solution of the differential equation in (6.3.2) is of the form

$$\widehat{f}(x) = e^{x\lambda B} u + e^{-x\lambda B} v, \quad u, v \in \mathbb{C}^n. \tag{6.3.3}$$

Substituting this into the boundary conditions in (6.3.2) leads to

$$\begin{cases} (B + \mathbb{T}^{-1/2} \Gamma \mathbb{T}^{-1/2}) e^{\lambda B} u + (\mathbb{T}^{-1/2} \Gamma \mathbb{T}^{-1/2} - B) e^{-\lambda B} v \\ = \mathbb{T}^{-1/2} C^T \mathbb{T}^{1/2} B (u - v), \\ (u + v) = \mathbb{T}^{1/2} C \mathbb{T}^{-1/2} (e^{\lambda B} u + e^{-\lambda B} v). \end{cases} \tag{6.3.4}$$

Above algebraic equations has a pair (u, v) of nonzero solution, this implies that the determinant of the coefficient matrix vanishes, i.e.,

$$D(\lambda) = \det \begin{bmatrix} I - \widehat{C} e^{\lambda B} & I - \widehat{C} e^{-\lambda B} \\ (B + \widehat{\Gamma}) e^{\lambda B} - \widehat{C}^T B & (\widehat{\Gamma} - B) e^{-\lambda B} + \widehat{C}^T B \end{bmatrix} = 0, \tag{6.3.5}$$

where $\widehat{C} = \mathbb{T}^{1/2} C \mathbb{T}^{-1/2}, \widehat{\Gamma} = \mathbb{T}^{-1/2} \Gamma \mathbb{T}^{-1/2}$.

Conversely, if $\lambda \in \mathbb{C}$ such that $D(\lambda) = 0$, then (6.3.4) has a non-zero solution (u, v). We can define function $\widehat{f}$ as same as (6.3.3). Obviously, $\widehat{f}$ satisfies equation

$$\widehat{f}''(x) = \lambda^2 B^2 \widehat{f}(x),$$

Eqs.(6.3.4) implies that $\widehat{f}$ satisfies the boundary conditions in (6.3.2). Consequently, functions

$$f(x) = \mathbb{T}^{-1/2} \widehat{f}(x), \quad g(x) = \lambda \mathbb{T}^{-1/2} \widehat{f}(x)$$

satisfy the equation (6.3.1). Therefore, λ is an eigenvalue of $\mathcal{A}$.

Since

$$\begin{aligned} D(\lambda) &= \det \begin{bmatrix} I - \widehat{C} e^{\lambda B} & I - \widehat{C} e^{-\lambda B} \\ (B + \widehat{\Gamma}) e^{\lambda B} - \widehat{C}^T B & (\widehat{\Gamma} - B) e^{-\lambda B} + \widehat{C}^T B \end{bmatrix} \\ &= \det \begin{bmatrix} e^{-\lambda B} - \widehat{C} & I - \widehat{C} e^{-\lambda B} \\ (B + \widehat{\Gamma}) - \widehat{C}^T B e^{-\lambda B} & (\widehat{\Gamma} - B) e^{-\lambda B} + \widehat{C}^T B \end{bmatrix} \det \begin{bmatrix} e^{\lambda B} & 0 \\ 0 & I \end{bmatrix} \\ &= \det \begin{bmatrix} I - \widehat{C} e^{\lambda B} & e^{\lambda B} - \widehat{C} \\ (B + \widehat{\Gamma}) e^{\lambda B} - \widehat{C}^T B & (\widehat{\Gamma} - B) + \widehat{C}^T B e^{\lambda B} \end{bmatrix} \det \begin{bmatrix} I & 0 \\ 0 & e^{-\lambda B} \end{bmatrix}, \end{aligned}$$

when $\Re\lambda \to \pm\infty$, we have

$$\lim_{\Re\lambda \to +\infty} \frac{D(\lambda)}{\det(e^{\lambda B})} = \det \begin{bmatrix} -\widehat{C} & I \\ (B + \widehat{\Gamma}) & \widehat{C}^T B \end{bmatrix} = (-1)^n \det[\widehat{\Gamma} + B + \widehat{C}^T B \widehat{C}], \tag{6.3.6}$$

and

$$\lim_{\Re\lambda \to -\infty} \frac{D(\lambda)}{\det(e^{-\lambda B})} = \det \begin{bmatrix} I & -\widehat{C} \\ -\widehat{C}^T B & (\widehat{\Gamma} - B) \end{bmatrix} = \det[\widehat{\Gamma} - B - \widehat{C}^T B \widehat{C}]. \tag{6.3.7}$$

Therefore, we have the following result.

THEOREM 6.3.1 *Let $\mathcal{A}$ be defined as (6.2.2) and (6.2.3) and let*

$$B^2 = \mathbb{T}^{-1/2} \mathbb{M} \mathbb{T}^{-1/2}, \quad \widehat{C} = \mathbb{T}^{1/2} C \mathbb{T}^{-1/2}, \quad \widehat{\Gamma} = \mathbb{T}^{-1/2} \Gamma \mathbb{T}^{-1/2} \tag{6.3.8}$$

and $D(\lambda)$ be defined by (6.3.5). Then

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid D(\lambda) = 0\}. \tag{6.3.9}$$

When $\det[\widehat{\Gamma} - B - \widehat{C}^T B \widehat{C}] \neq 0$, there is a positive constant $h > 0$ such that

$$\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} \mid -h \leq \Re\lambda < 0\}, \tag{6.3.10}$$

in this case, $\sigma(\mathcal{A})$ is a union of finite many separable sets.

Proof Let $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. For any given $(u, v) \in \mathcal{H}$, we consider the resolvent equation $(\lambda I - \mathcal{A})(f, g) = (u, v)$, i.e.,

$$\begin{cases} \lambda f - g = u, \\ \lambda g - \mathbb{M}^{-1}\mathbb{T}f'' = v. \\ f(0) = Cf(1), \\ \mathbb{T}f'(1) - C^T\mathbb{T}f'(0) = -\Gamma g(1). \end{cases} \tag{6.3.11}$$

So, $g = \lambda f - u$ and f satisfies the following differential equation

$$\begin{cases} \lambda^2 f(x) - \mathbb{M}^{-1}\mathbb{T}f''(x) = \lambda u(x) + v(x), \quad x \in (0,1), \\ f(0) = Cf(1), \\ \mathbb{T}f'(1) - C^T\mathbb{T}f'(0) + \lambda\Gamma f(1) = \Gamma u(1). \end{cases} \tag{6.3.12}$$

Set

$$\widehat{f}(x) = \mathbb{T}^{1/2}f(x), \quad \widehat{g}(x) = \mathbb{T}^{1/2}g(x), \quad \widehat{u}(x) = \mathbb{T}^{1/2}u(x), \quad \widehat{v}(x) = \mathbb{T}^{1/2}v(x).$$

Then Eqs.(6.3.12) is changed into

$$\begin{cases} \lambda^2 B^2\widehat{f}(x) - \widehat{f}''(x) = \lambda B^2\widehat{u}(x) + B^2\widehat{v}(x), \quad x \in (0,1), \\ \widehat{f}(0) = \widehat{C}\widehat{f}(1), \\ \widehat{f}'(1) - \widehat{C}^T\widehat{f}'(0) + \lambda\widehat{\Gamma}\widehat{f}(1) = \widehat{\Gamma}\widehat{u}(1), \end{cases} \tag{6.3.13}$$

where B, $\widehat{C}$ and $\widehat{\Gamma}$ are defined as (6.3.8).

Clearly, the differential equation in (6.3.13) has the general solution

$$\widehat{f}(x) = e^{x\lambda B}y + e^{-x\lambda B}z - \int_0^x \sinh(\lambda(x-s)B)[B\widehat{u}(s) + \lambda^{-1}B\widehat{v}(s)]ds \tag{6.3.14}$$

where $y, z \in \mathbb{C}^n$. Substituting (6.3.14) into the boundary conditions in (6.3.13) yields

$$\begin{cases} (I - \widehat{C}e^{\lambda B})y + (I - \widehat{C}e^{-\lambda B})z \\ = -\widehat{C}\displaystyle\int_0^1 \sinh(\lambda(1-s)B)[B\widehat{u}(s) + \lambda^{-1}B\widehat{v}(s)]ds, \\ \left[(\widehat{\Gamma} + B)e^{\lambda B} - \widehat{C}^T B\right]y + \left[(\widehat{\Gamma} - B)e^{-\lambda B} + \widehat{C}^T B\right]z \\ = \displaystyle\int_0^1 \left[B\cosh(\lambda(1-s)B) + \widehat{\Gamma}\sinh(\lambda(1-s)B)\right][B\widehat{u}(s) + \lambda^{-1}B\widehat{v}(s)]ds \\ +\lambda^{-1}\Gamma u(1). \end{cases} \tag{6.3.15}$$

Since the coefficient matrix of above algebraic equations is

$$\widetilde{G}(\lambda) = \begin{pmatrix} I - \widehat{C}e^{\lambda B} & I - \widehat{C}e^{-\lambda B} \\ (\widehat{\Gamma} + B)e^{\lambda B} - \widehat{C}^T B & (\widehat{\Gamma} - B)e^{-\lambda B} + \widehat{C}^T B \end{pmatrix}, \tag{6.3.16}$$

when $D(\lambda) = \det\widetilde{G}(\lambda) \neq 0$, we have

$$\widetilde{G}^{-1}(\lambda) = \frac{adj\widetilde{G}(\lambda)}{D(\lambda)} = \frac{1}{D(\lambda)}\begin{pmatrix} \widehat{G}_{11}(\lambda) & \widehat{G}_{12}(\lambda) \\ \widehat{G}_{21}(\lambda) & \widehat{G}_{22}(\lambda) \end{pmatrix} \tag{6.3.17}$$

where $adj(S)$ denotes the adjoint matric of S. We define functionals on $\mathcal{H}$ by

$$\begin{aligned}&F_1(u,v,\lambda) = -\widehat{G}_{11}(\lambda)\widehat{C}\int_0^1 \sinh(1-s)[B\widehat{u}(s)+\lambda^{-1}B\widehat{v}(s)]ds\\&+\widehat{G}_{12}(\lambda)\int_0^1 \Big[\widehat{\Gamma}\sinh(\lambda(1-s)B)+B\cosh(\lambda(1-s)B)\Big][B\widehat{u}(s)+\lambda^{-1}B\widehat{v}(s)]ds\\&\quad+\widehat{G}_{12}(\lambda)\lambda^{-1}\widetilde{\Gamma}\widehat{u}(1),\end{aligned} \tag{6.3.18}$$

$$\begin{aligned}&F_2(u,v,\lambda) = -\widehat{G}_{21}(\lambda)\widehat{C}\int_0^1 \sinh(\lambda((1-s)B)[B\widehat{u}(s)+\lambda^{-1}B\widehat{v}(s)]ds\\&+\widehat{G}_{22}(\lambda)\int_0^1 \Big[B\cosh(\lambda(1-s)B+\widehat{\Gamma}\sinh(\lambda(1-s)B)\Big][B\widehat{u}(s)+\lambda^{-1}B\widehat{v}(s)]ds\\&\quad+G_{22}(\lambda)\lambda^{-1}\widehat{\Gamma}\widehat{u}(1).\end{aligned} \tag{6.3.19}$$

Obviously, F_1 and F_2 defined by (6.3.18) and (6.3.19) respectively are bounded linear functionals on $\mathcal{H}$, and the solution to (6.3.15) is given by

$$(y,z) = D^{-1}(\lambda)(F_1(u,v,\lambda),F_2(u,v,\lambda)).$$

Therefore, when $\lambda\in\mathbb{C}$ with $D(\lambda)\neq 0$, we have

$$\begin{aligned}\widehat{f}(x) &= D^{-1}(\lambda)\left[e^{x\lambda B}F_1(u,v,\lambda)+e^{-x\lambda B}F_2(u,v,\lambda)\right]\\&\quad-\int_0^x \sinh((x-s)\lambda B)[B\widehat{u}(s)+\lambda^{-1}B\widehat{v}(s)]ds.\end{aligned} \tag{6.3.20}$$

Set

$$f(x) = \mathbb{T}^{-1/2}\widehat{f}(x),\quad g(x) = \lambda\mathbb{T}^{-1/2}\widehat{f}(x)-u(x). \tag{6.3.21}$$

A straightforward calculation shows $(f,g)\in\mathcal{D}(\mathcal{A})$ and

$$(\lambda I-\mathcal{A})(f,g) = (u,v).$$

So we have $\lambda\in\rho(\mathcal{A})$. The first assertion follows.

Now let $D(\lambda)$ be defined by (6.3.5), then $D(\lambda)$ is an entire function of finite exponential type on complex plane $\mathbb{C}$. From (6.3.6) and (6.3.7) we can see that, when $\det[\widehat{\Gamma}-B-\widehat{C}^TB\widehat{C}]\neq 0$, there are positive constants c_1, c_2 and h such that, as $|\Re\lambda|\geq h$,

$$c_1\det(e^{|\lambda|B})\leq|D(\lambda)|\leq c_2\det(e^{|\lambda|B}). \tag{6.3.22}$$

This means that $D(\lambda)$ is a sine type function on $\mathbb{C}$ (see, [9, Definition II,1.27, pp-61]). The Levin's theorem (see, [9, Propostion II.1.28]) asserts that set of zeros of $D(\lambda)$ is a union of finite separable sets. So is $\sigma(\mathcal{A})$. The proof is then complete. $\square$

In what follows we shall discuss the completeness of eigenvectors and generalized eigenvectors of $\mathcal{A}$. For this purpose, we begin with the following proposition.

PROPOSITION 6.3.1 *Let $\mathcal{H}$ be defined as before. Define an operator $\mathcal{A}_0$ in $\mathcal{H}$ by*

$$\mathcal{D}(\mathcal{A}_0) = \left\{(f,g)\in V_E^2(0,1)\times V_E^1(0,1)\;\middle|\;\mathbb{T}f'(1)-C^T\mathbb{T}f'(0)=0\right\},$$

$$\mathcal{A}_0\,(f,g)) = \left(g(x),\mathbb{M}^{-1}\mathbb{T}f''(x)\right).$$

Then $\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$, and for any $(u,v) \in \mathcal{H}$, $\lambda \in \mathbb{R}$, the solution (f_λ, g_λ) of the resolvent equation

$$\lambda(f,g) - \mathcal{A}_0(f,g) = (u,v)$$

satisfies

$$||g_\lambda(1)|| \leq M\|(u,v)\|_{\mathcal{H}},$$

where $M > 0$ is a constant.

Proof For any $(f_i, g_i) \in \mathcal{D}(\mathcal{A}_0), i = 1,2$, it holds that

$$\begin{aligned}
&\langle \mathcal{A}_0(f_1,g_1),(f_2,g_2)\rangle_{\mathcal{H}} = \int_0^1 (\mathbb{T}g_1'(x), f_2'(x))dx + \int_0^1 (\mathbb{M}\mathbb{M}^{-1}\mathbb{T}f_1''(x), g_2(x))dx \\
= \;& (\mathbb{T}g_1(x), f_2'(x))\Big|_0^1 - \int_0^1 (\mathbb{T}g_1(x), f_2''(x))dx \\
& + (\mathbb{T}f_1'(x), g_2(x))\Big|_0^1 - \int_0^1 ((\mathbb{T}f_1'(x), g_2'(x))dx \\
= \;& (\mathbb{T}g_1(x), f_2'(x))\Big|_0^1 + (\mathbb{T}f_1'(x), g_2(x))\Big|_0^1 - \langle (f_1,g_1), \mathcal{A}_0(f_2,g_2)\rangle_{\mathcal{H}} \\
= \;& (g_1(1), \mathbb{T}f_2'(1)) - (g_1(0), \mathbb{T}f_2'(0)) + (\mathbb{T}f_1'(1), g_2(1)) - (\mathbb{T}f_1'(0), g_2(0)) \\
& - \langle (f_1,g_1), \mathcal{A}_0(f_2,g_2)\rangle_{\mathcal{H}} \\
= \;& (g_1(1), \mathbb{T}f_2'(1) - C^T\mathbb{T}f'(0)) + (\mathbb{T}f_1'(1) - C^T\mathbb{T}f_1'(0), g_2(1)) - \langle (f_1,g_1), \mathcal{A}_0(f_2,g_2)\rangle_{\mathcal{H}} \\
= \;& -\langle (f_1,g_1), \mathcal{A}_0(f_2,g_2)\rangle_{\mathcal{H}}.
\end{aligned}$$

So, $\mathcal{A}_0^* = -\mathcal{A}_0$.

Now let $(u,v) \in \mathcal{H}$ be given and $\lambda \in \mathbb{R}$. Let (f_λ, g_λ) satisfy the resolvent equation

$$(\lambda I - \mathcal{A}_0)(f,g) = (u,v), \quad (f,g) \in \mathcal{D}(\mathcal{A}_0),$$

i.e.,

$$\lambda f_\lambda(x) - g_\lambda(x) = u(x), \quad \lambda g_\lambda(x) - \mathbb{M}^{-1}\mathbb{T}f_\lambda''(x) = v(x),$$

and

$$f_\lambda(0) = Cf_\lambda(1), \quad \mathbb{T}f_\lambda'(1) - C^T\mathbb{T}f_\lambda'(0) = 0.$$

Since

$$f_\lambda(1) = \int_0^1 f_\lambda'(x)dx + f_\lambda(0) = \int_0^1 f_\lambda'(x)dx + Cf_\lambda(1),$$

so we have

$$f_\lambda(1) = (I-C)^{-1}\int_0^1 f_\lambda'(x)dx.$$

Similarly, we have

$$u(1) = (I-C)^{-1}\int_0^1 u'(x)dx.$$

So,

$$g_\lambda(1) = \lambda f_\lambda(1) - u(1) = (I-C)^{-1}\mathbb{T}^{-1/2}\left[\lambda\int_0^1 \mathbb{T}^{1/2}f'(x)dx - \int_0^1 \mathbb{T}^{1/2}u'(x)dx\right].$$

Consequently,

$$\begin{aligned}
\|g_\lambda(1)\| &\le \left\|(I-C)^{-1}\mathbb{T}^{-1/2}\right\| \left[|\lambda|\int_0^1 (\mathbb{T}f'(x), f'(x))dx + \int_0^1 (\mathbb{T}u'(x), u'(x))dx\right] \\
&\le \left\|(I-C)^{-1}\mathbb{T}^{-1/2}\right\| \left[|\lambda|\|R(\lambda,\mathcal{A}_0)(u,v)\|_{\mathcal{H}} + \|(u,v)\|_{\mathcal{H}}\right].
\end{aligned}$$

Since $\mathcal{A}_0$ is a skew adjoint operator, $||\lambda R(\lambda,\mathcal{A}_0)|| \le 1, \lambda\in\mathbb{R}$, we have

$$\|g_\lambda(1)\| \quad\le\quad 2\|(I-C)^{-1}\mathbb{T}^{-1/2}\|\|(u,v)\|_{\mathcal{H}}, \quad \forall\lambda\in\mathbb{R}.$$

The desired result follows. □

THEOREM 6.3.2 *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. If* $\det(\widehat{\Gamma} - B - \widehat{C}^T B\widehat{C}) \neq 0$, *then the system of eigenvectors and generalized eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$.*

Proof Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, and $\mathcal{A}_0$ be defined as in proposition 6.3.1. Denote by

$$Sp(\mathcal{A}) = \overline{\operatorname{span}\left\{\sum y_k, y_k\in E(\lambda_k,\mathcal{A})\mathcal{H}, \forall\lambda_k\in\sigma(\mathcal{A})\right\}},$$

where $E(\lambda_k,\mathcal{A})$ is the Riesz projector corresponding to λ_k. We shall prove $Sp(\mathcal{A}) = \mathcal{H}$.

Let $(u_0,v_0)\in\mathcal{H}$ and $(u_0,v_0)\perp Sp(\mathcal{A})$. Then $R^*(\lambda,\mathcal{A})(u_0,v_0)$ is an entire function on $\mathbb{C}$ valued in $\mathcal{H}$. For any $(u,v)\in\mathcal{H}$, we define a function on complex plane $\mathbb{C}$ by

$$F(\lambda) = \langle (u,v), R^*(\lambda,\mathcal{A})(u_0,v_0)\rangle_{\mathcal{H}}.$$

Clearly, $F(\lambda)$ is an entire function of finite exponential type, and

$$|F(\lambda)| \le (\Re\lambda)^{-1}\|(u,v)||_{\mathcal{H}}||(u_0,v_0)\|_{\mathcal{H}}, \quad \Re\lambda > 0,$$

and hence $\lim\limits_{\Re\lambda\to+\infty}|F(\lambda)| = 0$.

Now we consider the solution of equations

$$(\lambda I - \mathcal{A})(f_{1\lambda}, g_{1\lambda}) = (u,v), \quad (\lambda I - \mathcal{A}_0)(f_{2\lambda}, g_{2\lambda}) = (u,v), \quad \lambda\in\rho(\mathcal{A})\cap\rho(\mathcal{A}_0)\cap\mathbb{R}_-.$$

Set

$$\varphi(x) = f_{1\lambda}(x) - f_{2\lambda}(x), \quad \psi(x) = g_{1\lambda}(x) - g_{2\lambda}(x).$$

Then we have

$$R(\lambda,\mathcal{A})(u,v) = (f_{1\lambda}, g_{1\lambda}) = (f_{2\lambda}, g_{2\lambda}) + (\varphi,\psi) = R(\lambda,\mathcal{A}_0)(u,v) + (\varphi,\psi),$$

$\psi(x) = \lambda\varphi(x)$, and φ satisfies differential equation

$$\begin{cases}
\lambda^2\mathbb{M}\varphi(x) = \mathbb{T}\varphi''(x), \quad x\in(0,1), \\
\varphi(0) = C\varphi(1), \\
\mathbb{T}\varphi'(1) - C^T\mathbb{T}\varphi'(0) + \lambda\Gamma\varphi(1) = \Gamma g_{2\lambda}(1).
\end{cases}$$

Setting $\widehat{\varphi}(x) = \mathbb{T}^{1/2}\varphi(x)$, using previous notation, we have

$$\widehat{\varphi}(x) = e^{x\lambda B}y + e^{-x\lambda B}z, \quad y,z\in\mathbb{C}^n,$$

where y and z solve algebraic equations

$$\begin{cases} (I-\widehat{C}e^{\lambda B})y+(I-\widehat{C}e^{-\lambda B})z=0, \\ \left[(\widehat{\Gamma}+B)e^{\lambda B}-\widehat{C}^T B\right]y+\left[(\widehat{\Gamma}-B)e^{-\lambda B}+\widehat{C}^T B\right]z=\lambda^{-1}\mathbb{T}^{1/2}\Gamma g_{2\lambda}(1). \end{cases}$$

Thus

$$\begin{cases} y=(I-\widehat{C}e^{\lambda B})^{-1}(\widehat{C}-e^{\lambda B})e^{-\lambda B}z=(\widehat{C}-O_1(\lambda))e^{-\lambda B}z, \\ \left[\widehat{\Gamma}-B-\widehat{C}^T B\widehat{C}-O_2(\lambda)\right]e^{-\lambda B}z=\lambda^{-1}\mathbb{T}^{1/2}\Gamma g_{2\lambda}(1), \end{cases}$$

where $\|O_j(\lambda)\|=o(\lambda^{-1}), j=1,2$, as $\lambda\to-\infty$. This means that

$$e^{-\lambda B}z=\lambda^{-1}(\widehat{\Gamma}-B-\widehat{C}^T B\widehat{C}+O_2(\lambda))^{-1}\mathbb{T}^{1/2}\Gamma g_{2\lambda}(1).$$

Therefore, when $|\lambda|$ is sufficiently large, we have

$$\begin{aligned} \widehat{\varphi}(1) &= e^{\lambda B}y+e^{-\lambda B}z=e^{\lambda B}(\widehat{C}-O_1(\lambda))e^{-\lambda B}z+e^{-\lambda B}z \\ &= (I+O_3(\lambda))e^{-\lambda B}z \\ &= \lambda^{-1}(I+O_3(\lambda))(\widehat{\Gamma}-B-\widehat{C}^T B\widehat{C}+O_2(\lambda))^{-1}\mathbb{T}^{1/2}\Gamma g_{2\lambda}(1). \end{aligned}$$

So there is a constant $M_1>0$ such that

$$\|\widehat{\varphi}(1)\|\le M_1|\lambda^{-1}|\|g_{2\lambda}(1)\|.$$

Thus,

$$\begin{aligned} \|(\varphi,\psi)\|_{\mathcal{H}}^2 &= \int_0^1(\mathbb{T}\varphi'(x),\varphi'(x))dx+\int_0^1(\mathbb{M}\psi(x),\psi(x))dx \\ &= (\mathbb{T}\varphi'(1),\varphi(1))-(\mathbb{T}\varphi'(0),\varphi(0)) \\ &= -\lambda(\Gamma\varphi(1),\varphi(1))+(\Gamma g_{2\lambda},\varphi(1)) \\ &= -\lambda(\widehat{\Gamma}\widetilde{\varphi}(1),\widetilde{\varphi}(1))+(\mathbb{T}^{-1/2}\Gamma g_{2\lambda}(1),\widetilde{\varphi}(1)) \\ &\le -\lambda\|\widehat{\Gamma}\|\|\widetilde{\varphi}(1)\|^2+\|\mathbb{T}^{-1/2}\Gamma\|\|g_{2\lambda}\|\|\widetilde{\varphi}(1)\| \\ &\le |\lambda^{-1}|\|\Gamma\|M_1^2\|g_{2\lambda}(1)\|^2+\|\mathbb{T}^{-1/2}\Gamma\|\|g_{2\lambda}(1)\|M_1|\lambda^{-1}|\|g_{2\lambda}\| \\ &\le M_2|\lambda^{-1}|\|g_{2\lambda}(1)\|^2 \end{aligned}$$

where M_2 is a positive constant. According to proposition 6.3.1, we have $\|g_{2\lambda}(1)\|\le M\|(u,v)\|_{\mathcal{H}}$, and hence there is a positive constant M_3 such that

$$\|(\varphi,\psi)\|_{\mathcal{H}}\le M_3\sqrt{|\lambda^{-1}|}\|(u,v)\|_{\mathcal{H}}.$$

Therefore, we get that, for $\lambda\in\rho(\mathcal{A})\cap\mathbb{R}_-$ with $|\lambda|$ large enough,

$$\begin{aligned} |F(\lambda)| &= |\langle R(\lambda,\mathcal{A})(u,v),(u_0,v_0)\rangle_{\mathcal{H}}|=|\langle R(\lambda,\mathcal{A}_0)(u,v),(u_0,v_0)\rangle_{\mathcal{H}}+\langle\varphi,\psi),(u_0,v_0)\rangle_{\mathcal{H}}| \\ &\le |\lambda^{-1}|\|(u,v)\|_{\mathcal{H}}\|u_0,v_0)\|_{\mathcal{H}}+\|(\varphi,\psi)\|_{\mathcal{H}}\|(u_0,v_0)\|_{\mathcal{H}} \\ &\le |\lambda^{-1}|\|(u,v)\|_{\mathcal{H}}\|u_0,v_0)\|_{\mathcal{H}}+M_3\sqrt{|\lambda^{-1}|}\|(u,v)\|_{\mathcal{H}}\|(u_0,v_0)\|_{\mathcal{H}}. \end{aligned}$$

Note that $F(\lambda)$ is an entire function of finite exponential type, the above inequality together with Phragmén-Lindelöf theorem (cf. [127]) implies $F(\lambda)\equiv 0$. So $R^*(\lambda,\mathcal{A})(u_0,v_0)\equiv 0$, which implies $(u_0,v_0)=0$. So $Sp(\mathcal{A})=\mathcal{H}$. The proof is then complete. □

In order to obtain the Riesz basis property of eigenvectors and generalized eigenvectors of $\mathcal{A}$, we need the following theorem, which comes from [119] and is an extension of the result in [117].

THEOREM 6.3.3 *Let $\mathcal{A}$ be the generator of a C_0 semigroup $T(t)$ on a separable Hilbert space $\mathcal{H}$. Suppose that the following conditions are satisfied:*

1). The spectrum of $\mathcal{A}$ has a decomposition

$$\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A}) \tag{6.3.23}$$

where $\sigma_2(\mathcal{A})$ consists of the isolated eigenvalues of $\mathcal{A}$ of finite multiplicity (repeated many times according to its algebraic multiplicity).

2). There exists a real number $\alpha \in \mathbb{R}$ such that

$$\sup\{\Re\lambda, \lambda \in \sigma_1(\mathcal{A})\} \le \alpha \le \inf\{\Re\lambda, \lambda \in \sigma_2(\mathcal{A})\} \tag{6.3.24}$$

3). The set $\sigma_2(\mathcal{A})$ is a union of finite many separable sets.

Then the following statements are true:

i). There exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with property that $\sigma(\mathcal{A}\big|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$, $\sigma(\mathcal{A}\big|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$, and there exists a finite combination, $E(\Omega_k, \mathcal{A})$, of some $\{E(\lambda_k, \mathcal{A})\}_{k=1}^{\infty}$:

$$E(\Omega_k, \mathcal{A}) = \sum_{\lambda \in \Omega_k \cap \sigma_2(\mathcal{A})} E(\lambda, \mathcal{A}) \tag{6.3.25}$$

such that $\{E(\Omega_k, \mathcal{A})\mathcal{H}_2\}_{k\in\mathbb{N}}$ forms a Riesz basis of subspaces for $\mathcal{H}_2$(the definition see [48, pp,332]). Furthermore,

$$\mathcal{H} = \overline{\mathcal{H}_1 \oplus \mathcal{H}_2}.$$

ii). If $\sup_{k\ge1} ||E(\lambda_k, \mathcal{A})|| < \infty$, then

$$\mathcal{D}(\mathcal{A}) \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}. \tag{6.3.26}$$

iii). $\mathcal{H}$ has a decomposition of the topological direct sum, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, if and only if

$$\sup_{n\ge1} \left\| \sum_{k=1}^{n} E(\Omega_k, \mathcal{A}) \right\| < \infty. \tag{6.3.27}$$

Combining Theorems 6.3.1, 6.3.2 and 6.3.3, we can get the following result.

THEOREM 6.3.4 *Let $\mathcal{A}$ be defined by (6.2.2) and (6.2.3) and $S(t)$ be the C_0 semigroup associated with $\mathcal{A}$, and let $B, \widehat{\Gamma}$ and $\widehat{C}$ be defined as in Theorem 6.3.1. If $\det(\widehat{\Gamma} - B - \widehat{C}^T B \widehat{C}) \neq 0$, then there is a sequence of eigenvectors and generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$. Therefore, $S(t)$ satisfies the spectrum determined growth assumption. In addition, $S(t)$ is in fact a C_0 group on $\mathcal{H}$.*

Proof Let $\mathcal{A}$ be defined by (6.2.2) and (6.2.3) and $S(t)$ be the C_0 semigroup associated with $\mathcal{A}$. Set $\sigma_1(\mathcal{A}) = \{-\infty\}$, $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$. Theorem 6.3.1 shows that all conditions in Theorem 6.3.3 are fulfilled, so the results of Theorem 6.3.3 are true. Thus there is a sequence of eigenvectors and generalized eigenvector of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}_2$. Theorem 6.3.2 says that the eigenvectors and generalized eigenvectors sequence is complete in $\mathcal{H}$, that is $\mathcal{H}_2 = \mathcal{H}$. Therefore the sequence is also a Riesz basis with parentheses for $\mathcal{H}$. The basis property together with the uniform boundedness of the multiplicities of eigenvalues of $\mathcal{A}$ implies that $S(t)$ satisfies the spectrum determined growth assumption. Also, the basis property of eigenvectors and generalized eigenvectors together with distribution of spectrum of $\mathcal{A}$ asserts that $\mathcal{A}$ generates a C_0 group on $\mathcal{H}$. The proof is then complete. $\square$

As a consequence of Riesz basis property, we have the following stability result of the system.

THEOREM 6.3.5 *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, and $B, \widehat{\Gamma}$ and $\widehat{C}$ be defined as in Theorem 6.3.1. Let $\det(\widehat{\Gamma} - B - \widehat{C}^T B\widehat{C}) \neq 0$ and $D(\lambda)$ be defined as (6.3.5). Suppose that $\Gamma > 0$ and $-1 \notin \sigma(C)$. Then the following statements are true:*

1). If $\inf\limits_{\lambda\in i\mathbb{R}} |D(\lambda)| \neq 0$, then the system (6.2.4) is exponentially stable;

2). If $\inf\limits_{\lambda\in i\mathbb{R}} |D(\lambda)| = 0$, then the system (6.2.4) is asymptotically stable but not exponentially stable;

Proof Under above assumptions, Theorem 6.3.4 shows that the system (6.2.4) is a Riesz system and satisfies the spectrum determined growth condition. Note that

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid D(\lambda) = 0\}.$$

If $\inf\limits_{\lambda\in i\mathbb{R}} |D(\lambda)| \neq 0$, then the imaginary axis is not an asymptote of $\sigma(\mathcal{A})$, which implies the system is exponentially stable. Since $\pm 1 \notin \sigma(C)$, Corollary 6.2.1 shows that there is no eigenvalue of $\mathcal{A}$ on the imaginary axis. If $\inf\limits_{\lambda\in i\mathbb{R}} |D(\lambda)| = 0$, then the imaginary axis is an asymptote of $\sigma(\mathcal{A})$, and hence the system is asymptotically stable but not exponential stable. The proof is then complete. $\square$

REMARK 6.3.1 *In Theorem 6.3.5, The condition that $\Gamma > 0$ and $\pm 1 \notin \sigma(C)$ is used to ensure that there is no eigenvalue on imaginary axis. The restriction can be relaxed, indeed, if $\Gamma \geq 0$ such that there is no eigenvalue on the imaginary axis, then the results in Theorem 6.3.5 is still right.*

6.4 Applications

In this section we shall give two examples in actual problem. One is the tree-shaped network of strings, another is n connected strings. We shall prove that these systems are exponentially stable.

6.4.1 Tree-shaped network of 7-strings

In this subsection we discuss the tree-shaped network of strings whose configuration is a simple and connected graph without circuit.

In past decade, the controllability and observability as well as stabilization of networks of strings have been a hot topic, many authors have made effort and obtained some nice results, for example, the authors of [33] [74][31] and [32] discussed the problems of observability and controllability of tree-shaped network of strings, authors of [5] and [6] discussed the stabilization problem of star-shaped and generic trees of strings. More general discussion for networks of strings and a list of entire literatures in this aspect we refer to a book [34] recent published.

Here, as an example, we give the Riesz basis property and exponential stability of this system for $n = 7$ with shaped as below figure.

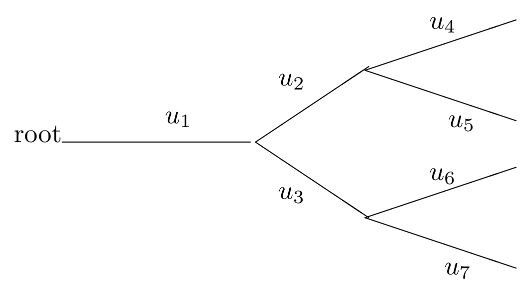

Fig 6.4.1

Firstly, we formulate the model into a normal form. We denote the strings by u_j, $j = 1, 2, \cdots, 7$. For these strings, we normalize the length into the unit, then the strings satisfy the equation

$$m_j u_{j,tt}(x,t) = T_j u_{j,xx}(x,t), \quad x \in (0,1), j = 1, 2, \cdots, 7, \tag{6.4.1}$$

the root and the geometric connected condition are given by

$$\begin{aligned} &u_1(0,t) = 0, \quad u_1(1,t) = u_2(0,t) = u_3(0,t) \\ &u_2(1,t) = u_4(0,t) = u_5(0,t), \quad u_3(1,t) = u_6(0,t) = u_7(0,t), \end{aligned} \tag{6.4.2}$$

and the dynamic conditions at the nodes are

$$\begin{aligned} &T_1 u_{1,x}(1,t) - [T_2 u_{2,x}(0,t) + T_3 u_{3,x}(0,t)] = -\alpha_1 u_{1,t}(1,t) \\ &T_2 u_{2,x}(1,t) - [T_4 u_{4,x}(0,t) + T_5 u_{5,x}(0,t)] = -\alpha_2 u_{2,t}(1,t), \\ &T_3 u_{3,x}(1,t) - [T_6 u_{6,x}(0,t) + T_7 u_{7,x}(0,t)] = -\alpha_3 u_{3,t}(1,t), \\ &T_4 u_{4,x}(1,t) = -\alpha_4 u_{4,t}(1,t), \quad T_5 u_{5,x}(1,t) = -\alpha_5 u_{5,t}(1,t), \\ &T_6 u_{6,x}(1,t) = -\alpha_6 u_{6,t}(1,t), \quad T_7 u_{7,x}(1,t) = -\alpha_7 u_{7,t}(1,t). \end{aligned} \tag{6.4.3}$$

Set

$$Y(x,t) = [u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t), u_5(x,t), u_6(x,t), u_7(x,t)]^T,$$

$$\mathbb{M} = \mathrm{diag}[m_1, m_2, \cdots, m_7], \quad \mathbb{T} = \mathrm{diag}[T_1, T_2, \cdots, T_7], \quad \Gamma = \mathrm{diag}[\alpha_1, \alpha_2, \cdots, \alpha_7] \tag{6.4.4}$$

and

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.4.5}$$

Then Eqs.(6.4.1)–(6.4.3) can be rewritten into

$$\begin{cases} \mathbb{M}Y_{tt}(x,t) = \mathbb{T}Y_{xx}(x,t), x \in (0,1), t > 0, \\ Y(0,t) = CY(1,t), t > 0, \\ \mathbb{T}Y_x(1,t) - C^T\mathbb{T}Y_x(0,t) = -\Gamma Y_t(1,t), t > 0, \\ Y(x,0) = Y_0(x), \quad Y_t(x,0) = Y_1(x) \end{cases} \tag{6.4.6}$$

where $Y_0(x)$ and $Y_1(x)$ are given suitable initial value condition.

REMARK 6.4.1 *The incidence matrix of graph G as shown Fig. 6.4.1 is given by*

$$\Phi = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The continuity condition can write as

$$Y(0,t) = (\Phi^-)^T d, \quad d = (d(root), d(a_1), d(a_2), \cdots, d(a_7)) \in \mathbb{C}^8.$$

The incidence matrix implies $d = \Phi^+ Y(1,t)$. Thus the matrix C just is $C = (\Phi^-)^T\Phi^+$

Clearly, $\pm 1 \notin \sigma(C)$. According to Corollary 6.2.1, we have the following result.

THEOREM 6.4.1 *The system (6.4.6) is well posed and asymptotically stable.*

To obtain exponential stability, we discuss the eigenvalue problem of the system.

Let $\lambda \in \mathbb{C}$, $Y(x,t) = e^{\lambda t}Y(x)$, then we have

$$\begin{cases} \lambda^2\mathbb{M}Y(x) = \mathbb{T}Y_{xx}(x), x \in (0,1), \\ Y(0) = CY(1), \\ \mathbb{T}Y_x(1) - C^T\mathbb{T}Y_x(0) = -\lambda\Gamma Y(1). \end{cases} \tag{6.4.7}$$

Let $B^2 = \mathrm{diag}[\rho_1^2, \rho_2^2, \cdots, \rho_7^2]$ where $\rho_j^2 = \frac{m_j}{T_j}$, then we have

$$\begin{cases} \lambda^2 B^2 Y(x) = Y_{xx}(x), x \in (0,1), \\ Y(0) = CY(1), \\ \mathbb{T}Y_x(1) - C^T\mathbb{T}Y_x(0) = -\lambda\Gamma Y(1). \end{cases} \tag{6.4.8}$$

Thus $Y(x)$ has the following form

$$Y(x) = \sinh(x\lambda B)u + \cosh(x\lambda B)v, \quad u, v \in \mathbb{C}^7.$$

Substituting it into the boundary conditions leads to

$$v = C[\sinh \lambda B u + \cosh \lambda B v],$$

$$\mathbb{T}B[\cosh \lambda B u + \sinh \lambda B v] - C^T \mathbb{T}B u = -\Gamma[\sinh \lambda B u + \cosh \lambda B v].$$

So λ is an eigenvalue if and only if

$$D(\lambda) = \det \begin{pmatrix} I - C\cosh \lambda B & -C \sinh \lambda B \\ \mathbb{T}B \sinh \lambda B + \Gamma \cosh \lambda B & \mathbb{T}B \cosh \lambda B + \Gamma \sinh \lambda B - C^T \mathbb{T}B \end{pmatrix} = 0 \qquad (6.4.9)$$

When $\det[\Gamma - \mathbb{T}B - C^T \mathbb{T}BC] \neq 0$, all eigenvalues are located in a strip. Note that

$$\mathbb{T}B = \begin{pmatrix} T_1\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_2\rho_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_3\rho_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_4\rho_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_5\rho_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_6\rho_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & T_7\rho_7 \end{pmatrix}, \qquad (6.4.10)$$

$$C^T \mathbb{T}BC = \begin{pmatrix} T_2\rho_2 + T_3\rho_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_4\rho_4 + T_5\rho_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_6\rho_6 + T_7\rho_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (6.4.11)$$

The condition $\det[\Gamma - \mathbb{T}B - C^T \mathbb{T}BC] \neq 0$ means that

$$\alpha_1 \neq T_1\rho_1 + T_2\rho_2 + T_3\rho_3, \quad \alpha_2 \neq T_2\rho_2 + T_4\rho_4 + T_5\rho_5,$$

$$\alpha_3 \neq T_3\rho_3 + T_6\rho_6 + T_7\rho_7, \quad \alpha_j \neq T_j\rho_j, \quad j = 4, 5, 6, 7.$$

Set

$$\begin{cases} w_j(\lambda) = T_j\rho_j \cosh \lambda\rho_j + \alpha_j \sinh \lambda\rho_j, \\ v_j(\lambda) = T_j\rho_j \sinh \lambda\rho_j + \alpha_j \cosh \lambda\rho_j, \end{cases} \quad j = 1, 2, 3, 4, 5, 6, 7, \qquad (6.4.12)$$

$$\begin{cases} F_2(\lambda) = \frac{1}{T_2\rho_2}\left[(w_5 v_4 T_4\rho_4 + w_4 v_5 T_5\rho_5)\sinh \lambda\rho_2 + w_2 w_4 w_5\right], \\ G_2(\lambda) = \frac{1}{T_2\rho_2}\left[(w_5 v_4 T_4\rho_4 + w_4 v_5 T_5\rho_5)\cosh \lambda\rho_2 + v_2 w_4 w_5\right], \end{cases} \qquad (6.4.13)$$

and

$$\begin{cases} F_3(\lambda) = \frac{1}{T_3\rho_3}\left[(w_7 v_6 T_6\rho_6 + w_6 v_7 T_7\rho_7)\sinh \lambda\rho_3 + w_3 w_6 w_7\right], \\ G_3(\lambda) = \frac{1}{T_3\rho_3}\left[(w_7 v_6 T_6\rho_6 + w_6 v_7 T_7\rho_7)\cosh \lambda\rho_3 + v_3 w_6 w_7\right]. \end{cases} \qquad (6.4.14)$$

A complicated calculation shows that

$$D(\lambda) = [T_2\rho_2 G_2(\lambda)F_3(\lambda) + T_3\rho_3 G_3(\lambda)F_2(\lambda)]\sinh\lambda\rho_1 + w_1(\lambda)F_2(\lambda)F_3(\lambda). \tag{6.4.15}$$

We are now in a position to calculate $\inf_{\sigma\in\mathbb{R}} |D(i\sigma)|$. Note that

$$\inf_{\sigma\in\mathbb{R}} |w_j(i\sigma)| \neq 0, \quad \inf_{\sigma\in\mathbb{R}} |v_j(i\sigma)| \neq 0, \quad j = 1,2,3,4,5,6,7,$$

$$\frac{v_j(i\sigma)}{w_j(i\sigma)} = \frac{v_j(i\sigma)\overline{w_j(i\sigma)}}{|w_j(i\sigma)|^2} = \frac{\alpha_j T_j\rho_j + i\frac{T_j^2\rho_j^2-\alpha_j^2}{2}\sin 2\sigma\rho_j}{T_j^2\rho_j^2\cos^2\sigma\rho_j + \alpha_j^2\sin^2\sigma\rho_j}$$

and

$$\frac{i\sin\sigma\rho_j}{w_j(i\sigma)} = \frac{i\sin\sigma\rho_j(T_j\rho_j\cos\sigma\rho_j - i\alpha_j\sin\sigma\rho_j)}{|w_j(i\sigma)|^2} = \frac{\alpha_j\sin^2\sigma\rho_j + i\frac{T_j\rho_j}{2}\sin 2\sigma\rho_j}{T_j^2\rho_j^2\cos^2\sigma\rho_j + \alpha_j^2\sin^2\sigma\rho_j},$$

where $j = 1,2,3$. From (6.4.13)–(6.4.14) we can get

$$\inf_{\sigma\in\mathbb{R}} |F_2(i\sigma)| > 0, \quad \inf_{\sigma\in\mathbb{R}} |F_3(i\sigma)| > 0,$$

$$\begin{aligned}\Re(G_2(i\sigma)\overline{F_2(i\sigma)}) &= [|w_5(i\sigma)|^2\alpha_4(T_4\rho_4)^2 + |w_4(i\sigma)|^2\alpha_5(T_5\rho_5)^2]T_2\rho_2\\ &\quad +\alpha_2 T_2\rho_2|w_4(i\sigma)|^2|w_5(i\sigma)|^2 > 0\end{aligned}$$

and

$$\begin{aligned}\Re(G_3(i\sigma)\overline{F_3(i\sigma)}) &= [|w_7(i\sigma)|^2\alpha_6(T_6\rho_6)^2 + |w_6(i\sigma)|^2\alpha_7(T_7\rho_7)^2]T_3\rho_3\\ &\quad +\alpha_3 T_3\rho_3|w_6(i\sigma)|^2|w_7(i\sigma)|^2 > 0.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\frac{D(i\sigma)}{w_1(i\sigma)F_2(i\sigma)F_3(i\sigma)} &= \left[T_2\rho_2\frac{G_2(i\sigma)\overline{F_2(i\sigma)}}{|F_2(i\sigma)|^2} + T_3\rho_3\frac{G_3(i\sigma)\overline{F_3(i\sigma)}}{|F_3(i\sigma)|^2}\right]\frac{i\sin\sigma\rho_1\overline{w_1(i\sigma)}}{|w_1(i\sigma)|^2} + 1\\ &= \left[T_2\rho_2\frac{\Re(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2} + T_3\rho_3\frac{\Re(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\frac{\alpha_1\sin^2\sigma\rho_1}{|w_1(i\sigma)|^2} + 1\\ &\quad -\left[T_2\rho_2\frac{\Im(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2} + T_3\rho_3\frac{\Im(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\frac{T_1\rho_1\sin 2\sigma\rho_1}{2|w_1(i\sigma)|^2}\\ &\quad +i\left[T_2\rho_2\frac{\Re(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2} + T_3\rho_3\frac{\Re(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\frac{T_1\rho_1\sin 2\sigma\rho_1}{2|w_1(i\sigma)|^2}\\ &\quad +i\left[T_2\rho_2\frac{\Im(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2} + T_3\rho_3\frac{\Im(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\frac{\alpha_1\sin^2\sigma\rho_1}{|w_1(i\sigma)|^2}.\end{aligned}$$

When $\sin\sigma\rho_1 \to 0$, we have $D(i\sigma) \not\to 0$. Now we assume that $\sin\sigma\rho_1 \not\to 0$, then

$$\begin{aligned}&\left[T_2\rho_2\frac{\Re(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2} + T_3\rho_3\frac{\Re(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\frac{T_1\rho_1\sin 2\sigma\rho_1}{2|w_1(i\sigma)|^2}\\ &+\left[T_2\rho_2\frac{\Im(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2} + T_3\rho_3\frac{\Im(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\frac{\alpha_1\sin^2\sigma\rho_1}{|w_1(i\sigma)|^2} \to 0\end{aligned}$$

if and only if

$$\begin{aligned}
&-\left[T_2\rho_2\frac{\Im(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2}+T_3\rho_3\frac{\Im(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\\
=&\left[T_2\rho_2\frac{\Re(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2}+T_3\rho_3\frac{\Re(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\frac{T_1\rho_1\cos\sigma\rho_1}{\alpha_1\sin\sigma\rho_1}+o(1).
\end{aligned}$$

From above we get

$$\begin{aligned}
\frac{D(i\sigma)}{w_1(i\sigma)F_2(i\sigma)F_3(i\sigma)} &= \left[T_2\rho_2\frac{\Re(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2}+T_3\rho_3\frac{\Re(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\frac{\alpha_1\sin^2\sigma\rho_1}{|w_1(i\sigma)|^2}+1\\
&\quad+\left[T_2\rho_2\frac{\Re(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2}+T_3\rho_3\frac{\Re(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\frac{T_1^2\rho_1^2\cos^2\sigma\rho_1}{\alpha_1|w_1(i\sigma)|^2}\\
&\quad+o(1)+io(1)\\
&= \left[T_2\rho_2\frac{\Re(G_2(i\sigma)\overline{F_2(i\sigma)})}{|F_2(i\sigma)|^2}+T_3\rho_3\frac{\Re(G_3(i\sigma)\overline{F_3(i\sigma)})}{|F_3(i\sigma)|^2}\right]\frac{1}{\alpha_1}+1+o(1)+io(1).
\end{aligned}$$

Therefore,

$$\inf_{\sigma\in\mathbb{R}}|D(i\sigma)|\neq 0.$$

According to Theorem 6.3.4 and 6.3.5, we have the following result.

THEOREM 6.4.2 *Let $\mathcal{H}$ be defined as in section 3 and let $\alpha_1\neq T_1\rho_1+T_2\rho_2+T_3\rho_3$,$\alpha_2\neq T_2\rho_2+T_4\rho_4+T_5\rho_5$,$\alpha_3\neq T_3\rho_3+T_6\rho_6+T_7\rho_7$,$\alpha_j\neq T_j\rho_j, j=4,5,6,7$. Then there is a sequence of eigenvector and generalized eigenvectors of the system that forms a Riesz basis with parentheses for the state space $\mathcal{H}$. The system (6.4.6) is exponentially stable.*

REMARK 6.4.2 *As mentioned in Remark 6.3.1, Γ being a positive definite matrix is only a sufficient condition for the system (6.4.1–6.4.3) decays exponentially. Similar to proof of Theorem 6.4.1, we can prove that $\Gamma=\mathrm{diag}(0,0,0,\alpha_4,\alpha_5,\alpha_6,\alpha_7)$ stabilizes exponentially the system. Further, if we suppose that the network of strings satisfy $T_j=m_j=1$, then we can use three controllers to stabilize exponentially the network. The controllers are setup as follows:*

1). $\alpha_1\neq 0$, $\alpha_5\alpha_7\neq 0$;

2). $\alpha_1=\alpha_1=\alpha_3=0$, one of α_j,$j=4,5,6,7$, is 0.

In above both cases, it always holds that $\inf\limits_{\sigma\in\mathbb{R}}|D(i\sigma)|>0$. Here we omit the details of calculation.

6.4.2 n-serially connected strings

In this subsection we discuss the n serially connected strings with internal node and boundary controls. This problem was proposed and discussed for $n=2$ in [16]. The exponential stability was investigated in [71]. Here, we give the exponential decay rate of the system and Riesz basis property. Let us recall the model.

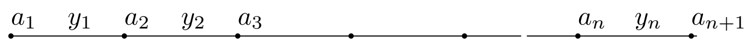

Fig. 6.4.2. A long chain graph G

Let $y(x,t)$, the transverse displacement of n connected strings at location x at time t, satisfy

$$m_i \frac{\partial^2 y(x,t)}{\partial t^2} - T_i \frac{\partial^2 y(x,t)}{\partial x^2} = 0, \quad i-1 < x < i, \quad i = 1, 2, \cdots, n, \ t > 0 \tag{6.4.16}$$

on the span $[0, n]$. We assume Dirichlet condition at the left hand $x = 0$ and Neumann boundary condition at the right hand $x = n$ where a control force $u_n(t)$ is applied, i.e.,

$$y(0,t) = 0, \quad \frac{\partial y(n,t)}{\partial x} = u_n(t), \quad t > 0. \tag{6.4.17}$$

At the i-th intermediate node $x = i$, we assume the continuity of displacement

$$y(i^+, t) = y(i^-, t), \quad i = 1, 2, \cdots, n-1, \quad t > 0, \tag{6.4.18}$$

and discontinuity of vertical force component

$$T_i \frac{\partial y(i^-, t)}{\partial x} - T_{i+1} \frac{\partial y(i^+, t)}{\partial x} = u_i(t), \quad i = 1, 2, \cdots, n-1, \quad t > 0, \tag{6.4.19}$$

where $u_j(t)$,$j = 1, 2, \cdots, n-1$, are applied external forces.

An important task in engineering is to design controllers $U = (u_1(t), u_2(t), \cdots, u_n(t))$ such that the system comes back to its equilibrium. The authors of [16] designed the following feedback controllers at the intermediated point $x = i$ and the right end $x = n$,

$$u_i(t) = -\alpha_i \frac{\partial y(i,t)}{\partial t}, \quad \alpha_i > 0, \quad i = 1, 2, \cdots, n. \tag{6.4.20}$$

Then the equation (6.4.16) together with (6.4.17)–(6.4.20) forms a closed loop system. In what following, we shall prove that this closed loop system is a Riesz system and decays exponentially.

Let $y_i(x,t) = y(i-1+x, t), x \in (0,1)$, and

$$Y(x,t) = (y_1(x,t), y_2(x,t), \cdots, y_n(x,t)), \quad x \in (0,1), \quad t > 0.$$

Then Eqs. (6.4.16) is equivalent to an equation in $\mathbb{C}^n$

$$\mathbb{M} \frac{\partial^2 Y(x,t)}{\partial t^2} = \mathbb{T} \frac{\partial^2 Y(x,t)}{\partial x^2}, \quad x \in (0,1), \quad t > 0 \tag{6.4.21}$$

where

$$\mathbb{M} = \mathrm{diag}(m_1, m_2, \cdots, m_n), \quad \mathbb{T} = \mathrm{diag}(T_1, T_2, \cdots, T_n). \tag{6.4.22}$$

The continuity conditions at intermediate nodes together with the condition at the left hand endpoint can be written into

$$Y(0,t) = CY(1,t), \tag{6.4.23}$$

where

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & \ddots & & & \ddots & \vdots \\ \vdots & & \ddots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{6.4.24}$$

The discontinuity condition of vertical force at the intermediate nodes together with the condition at the right hand endpoint can be written as

$$\mathbb{T}\frac{\partial Y(1,t)}{\partial x} - C^T\mathbb{T}\frac{\partial Y(0,t)}{\partial x} = U(t) = -\Gamma\frac{\partial Y(1,t)}{\partial t}, \tag{6.4.25}$$

here C^T denotes the transpose matrix of C and

$$\Gamma = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n) \tag{6.4.26}$$

is a $n \times n$ positive definite matrix. Thus the closed loop system is written into

$$\begin{cases} \mathbb{M}Y_{tt}(x,t) = \mathbb{T}Y_{xx}(x,t), \quad x \in (0,1),\ t > 0, \\ Y(0,t) = CY(1,t), \quad t > 0, \\ \mathbb{T}Y_x(1,t) - C^T\mathbb{T}Y_x(0,t) = -\Gamma Y_t(1,t), \quad t > 0, \\ Y(x,0) = Y_0(x), \quad Y_t(x,0) = Y_1(x), \quad x \in (0,1). \end{cases} \tag{6.4.27}$$

where $Y_0(x)$ and $Y_1(x)$ are given suitable initial data.

In this case, we can take matrix

$$B^2 = \mathrm{diag}\left[\frac{m_1}{T_1}, \frac{m_2}{T_2}, \cdots, \frac{m_n}{T_n}\right],$$

then the condition (6.3.5) in previous section is equivalent to

$$D(\lambda) = \det\begin{pmatrix} (I-C)e^{\lambda B} & (I-C)e^{-\lambda B} \\ (\Gamma + \mathbb{T}B)e^{\lambda B} - C^T\mathbb{T}B & (\Gamma + TB)e^{-\lambda B} + C^TTB \end{pmatrix},$$

and the condition $\det[\widehat{\Gamma} - B - \widehat{C}^T B\widehat{C}] \neq 0$ is equivalent to $\det[\Gamma - \mathbb{T}B - C^T\mathbb{T}BC] \neq 0$.

Now let us determine $D(\lambda)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue, $Y = (y_1, y_2, \cdots, y_n)$ be corresponding an eigenfunction, then

$$m_j\lambda^2 y_j(x) = T_j y_{j,xx}$$

$$y_1(0) = 0, \quad y_j(1) = y_{j+1}(0), \quad j = 1, 2, \cdots, n. \tag{6.4.28}$$

$$T_j y_{j,x}(1) - T_{j+1}y_{j+1,x}(0) = -\alpha_j\lambda y_j(1), j = 1, 2, \cdots, n-1, \quad T_n y_{n,x}(1) = -\alpha_n\lambda y_n(1). \tag{6.4.29}$$

Set $\rho_j^2 = \frac{m_j}{T_j}$ and

$$y_1(x) = a_1 \sinh\lambda\rho_1, \quad y_j(x) = a_j\sinh\lambda\rho_j + b_j\cosh\lambda\rho_j, \quad j = 2, 3, \cdots, n.$$

From (6.4.28) and (6.4.29) we get

$$\begin{cases} [T_n\rho_n \cosh\lambda\rho_n + \alpha_n \sinh\lambda\rho_n]a_n + [T_n\rho_n \sinh\lambda\rho_n + \alpha_n \cosh\lambda\rho_n]b_n = 0, \\ [T_j\rho_j \cosh\lambda\rho_j + \alpha_j \sinh\lambda\rho_j]a_j + [T_j\rho_j \sinh\lambda\rho_j + \alpha_j \cosh\lambda\rho_j]b_j = T_{j+1}\rho_{j+1}a_{j+1}, \\ \quad j = 2, \cdots, n-1, \\ \sinh\lambda\rho_j a_j + \cosh\lambda\rho_j b_j = b_{j+1} \quad j = 2, \cdots, n-1, \\ [T_1\rho_1 \cosh\lambda\rho_1 + \alpha_1 \sinh\lambda\rho_1]a_1 = T_2\rho_2 a_2 \\ [\sinh\lambda\rho_1]a_1 = b_2. \end{cases} \tag{6.4.30}$$

Set $T_{n+1}\rho_{n+1} = 1$ and

$$\begin{aligned} w_j(\lambda) &= \frac{1}{T_{j+1}\rho_{j+1}}[T_j\rho_j \cosh\lambda\rho_j + \alpha_j \sinh\lambda\rho_j], \\ v_j(\lambda) &= \frac{1}{T_{j+1}\rho_{j+1}}[T_j\rho_j \sinh\lambda\rho_j + \alpha_j \cosh\lambda\rho_j]. \end{aligned} \tag{6.4.31}$$

We can rewrite (6.4.30) into

$$\begin{aligned} (1,0)\begin{pmatrix} w_n(\lambda) & v_n(\lambda) \\ \sinh\lambda\rho_n & \cosh\lambda\rho_n \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix} &= 0, \\ \begin{pmatrix} w_j(\lambda) & v_j(\lambda) \\ \sinh\lambda\rho_j & \cosh\lambda\rho_j \end{pmatrix}\begin{pmatrix} a_j \\ b_j \end{pmatrix} &= \begin{pmatrix} a_{j+1} \\ b_{j+1} \end{pmatrix}, \\ \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} &= \begin{pmatrix} w_1(\lambda) & v_1(\lambda) \\ \sinh\lambda\rho_1 & \cosh\lambda\rho_1 \end{pmatrix}\begin{pmatrix} a_1 \\ 0 \end{pmatrix}. \end{aligned} \tag{6.4.32}$$

Therefore, for $1 \le k \le n-1$, we have

$$\begin{pmatrix} a_{k+1} \\ b_{k+1} \end{pmatrix} = \left[\prod_{j=0}^{k-1}\begin{pmatrix} w_{k-j}(\lambda) & v_{k-j}(\lambda) \\ \sinh\lambda\rho_{k-j} & \cosh\lambda\rho_{k-j} \end{pmatrix}\right]\begin{pmatrix} a_1 \\ 0 \end{pmatrix}. \tag{6.4.33}$$

Note that for $j = 1, 2, \cdots, n$, the matrices

$$\begin{pmatrix} w_j(\lambda) & v_j(\lambda) \\ \sinh\lambda\rho_j & \cosh\lambda\rho_j \end{pmatrix} = \begin{pmatrix} \frac{T_j\rho_j}{T_{j+1}\rho_{j+1}} & \frac{\alpha_j}{T_{j+1}\rho_{j+1}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cosh\lambda\rho_j & \sinh\lambda\rho_j \\ \sinh\lambda\rho_j & \cosh\lambda\rho_j \end{pmatrix}$$

are invertible. So $\lambda \in \mathbb{C}$ is an eigenvalue if and only if

$$D(\lambda) = (1,0)\left[\prod_{j=0}^{n-1}\begin{pmatrix} w_{n-j}(\lambda) & v_{n-j}(\lambda) \\ \sinh\lambda\rho_{n-j} & \cosh\lambda\rho_{n-j} \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \tag{6.4.34}$$

or

$$D(\lambda) = D(\lambda)^T = (1,0)\left[\prod_{j=1}^{n}\begin{pmatrix} w_j(\lambda) & \sinh\lambda\rho_j \\ v_j(\lambda) & \cosh\lambda\rho_j \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0. \tag{6.4.35}$$

Similar to previous subsection, a complicated calculation shows that

$$\inf_{\sigma\in\mathbb{R}} |D(i\sigma)| > 0.$$

The condition $\det[\Gamma - \mathbb{T}B - C^T\mathbb{T}BC] \neq 0$ becomes

$$\alpha_j \neq T_j\rho_j + T_{j+1}\rho_{j+1}, \quad j = 1, 2, \cdots, n-1, \quad \alpha_n \neq T_n\rho_n. \tag{6.4.36}$$

Combining Theorem 6.3.4, we have the following result.

THEOREM 6.4.3 *Let (6.4.36) hold, then system (6.4.27) is exponentially stable. There is a sequence of eigenvector and generalized eigenvectors of the system that forms a Riesz basis with parentheses for the space $\mathcal{H}$.*

REMARK 6.4.3 *In the long chains case, if it has continuous displacement, then we have $Y(0,t) = (\Phi^-)^T Y(v,t)$ and $Y(1,t) = (\Phi^+)^T Y(v,t)$. At same time, it also satisfies the flow continuous condition,*

$$Y(v,t) = \Phi^- Y(0,t), \quad Y(v,t) = \Phi^+ Y(1,t).$$

So $Y(0) = (\Phi^-)^T\Phi^+ Y(1)$. Denote $C = (\Phi^-)^T\Phi^+$. We get $Y(0) = CY(1)$.

$$(\Phi^-)^T\Phi^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

The dynamic condition

$$\Phi^{+}\mathbb{T}Y_x(1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 y_1'(1) \\ T_2 y_2'(1) \\ \vdots \\ \vdots \\ T_{n-1} y_{n-1}'(1) \\ T_n y_n'(1) \end{bmatrix}$$

and

$$\Phi^{-}\mathbb{T}Y_x(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 y_1'(0) \\ T_2 y_2'(0) \\ \vdots \\ \vdots \\ T_{n-1} y_{n-1}'(0) \\ T_n y_n'(0) \end{bmatrix}$$

Define the project operator $P : \mathbb{C}^{n+1} \to \mathbb{C}^n$ by

$$P(v_1, v_2, \cdots, v_n, v_{n+1}) = (v_2, \cdots, v_n, v_{n+1})$$

which has matrix representation

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The dynamic condition can write into

$$P\Phi^{+}TY_x(1,t) - P\Phi^{-}TY_x(0,t) + \Gamma Y(1,t) = 0.$$

A direct verification shows that $P\Phi^{+} = I_n$ and $P\Phi^{-} = C^T$. Therefore, we have

$$\mathbb{T}Y_x(1,t) - C^T\mathbb{T}Y_x(0,t) = -\Gamma Y(1,t).$$

6.5 Conclusion remark

In this chapter, we discussed the abstract hyperbolic equation

$$\begin{cases} \mathbb{M}Y_{tt}(x,t) = \mathbb{T}Y_{xx}(x,t), \quad x \in (0,1),\ t>0, \\ Y(0,t) = CY(1,t), \quad t>0, \\ \mathbb{T}Y_x(1,t) - C^T\mathbb{T}Y_x(0,t) = -\Gamma Y_t(1,t), \quad t>0, \\ Y(x,0) = Y_0(x), \quad Y_t(x,0) = Y_1(x), \quad x \in (0,1). \end{cases} \tag{6.5.1}$$

where $\mathbb{M}, \mathbb{T}$ are positive definite $n \times n$ matrices, Γ is a nonnegative matrix, and C is a real $n \times n$ matrix, and C^T denotes the transpose of matrix C. The obtained main results are as follows:

1) If C satisfies the condition $\det(I \pm C) \neq 0$ and Γ is positive definite matrix, then the system is asymptotically stable;

2) if $\Gamma \geq 0$ satisfies the condition $\det[\widehat{\Gamma} - B - \widehat{C}^T B \widehat{C}] \neq 0$, spectra of the system is distributed in a strip parallel to the imaginary axis, and the eigenvector and generalized eigenvectors form a Riesz basis with parentheses for the Hilbert state space;

3) Suppose that $\det(I \pm C) \neq 0$. If $\inf_{\sigma\in\mathbb{R}} |D(i\sigma)| > 0$, then the system is exponentially stable. If $\inf_{\sigma\in\mathbb{R}} |D(i\sigma)| = 0$, the system is at most asymptotically stable. In particular, when $\Gamma > 0$, we have simple test of asymptotic stability, i.e, $-1 \notin \sigma(C)$.

Note that in the system assumptions, we only require that $\mathbb{M}$ and $\mathbb{T}$ are positive definite $n \times n$ matrices, they need not to be the diagonal matrices. So these results can be applied to more complex systems, for instance, the system with coupling equations, i.e.,

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} y_{1,tt} \\ y_{2,tt} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} y_{1,xx} \\ y_{2,xx} \end{pmatrix}, \quad x \in (0,1), \tag{6.5.2}$$

where $T_{kk} > 0$ and $T_{11}T_{22} - T_{12}T_{21} > 0$. Here we do not study such systems.

Chapter 7

Continuous Network of Strings

In this chapter, we shall discuss the continuous network of strings:

$$\begin{cases} m_j(s)u_{j,tt}(s,t) = (T_j(s)u_{j,s}(s,t))_s - q_j(s)u_j(s), \quad s \in (0,1) \\ u(a_j,t) = u_k(0,t) = u_i(1,t) = 0, \quad \forall k \in J^-(a_j), (or\ i \in J^+(a_j)), \quad a_j \in \partial G_D, \\ u(a_j,t) = u_k(0,t) = u_i(1,t), \quad \forall k \in J^-(a_j), i \in J^+(a_j), \quad a_j \in V \backslash \partial G_D \\ \sum\limits_{j \in J^+(a)} T_j(1)u_{j,s}(1,t) - \sum\limits_{k \in J^-(a)} T_k(0)u_{k,s}(0,t) + k(a)u(a,t) = -\alpha(a)s(a)u_t(a,t), a \in V \backslash \partial G_D \\ u_j(s,0) = u_{j0}(s), \quad u_{jt}(s,0) = u_{j1}(s), \quad s \in (0,1) \end{cases}$$

where $T_j(s)$ and $m_j(s)$ are positive and twice continuous differentiable functions and $q_j(s)$ are nonnegative continuous functions (or called potentials), $s(a) = 0$ or 1 is the function of vertex selection and $\alpha(a)s(a) \geq 0$. Its vectorization form is

$$\begin{cases} \mathbb{M}(x)U_{tt}(x,t) = (\mathbb{T}(x)U_x(x,t))_x - \mathbb{Q}(x)U(x,t), \quad x \in (0,1) \\ \exists U(v,t) \in \mathbb{V},\ s.t.\ U(1,t) = (\Phi^+)^T U(v,t), \quad U(0,t) = (\Phi^-)^T U(v,t) \\ \mathcal{P}\Phi^+\mathbb{T}(1)U_x(1,t) - \mathcal{P}\Phi^-\mathbb{T}(0)U_x(0,t) + \mathbb{K}(v)U(v,t) = -\Gamma\mathcal{S}U_t(v,t) \in \mathbb{V} \\ U(x,0) = U_0(x), \quad U_t(x,0) = U_1. \end{cases} \tag{7.0.1}$$

where $\mathcal{P} : \mathbb{C}^m \to \mathbb{V} = \{(w(a_1), w(a_2), \cdots, w(a_m)) \mid w(a_k) = 0, a_k \in \partial G_D\}$, We shall study the asymptotic property of this network.

7.1 Well-posedness of the continuous network

Let $C_d(G)$ denote the set of all continuous functions on G with zero values at ∂G_D. Then $u \in C_d(G)$ implies that there is $d \in \mathbb{V}$ such that $U(1) = (\Phi^+)^T d$, $U(0) = (\Phi^-)^T d$. For simplicity, we set

$$H_e^1(G) = \{f \in C_d(G) \mid f' \in H^1(E)\}.$$

Let $\mathcal{H}$ be the state space defined by

$$\mathcal{H} = H_e^1(G) \times L^2(E)$$

equipped with inner product

$$\begin{aligned}((f,g),(u,v))_{\mathcal{H}} &= \int_0^1 (\mathbb{T}(x)f'(x), u'(x))_{c^n} dx + \int_0^1 (\mathbb{M}(x)g(x), v(x))_{c^n} dx \\ &\quad + \int_0^1 (\mathbb{Q}(x)f(x), u(x))_{c^n} dx + (\mathbb{K}(v)f(v), u(v))_{\mathbb{V}}.\end{aligned}$$

Define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}(w,z) = (z, \mathbb{M}^{-1}(x)[(\mathbb{T}(x)w'(x))' - \mathbb{Q}(x)w(x)]), \quad (w,z) \in D(\mathcal{A}) \tag{7.1.1}$$

with domain

$$\begin{aligned}D(\mathcal{A}) = \{(w,z) \in H^2(E) \times H_e^1(G) \mid w(v), z(v) \in \mathbb{V} \\ \mathcal{P}\Phi^+\mathbb{T}(1)w'(1) - \mathcal{P}\Phi^-\mathbb{T}(0)w'(0) + \mathbb{K}(v)w(v) = -\Gamma\mathcal{S}z(v)\}.\end{aligned} \tag{7.1.2}$$

With this definition, we can rewrite (7.0.1) into an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \frac{dW(t)}{dt} = \mathcal{A}W(t), & t > 0 \\ W(0) = W_0 \end{cases} \tag{7.1.3}$$

where $W(t) = (U(x,t), U_t(x,t))^T$ and $W(0) = (U_0(x), U_1(x))^T$.

PROPOSITION 7.1.1 *Let $\mathcal{A}$ be defined by (7.1.1) and (7.1.2). Then $\mathcal{A}$ is a densely defined and closed linear operator in $\mathcal{H}$.*

Let us consider the dual operator of $\mathcal{A}$, $\mathcal{A}^*$. We shall prove that

$$\mathcal{A}^*(f,g) = -(g, \mathbb{M}^{-1}(x)[(\mathbb{T}(x)f'(x))' - \mathbb{Q}(x)f(x)]), \quad (f,g) \in D(\mathcal{A}^*) \tag{7.1.4}$$

with domain

$$\begin{aligned}D(\mathcal{A}^*) = \{(f,g) \in H^2(E) \times H_e^1(G) \mid f(v), g(v) \in \mathbb{V} \\ \mathcal{P}\Phi^+\mathbb{T}(0)f'(1) - \mathcal{P}\Phi^-\mathbb{T}(1)f'(0) + \mathbb{K}(v)f(v) = \Gamma\mathcal{S}g(v)\}\end{aligned} \tag{7.1.5}$$

Here we mainly find out the expression of $\mathcal{A}^*$. For any $(w,z) \in D(\mathcal{A})$, $(f,g) \in D(\mathcal{A}^*)$, we have

$$\begin{aligned}&((\mathcal{A}(w,z),(f,g))_{\mathcal{H}} = ((w,z), \mathcal{A}^*(f,g))_{\mathcal{H}} \\ =\ & \int_0^1 [(\mathbb{T}(x)z'(x), f'(x))_{c^n} + ((\mathbb{T}(x)w'(x))' - \mathbb{Q}(x)w(x), g(x))_{c^n}]dx \\ &+ \int_0^1 (\mathbb{Q}(x)z(x), f(x))_{c^n} dx + (\mathbb{K}(v)z(v), f(v))_{\mathbb{V}} \\ =\ & (z(1), \mathbb{T}(1)f'(1))_{c^n} - (z(0), \mathbb{T}(0)f'(0))_{c^n} - \int_0^1 (z(x), (\mathbb{T}(x)f'(x))')_{c^n} dx\end{aligned}$$

$$
\begin{aligned}
& +(\mathbb{T}(1)w'(1),g(1))_{c^n}-(\mathbb{T}(0)w'(0),g(0))_{c^n}-\int_0^1(\mathbb{T}(x)w'(x),g'(x))_{c^n}dx \\
& -\int_0^1(\mathbb{Q}(x)w(x),g(x))_{c^n}dx+\int_0^1(z(x),\mathbb{Q}(x)f(x))_{c^n}dx+(z(v),\mathbb{K}(v)f(v))_{\mathbb{V}} \\
= \; & ((\Phi^+)^Tz(v),\mathbb{T}(1)f'(1))_{c^n}-((\Phi^-)^Tz(v),\mathbb{T}(0)f'(0))_{c^n}+(z(v),\mathbb{K}(v)f(v))_{\mathbb{V}} \\
& +(\mathbb{T}(1)w'(1),(\Phi^+)^Tg(v))_{c^n}-(\mathbb{T}(0)w'(0),(\Phi^-)^Tg(v))_{c^n}+(\mathbb{K}(v)w(v),g(v))_{\mathbb{V}} \\
& -\int_0^1(\mathbb{T}(x)w'(x),g'(x))_{c^n}dx-\int_0^1(\mathbb{M}(x)z(x),\mathbb{M}^{-1}(x)((\mathbb{T}(x)f'(x))'-\mathbb{Q}(x)f(x)))_{c^n}dx \\
& -\int_0^1(\mathbb{Q}(x)w(x),g(x))_{c^n}dx-(\mathbb{K}(v)w(v),g(v))_{\mathbb{V}} \\
= \; & (z(v),\mathcal{P}\Phi^+\mathbb{T}(1)f'(1)-\mathcal{P}\Phi^-\mathbb{T}(0)f'(0)+\mathbb{K}(v)f(v))_{\mathbb{V}} \\
& +(\mathcal{P}\Phi^+\mathbb{T}(1)w'(1)-\mathcal{P}\Phi^-\mathbb{T}(0)w'(0)+\mathbb{K}(v)w(v),g(v))_{\mathbb{V}} \\
& -\int_0^1(\mathbb{T}(x)w'(x),g'(x))_{c^n}dx-\int_0^1(\mathbb{M}(x)z(x),\mathbb{M}^{-1}(x)((\mathbb{T}(x)f'(x))'-\mathbb{Q}(x)f(x)))_{c^n}dx \\
& -\int_0^1(\mathbb{Q}(x)w(x),g(x))_{c^n}dx-(\mathbb{K}(v)w(v),g(v))_{\mathbb{V}} \\
= \; & (z(v),\mathcal{P}\Phi^+\mathbb{T}(1)f'(1)-\mathcal{P}\Phi^-\mathbb{T}(0)f'(0)+\mathbb{K}(v)f(v))_{\mathbb{V}}-(\Gamma\mathcal{S}z(v),g(v))_{\mathbb{V}} \\
& -\int_0^1(\mathbb{T}(x)w'(x),g'(x))_{c^n}dx-\int_0^1(\mathbb{M}(x)z(x),\mathbb{M}^{-1}(x)((\mathbb{T}(x)f'(x))'-\mathbb{Q}(x)f(x)))_{c^n}dx \\
& -\int_0^1(\mathbb{Q}(x)w(x),g(x))_{c^n}dx-(\mathbb{K}(v)w(v),g(v))_{\mathbb{V}}
\end{aligned}
$$

where we have used the relations

$$g(1)=(\Phi^+)^Tg(v),\quad g(0)=(\Phi^-)^Tg(v);\quad z(1)=(\Phi^+)^Tz(v),\quad z(0)=(\Phi^-)^Tz(v).$$

Therefore, due to $\Gamma\mathcal{S}$ being a diagonal matrix,

$$\mathcal{A}^*(f,g)=-(g,\mathbb{M}^{-1}((\mathbb{T}(x)f'(x))'-\mathbb{Q}(x)f(x)))$$

and

$$D(\mathcal{A}^*)=\left\{(f,g)\in H^2(E)\times H_e^1(0,1)\;\middle|\;\begin{array}{l} f(v),g(v)\in\mathbb{V}, \\ \mathcal{P}\Phi^+\mathbb{T}(1)f'(1)-\mathcal{P}\Phi^-\mathbb{T}(0)f'(0)+\mathbb{K}(v)g(v)=\Gamma\mathcal{S}g(v)\end{array}\right\}.$$

THEOREM 7.1.1 *$\mathcal{A}$ and $\mathcal{A}^*$ are dissipative operators in $\mathcal{H}$. Hence $\mathcal{A}$ generates a C_0 semigroup of contraction on $\mathcal{H}$ and hence (7.1.3) is well-posed.*

Proof For any $(w,z)\in D(\mathcal{A})$, it holds that

$$
\begin{aligned}
(\mathcal{A}(w,z),(w,z))_{\mathcal{H}} &= \int_0^1[(\mathbb{T}(x)z'(x),w'(x))_{c^n}+((\mathbb{T}(x)w'(x))'-\mathbb{Q}(x)w(x),z(x))_{c^n}]dx \\
&\quad +\int_0^1(\mathbb{Q}(x)z(x),w(x))_{c^n}dx+(\mathbb{K}(v)z(v),w(v))_{\mathbb{V}} \\
&= (z(v),\mathcal{P}\Phi^+\mathbb{T}(1)w'(1)-\mathcal{P}\Phi^-\mathbb{T}(0)w'(0)+\mathbb{K}(v)w(v))_{c^n} \\
&\quad +\int_0^1(\mathbb{T}(x)z'(x),w(x))_{c^n}dx-\int_0^1(\mathbb{T}(x)w'(x),z'(x))_{c^n}dx
\end{aligned}
$$

$$-\int_0^1 (\mathbb{Q}(x)w(x), z(x))_{c^n} dx + \int_0^1 (\mathbb{Q}(x)z(x), w(x))_{c^n} dx,$$

and hence

$$\begin{aligned}
& 2\Re(\mathcal{A}(w,z),(w,z))_{\mathcal{H}} = (\mathcal{A}(w,z),(w,z))_{\mathcal{H}} + ((w,z),\mathcal{A}(w,z))_{\mathcal{H}} \\
= \; & (z(v), \mathcal{P}\Phi^+\mathbb{T}(1)w'(1) - \mathcal{P}\Phi^-\mathbb{T}(0)w'(0) + \mathbb{K}(v)w(v))_{\mathbb{V}} \\
& +(\mathcal{P}\Phi^+\mathbb{T}(1)w'(1) - \mathcal{P}\Phi^-\mathbb{T}(0)w'(0) + \mathbb{K}(v)w(v), z(v))_{\mathbb{V}} \\
= \; & -(z(v), \Gamma\mathcal{S}z(v))_V - (\Gamma\mathcal{S}z(v), z(v))_V.
\end{aligned}$$

The dissipatedness of $\mathcal{A}$ follows from the nonnegativity of $\Gamma\mathcal{S}$.

Similarly, for any $(f,g) \in D(\mathcal{A}^*)$,

$$\begin{aligned}
(\mathcal{A}^*(f,g),(f,g))_{\mathcal{H}} &= -\int_0^1 (\mathbb{T}g'(x), f'(x))_{c^n} dx - \int_0^1 ((\mathbb{T}f'(x))' - \mathbb{Q}(x)f(x), g(x))_{c^n} dx \\
&\quad - \int_0^1 (\mathbb{Q}(x)g(x), f(x))_{c^n} dx - (\mathbb{K}(v)g(v), f(v))_{\mathbb{V}} \\
&= -((f,g),\mathcal{A}^*(f,g))_{\mathcal{H}} - (g(v), \Gamma\mathcal{S}g(v))_{\mathbb{V}} - (\Gamma\mathcal{S}g(v), g(v))_{\mathbb{V}}.
\end{aligned}$$

Therefore, we have

$$\Re(\mathcal{A}^*(w,z),(w,z))_{\mathcal{H}} = -(\Gamma\mathcal{S}g(v), g(v))_{\mathbb{V}}.$$

Also $\mathcal{A}^*$ is dissipative. The Lumer-Phillips Theorem (e.g., see [92]) asserts that $\mathcal{A}$ generates a C_0 semigroup of contraction on $\mathcal{H}$. Hence the system (7.1.3) is well-posed. □

REMARK 7.1.1 *Define the energy function of (7.1.3) by*

$$\begin{aligned}
\mathcal{E}(t) &= \frac{1}{2}||(w(x,t), z(x,t))||^2_{\mathcal{H}} \\
&= \frac{1}{2}\int_0^1 (\mathbb{T}(x)w_x(x,t), w_x(x,t))_{c^n} + (\mathbb{Q}(x)w(x,t), w(x,t))_{c^n} dx \\
&\quad + \frac{1}{2}\int_0^1 (\mathbb{M}(x), z(x,t), z(x,t))_{c^n} dx + \frac{1}{2}(\mathbb{K}(v)w(v,t), w(v,t))_{\mathbb{V}},
\end{aligned}$$

then we have

$$\frac{d\mathcal{E}(t)}{dt} = -(\Gamma\mathcal{S}z(v,t), z(v,t))_{\mathbb{V}} \le 0 \tag{7.1.6}$$

Therefore, for any $t > 0$, it holds the equality

$$\mathcal{E}(t) + \int_0^t (\Gamma\mathcal{S}z(v,t), z(v,t))_{\mathbb{V}} dt = \mathcal{E}(0). \tag{7.1.7}$$

7.2 Asymptotic stability of the system

In this section, we discuss the asymptotic stability of the system (7.1.3). According to Theorem 7.1.1, the system (7.1.3) is dissipative, which implies the spectrum of $\mathcal{A}$ are lying in the left-half complex plane, i.e., $\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} \mid \Re\lambda \le 0\}$. Therefore, by the stability theorem in [70], we need only to discuss whether or not there is a spectral point of $\mathcal{A}$ on the imaginary axis.

7.2.1 $\lambda = 0 \in \rho(\mathcal{A})$

For any $(f, g) \in \mathcal{H}$, we consider the solvability of the equation: $\mathcal{A}(w, z) = (f, g)$, i.e.,

$$\begin{cases} z = f \\ \mathbb{M}^{-1}(x)[(\mathbb{T}(x)w'(x))' - \mathbb{Q}(x)w(x)] = g \\ w(1) = (\Phi^+)^T w(v), \quad w(0) = (\Phi^-)^T w(v) \\ \mathcal{P}\Phi^+\mathbb{T}(1)w'(1) - \mathcal{P}\Phi^-\mathbb{T}(0)w'(0) + \mathbb{K}(v)w(v) = -\Gamma\mathcal{S}z(v). \end{cases} \tag{7.2.1}$$

From the second equation in (7.2.1) we get

$$(\mathbb{T}(x)w'(x))' - \mathbb{Q}(x)w(x) = \mathbb{M}(x)g(x).$$

For any test function $\phi(x) \in C_d(G) \cap H^2(E)$, it holds that

$$\begin{aligned} &\int_0^1 [(\mathbb{T}(x)w'(x)), \phi'(x))_{c^n} + (\mathbb{Q}(x)w(x), \phi(x))_{c^n}]dx - (\mathbb{T}(x)w'(x), \phi(x))_{c^n}\big|_0^1 \\ = \quad & -\int_0^1 (\mathbb{M}(x)g(x), \phi(x))_{c^n} dx. \end{aligned}$$

Note that $\phi(v) \in \mathbb{V}$ and

$$\begin{aligned} (\mathbb{T}(x)w'(x), \phi(x))_{c^n}\big|_0^1 &= (\mathbb{T}(1)w'(1), \phi(1))_{c^n} - (\mathbb{T}(0)w'(0), \phi(0))_{c^n} \\ &= (\Phi^+\mathbb{T}(1)w'(1), \phi(v))_V - (\Phi^-\mathbb{T}(0)w'(0), \phi(v))_{\mathbb{V}} \\ &= (\mathcal{P}[(\Phi^+\mathbb{T}(1)w'(1) - \Phi^-\mathbb{T}(0)w'(0)), \phi(v))_{\mathbb{V}} \\ &= (-\mathbb{K}(v)w(v) - \Gamma\mathcal{S}z(v), \phi(v))_{\mathbb{V}} \end{aligned}$$

and hence

$$\begin{aligned} &\int_0^1 [(\mathbb{T}(x)w'(x)), \phi'(x))_{c^n} + (\mathbb{Q}(x)w(x), \phi(x))_{c^n}]dx + (\mathbb{K}(v)w(v), \phi(v))_V \\ = \quad & -(\Gamma\mathcal{S}f(v), \phi(v))_{\mathbb{V}} - \int_0^1 (\mathbb{M}(x)g(x), \phi(x))_{c^n} dx. \end{aligned} \tag{7.2.2}$$

The bilinear form $B(w, z)$ defined by, $\forall w, z \in C_d(G) \cap H^1(E) = H_e^1(G)$

$$B(w, z) = \int_0^1 [(\mathbb{T}(x)w'(x)), z'(x))_{c^n} + (\mathbb{Q}(x)w(x), z(x))_{c^n}]dx + (\mathbb{K}(v)w(v), z(v))_{\mathbb{V}}$$

is coercive and bounded, this is because $B(w, w) = \|w\|^2_{H^1(G)}$ can define a norm on $H_e^1(G)$ and $|B(w, z)| \leq \|w\|_{H_e^1(G)} \cdot \|z\|_{H_e^1(G)}$. The Lax-Milgram's Theorem asserts that there exists unique a $w(x) \in H_e^2(G)$ satisfying (7.2.2) and hence satisfying (7.2.1). Therefore, $(w, z) = (w, f) \in D(\mathcal{A})$ and $\mathcal{A}(w, z) = (f, g)$. So $0 \in \rho(\mathcal{A})$.

Observe that $D(\mathcal{A}) \subset H^2(E) \times H^1(E)$, the Sobolev Embedding Theorem ensures that $\mathcal{A}^{-1}$ is a compact operator on $\mathcal{H}$. Therefore, we have the following result.

THEOREM 7.2.1 *Let $\mathcal{A}$ be defined by (7.1.1) (7.1.2). Then $0 \in \rho(\mathcal{A})$, and $\mathcal{A}^{-1}$ is compact. Hence the spectrum of $\mathcal{A}$ consists of all isolated eigenvalues of finite multiplicity.*

7.2.2 Eigenvalue problem

In this subsection, we concentrate our attention on the eigenvalue problem of $\mathcal{A}$. It is easy to see that the eigenvalue problem of $\mathcal{A}$ is equivalent to the following boundary eigenvalue problem:

$$\begin{cases} \lambda^2 \mathbb{M}(x) f(x) = (\mathbb{T}(x) f'(x))' - \mathbb{Q}(x) f(x), \quad g(x) = \lambda f(x), \quad x \in (0,1) \\ \exists f(v) \in \mathbb{V} \; s.t. f(1) = (\Phi^+)^T f(v), f(0) = (\Phi^-)^T f(v) \\ \mathcal{P}\Phi^+ \mathbb{T}(1) f'(1) - \mathcal{P}\Phi^- \mathbb{T}(0) f'(0) + \mathbb{K}(v) f(v) = -\lambda \Gamma \mathcal{S} f(v). \end{cases} \tag{7.2.3}$$

For each $j \in \{1, 2, \cdots, n\}$, let $s_j(x,\lambda)$ and $c_j(x,\lambda)$ be the solutions to differential equation

$$\lambda^2 m_j(x) w_j(x) = (T_j(x) w_j'(x))' - q_j(x) w_j(x), \quad x \in (0,1)$$

with $s_j(0,\lambda) = 0, s_j'(0,\lambda) = 1$ and $c_j(0,\lambda) = 1, c_j'(0,\lambda) = 0$, respectively. Obviously they are the entire functions of finite exponential type with respect to λ (see, [85]).

Define the diagonal matrices by

$$\begin{aligned} S(x,\lambda) &= \mathrm{diag}(s_1(x,\lambda), s_2(x,\lambda), \cdots, s_n(x,\lambda)), \\ C(x,\lambda) &= \mathrm{diag}(c_1(x,\lambda), c_2(x,\lambda), \cdots, c_n(x,\lambda)), \end{aligned} \tag{7.2.4}$$

Obviously, $S(x,\lambda)$ and $C(x,\lambda)$ satisfy the differential equation in (7.2.3). Thus the general solution to the differential equation in (7.2.3) is of the form

$$f(x) = C(x,\lambda) f(0) + S(x,\lambda) f'(0),$$

so

$$f'(x) = C'(x,\lambda) f(0) + S'(x,\lambda) f'(0).$$

Substituting them into the boundary conditions in (7.2.3) lead to $f(0) = (\Phi^-)^T f(v), f(v) \in \mathbb{V}$ and

$$\begin{aligned} & f(1) = (\Phi^+)^T f(v) = C(1,\lambda)(\Phi^-)^T f(v) + S(1,\lambda) f'(0) \\ & \mathcal{P}\Phi^+ \mathbb{T}(1)[C'(1,\lambda)(\Phi^-)^T f(v) + S'(1,\lambda) f'(0)] - \mathcal{P}\Phi^- \mathbb{T}(0)[f'(0)] + \mathbb{K}(v) f(v) = -\lambda \Gamma \mathcal{S} f(v) \end{aligned}$$

i.e.,

$$\begin{cases} [C(1,\lambda)(\Phi^-)^T - (\Phi^+)^T] f(v) + S(1,\lambda) f'(0) = 0 \\ \mathcal{P}[\Phi^+ \mathbb{T}(1) C'(1,\lambda)(\Phi^-)^T + \mathbb{K}(v) + \lambda \Gamma \mathcal{S}] f(v) + \mathcal{P}[\Phi^+ \mathbb{T}(1) S'(1,\lambda) - \Phi^- \mathbb{T}(0)] f'(0) = 0. \end{cases} \tag{7.2.5}$$

Let $k = \dim \mathbb{V}$ and $\mathbb{U}$ be isomorphic mapping between $\mathbb{V}$ and $\mathbb{C}^k$. Then there exists $d \in \mathbb{C}^k$ such that $\mathbb{U} f(v) = d$. Hence

$$\begin{cases} [C(1,\lambda)(\Phi^-)^T - (\Phi^+)^T] \mathbb{U}^{-1} d + S(1,\lambda) f'(0) = 0 \\ \mathbb{U}\mathcal{P}[\Phi^+ \mathbb{T}(1) C'(1,\lambda)(\Phi^-)^T + \mathbb{K}(v) + \lambda \Gamma \mathcal{S}] \mathbb{U}^{-1} d + \mathbb{U}\mathcal{P}[\Phi^+ \mathbb{T}(1) S'(1,\lambda) - \Phi^- \mathbb{T}(0)] f'(0) = 0. \end{cases} \tag{7.2.6}$$

Obviously, the above algebraic equations has a nonzero solution if and only if the determinant of the coefficients matrix vanishes, i.e., $\Delta(\lambda)=0$, where $\Delta(\lambda)$ is defined by

$$\Delta(\lambda)=\det\begin{vmatrix} \mathbb{U}\mathcal{P}[\Phi^+\mathbb{T}(1)C'(1,\lambda)(\Phi^-)^T+\mathbb{K}(v)+\lambda\Gamma\mathcal{S}]\mathbb{U}^{-1} & \mathbb{U}\mathcal{P}[\Phi^+\mathbb{T}(1)S'(1,\lambda)-\Phi^-\mathbb{T}(0)] \\ [C(1,\lambda)(\Phi^-)^T-(\Phi^+)^T]\mathbb{U}^{-1} & S(1,\lambda) \end{vmatrix}. \tag{7.2.7}$$

Thus, one has the following result.

THEOREM 7.2.2 *Let $\mathcal{A}$ be defined as (7.1.1) and (7.1.2). Then we have*

$$\sigma(\mathcal{A})=\{\lambda\in\mathbb{C}\mid\Delta(\lambda)=0\}$$

where $\Delta(\lambda)$ is defined by (7.2.7).

7.2.3 Criterion on asymptotic stability

In this subsection we analyze the asymptotic stability of the system (7.1.3). Firstly we introduce an auxiliary operator $\mathcal{A}_0$ defined by

$$\mathcal{A}_0(w,z)=(z,\mathbb{M}^{-1}(x)[(\mathbb{T}(x)w'(x))'-\mathbb{Q}(x)w(x)]),\quad (w,z)\in D(\mathcal{A}_0) \tag{7.2.8}$$

with domain

$$\begin{aligned} D(\mathcal{A}_0)=\{(w,z)\in H^2(E)\times H_e^1(G)\mid w(v),z(v)\in\mathbb{V}\\ \mathcal{P}\Phi^+\mathbb{T}(1)w'(1)-\mathcal{P}\Phi^-\mathbb{T}(0)w'(0)+\mathbb{K}(v)w(v)=0\}. \end{aligned} \tag{7.2.9}$$

Obviously, $\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$.

Define the vertex observation operator $\widehat{\mathcal{S}}$ from $\mathcal{H}$ to $\mathbb{V}$ by

$$\widehat{\mathcal{S}}(f,g)=\mathcal{S}g(v),\quad \forall(f,g)\in H_e^1(G)\times C_d(G). \tag{7.2.10}$$

With these operators we can prove the following asymptotic stability result.

THEOREM 7.2.3 *Let $\mathcal{A}_0$ be defined by (7.2.8) and (7.2.9) and $\widehat{\mathcal{S}}:D(\widehat{\mathcal{S}})\to\mathbb{V}$ be the observation operator defined by (7.2.10). Then the system (7.1.3) is asymptotically stable if and only if*

$$\mathcal{N}(\lambda I-\mathcal{A}_0)\cap\mathcal{N}(\widehat{\mathcal{S}})=\{\theta\},\quad \forall\lambda\in\mathbb{C}. \tag{7.2.11}$$

Proof Since $\mathcal{A}$ is a dissipative operator, and Theorem 7.2.1 says that $\sigma(\mathcal{A})=\sigma_p(\mathcal{A})$, so the system (7.1.3) is asymptotically stable if and only if there is no eigenvalue of $\mathcal{A}$ on the imaginary axis according to the stability theorem [70].

We prove that there is an eigenvalue of $\mathcal{A}$ on the imaginary axis, i.e., $\sigma(\mathcal{A})\cap i\mathbb{R}\neq\emptyset$, if and only if

$$\mathcal{N}(\lambda I-\mathcal{A}_0)\cap\mathcal{N}(\widehat{\mathcal{S}})\neq\{\theta\}.$$

In fact, suppose that there are a $\lambda=ik,k\in\mathbb{R}$ and a nonzero vector $(w,z)\in D(\mathcal{A})$ such that $\mathcal{A}(w,z)=\lambda(w,z)$, then by the dissipation of $\mathcal{A}$, we have

$$\Re\lambda||(w,z)||^2=\Re\langle\mathcal{A}(w,z),(w,z)\rangle_{\mathcal{H}}=-(\Gamma\mathcal{S}z(v),z(v))_{\mathbb{V}}\le 0$$

which shows that $\mathcal{S}z(v) = 0$, i.e., $(w, z) \in \mathcal{N}(\widehat{\mathcal{S}})$. By the definition of $D(\mathcal{A}_0)$ in (7.2.9), we see that $(w, z) \in D(\mathcal{A}_0)$ and $\mathcal{A}_0(w, z) = \lambda(w, z)$. Therefore

$$(w, z) \in \mathcal{N}(\lambda I - \mathcal{A}_0) \cap \mathcal{N}(\widehat{\mathcal{S}}) \neq \{\theta\}.$$

Conversely, if there is a $\lambda \in \mathbb{C}$ such that $\mathcal{N}(\lambda I - \mathcal{A}_0) \cap \mathcal{N}(\widehat{\mathcal{S}}) \neq \{\theta\}$, then from $(\lambda I - \mathcal{A}_0)(w, z) = 0$ we get that $\lambda \in i\mathbb{R}$, this together with $\widehat{\mathcal{S}}(w, z) = \mathcal{S}z(v) = 0$ gives that $(w, z) \in D(\mathcal{A})$ and $(\lambda I - \mathcal{A})(w, z) = 0$. Therefore $\lambda \in \sigma(\mathcal{A}) \cap i\mathbb{R}$. The proof is then complete. □

7.3 Geometric method in analysis of asymptotic stability

In this section we explore a geometric approach in analysis of asymptotic stability of the system (7.1.3). From the proof of Theorem 7.2.3 we can see that, in order to check the condition (7.2.11), we only need to check it for $\lambda \in \sigma(\mathcal{A}_0)$. It is equivalent to check whether or not there exists a nonzero vector $(w(x), \lambda w(x)) \in D(\mathcal{A}_0)$, where $w(x)$ is a solution of the following second-order differential equations

$$\begin{cases} \lambda^2 m_j(x) w_j(x) + q_j(x) w_j(x) = (T_j(x) w_j'(x))' \\ w_j(1) = w_i(0) = w(a), \quad \forall j \in J^+(a), i \in J^-(a), \quad a \in V, \\ \sum_{j\in J^+(a)} T_j(1) w_j'(1) - \sum_{i \in J^-(a)} T_i(0) w_i'(0) + k(a) w(a) = 0, a \in V \backslash \partial G_D \\ w_j(1) = w_i(0) = w(a) = 0, \quad \forall j \in J^+(a), i \in J^-(a), a \in \partial G_D \\ T_j(1) w_j'(1) + k(a) w_j(a) = 0, \quad j \in J^+(a), \quad a \in \partial G \backslash \partial G_D \\ w(a) = 0, \quad a \in V_o. \end{cases} \tag{7.3.1}$$

where V_o denotes the set of all observation vertices, which are also the controlled vertices. Obviously if (7.3.1) has a nonzero solution, then there is at least an eigenvalue of $\mathcal{A}$ on the imaginary axis; if (7.3.1) has uniquely an zero solution, then there is no eigenvalue of $\mathcal{A}$ on the imaginary axis.

Observe that boundary eigenvalue problem (7.3.1) is defined on the graph G. We can reduce it into a boundary eigenvalue problem on a subgraph $G_1 \subset G$. The subgraph G_1 is obtained by cutting out all the controlled boundary edges and the resulted edges satisfying condition $w(a) = w_j'(1) = w_i'(0) = 0, j \in J^+(a), i \in J^-(a)$.

For the resulted subgraph $G_1 = (E_1, V_1)$, the Dirichlet vertex set V_{1d} consists of all such vertices $a \in V$ if $a \in V$ is an endpoint of the cut off edge, or $a \in \partial G_D \cup V_o$. The corresponding boundary eigenvalue problem of the second-order differential equations on G_1 is

$$\begin{cases} \lambda^2 m_j(x) w_j(x) + q_j(x) w_j(x) = (T_j(x) w_j'(x))', \quad x \in (0,1) \\ w_j(1) = w_i(0) = w(a), \quad \forall j \in J^+(a), i \in J^-(a), \quad a \in V_1, \\ \sum_{j\in J^+(a)} T_j(1) w_j'(1) - \sum_{i \in J^-(a)} T_i(0) w_j'(0) + k(a) w(a) = 0, a \in V_{1,int} \\ T_j(1) w_j'(1) + k(a) w_j(a) = 0, \quad j \in J^+(a), \quad a \in \partial G \backslash \partial G_d \\ w(a) = 0, \quad a \in V_{1d}. \end{cases} \tag{7.3.2}$$

Clearly, (7.3.1) has a nonzero solution if and only if (7.3.2) has.

Now we divide the resulted subgraph G_1 into several simpler subgraph G_{1j}, $j = 1, 2, \cdots, k$ such that $G_{1j} \cap G_{1i} \subset V_{1d}$, and then discuss the eigenvalue problem on each G_{1j}.

Let the second-order differential operator on each subgraph G_{1j} have eigenvalue set σ_{1j} for the Dirichlet Problem. Here we emphasize that we only take the Dirichlet boundary at the Dirichlet vertices.

Obviously if $\cap_{j=1}^{k} \sigma_j \neq \emptyset$, then the system is unstable. However, $\cap_{j=1}^{k} \sigma_j = \emptyset$ is not a sufficient condition for system stable. In this case, we process the subgraph G_1 according the following procedure:

1) If there are two indices i and j such that $\sigma_{1i} \cap \sigma_{1j} = \emptyset$, then we divide the subgraph G_1 into subgraph pairs

$$G_1 \backslash G_{1i} \sim G_{1i}, \quad \text{and} \quad (G_1 \backslash G_{1j} \sim G_{1j})$$

and take zero value of Neumann-Kirchhoff condition of G_{1i} and $G_1 \backslash G_{1i}$ respectively at the intersection vertices (correspondingly, G_{1j} and $G_1 \backslash G_{1j}$).

2) The resulted subgraph, for instance, $G_1 \backslash G_{1j}$, together with the Neumann-Kirchhoff condition at the intersection vertices composes a new eigenvalue problem. If one can see obviously that the eigenvalue problem has uniquely a zero solution, one can cut out this subgraph. If not, we repeat the first step and compare reminder σ_{1i} and σ_{1k}.

3) If for any i and j, it holds that $\sigma_i \cap \sigma_j \neq \emptyset$, then we can divide the subgraph G_1 into subgraph sequence $G_1 \backslash G_{1j}, j = 1, 2, \cdots, k$ since $\cap_{j=1}^{k} \sigma_j = \emptyset$, and take the zero value of Neumann-Kirchhoff condition of G_{1j} at the intersection vertices.

4) If for some $G_1 \backslash G_{1k}$ one can see obviously that the eigenvalue problem has no nonzero solution, one can cut out this subgraph from the subgraph sequence. If not, we repeat the third step for each subgraph $G_1 \backslash G_{1j}$ or the first step.

In this manner, if all edges of G are cut out, then the system is asymptotically stable. Otherwise, the system is unstable.

Based on previous analysis, we get an approach of analyzing stability–**Geometric method**, this is achieved by deleting edge (or subgraph). This approach can be used to analyze more complicated networks involving un-normalized networks.

To explain how to reduce a graph, we give two examples.

EXAMPLE 7.3.1 *Let G be a tree-shaped graph, whose structure is shown in Fig. 7.3.1*

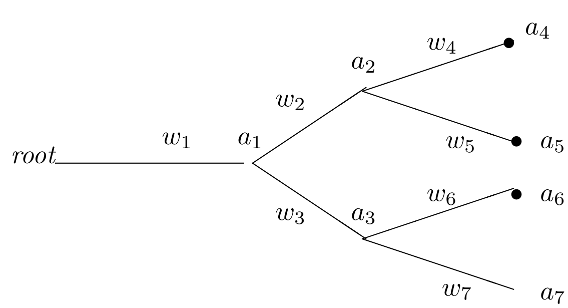

Fig 7.3.1. Tree shaped graph with controllers at a_4, a_5 and a_6

Consider a continuous network of strings on G

$$\begin{cases} m_j(x)w_{j,tt}(x,t) = (T_j(x)w_{j,x}(x,t))_x - q_j(x)w_j(x,t), \quad x \in (0,1) \\ w_1(0,t) = 0, \quad w_1(1,t) = w_2(0,t) = w_3(0,t) = w(a_1,t), \\ T_1(1)w_{1,x}(1,t) - T_2(0)w_{2,x}(0,t) - T_3(0)w_{3,x}(0,t) = 0 \\ w_2(1,t) = w_4(0,t) = w_5(0,t) = w(a_2,t), \\ T_2(1)w_{2,x}(1,t) - T_4(0)w_{4,x}(0,t) - T_5(0)w_{5,x}(0,t) = 0 \\ w_3(1,t) = w_6(0,t) = w_7(0,t) = w(a_3,t), \\ T_3(1)w_{3,x}(1,t) - T_6(0)w_{6,x}(0,t) - T_7(0)w_{7,x}(0,t) = 0 \\ T_j(1)w_{j,x}(1,t) = -\alpha_j w_{j,t}(1,t), \quad \alpha_j > 0, \quad j = 4,5,6 \\ T_7(1)w_{7,x}(1,t) = 0, \\ w_j(x,0) = w_{j0}(x), \quad w_{j,t}(x,0) = w_{j1}(x). \end{cases} \tag{7.3.3}$$

Step 1. The asymptotic stability is equivalent to verification of the following differential equations having no nonzero solution

$$\begin{cases} \lambda^2 m_j(x)w_j(x) = (T_j(x)w_{j,x}(x))_x - q_j(x)w_j(x), \quad x \in (0,1) \\ w_1(0) = 0, \quad w_1(1) = w_2(0) = w_3(0) = w(a_1), \\ T_1(1)w_{1,x}(1) - T_2(0)w_{2,x}(0) - T_3(0)w_{3,x}(0) = 0 \\ w_2(1) = w_4(0) = w_5(0) = w(a_2), \\ T_2(1)w_{2,x}(1) - T_4(0)w_{4,x}(0) - T_5(0)w_{5,x}(0) = 0 \\ w_3(1) = w_6(0) = w_7(0) = w(a_3), \\ T_3(1)w_{3,x}(1) - T_6(0)w_{6,x}(0) - T_7(0)w_{7,x}(0) = 0 \\ T_j(1)w_{j,x}(1) = 0, \quad w_j(1) = 0, \quad j = 4,5,6 \\ T_7(1)w_{7,x}(1) = 0. \end{cases} \tag{7.3.4}$$

Step 2. Cut out the boundary edges w_4, w_5 and w_6 from G. Since $w_{2,x}(1) = w_2(1) = 0$ at a_2, one can cut out w_2 again. So the resulted subgraph is shown in Fig.7.3.2, in which $V_{1d} = \{\text{root}, a_1, a_3\}$

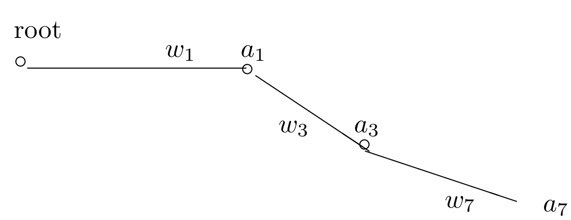

Fig 7.3.2. Resulted subgraph with Dirichlet vertices: the root, a_1, a_3

Then the asymptotic stability is equivalent to verification of the following differential equations having no nonzero solution

$$\begin{cases} \lambda^2 m_j(x)w_j(x) = (T_j(x)w_{j,x}(x))_x - q_j(x)w_j(x), \quad x \in (0,1), j = 1,3,7 \\ w_1(0) = w_1(1) = w_3(0) = w_3(1) = w_7(0) = 0, \\ T_1(1)w_{1,x}(1) - T_3(0)w_{3,x}(0) = 0 \\ T_3(1)w_{3,x}(1) - T_7(0)w_{7,x}(0) = 0 \\ T_7(1)w_{7,x}(1) = 0. \end{cases} \tag{7.3.5}$$

Step 3. Divide the graph G_1 into three subgraph G_{11}, G_{12} and G_{13}, where G_{11}, G_{12} and G_{13} consist only one of w_1,w_3 and w_7, respectively.

Let $s_j(x,\lambda)$ be unique a solution to equation

$$\begin{cases} \lambda^2 m_j(x)w_j(x) = (T_j(x)w_{j,x}(x))_x - q_j(x)w_j(x), \quad x \in (0,1), \\ w_j(0) = 0, \quad w_j'(0) = 1. \end{cases}$$

For G_{11}, the eigenvalue problem is

$$\begin{cases} \lambda^2 m_1(x)w_1(x) = (T_1(x)w_{1,x}(x))_x - q_1(x)w_1(x), \quad x \in (0,1), \\ w_1(0) = w_1(1) = 0. \end{cases}$$

The eigenvalues are gives by

$$\sigma_1 = \{\lambda \in \mathbb{C} \mid s_1(1,\lambda) = 0\}.$$

For G_{12}, the eigenvalue problem is

$$\begin{cases} \lambda^2 m_3(x)w_3(x) = (T_3(x)w_{3,x}(x))_x - q_3(x)w_3(x), \quad x \in (0,1), \\ w_3(0) = w_3(1) = 0. \end{cases}$$

The eigenvalues are gives by

$$\sigma_2 = \{\lambda \in \mathbb{C} \mid s_2(1,\lambda) = 0\}.$$

For G_{13}, the eigenvalue problem is

$$\begin{cases} \lambda^2 m_7(x)w_7(x) = (T_7(x)w_{7,x}(x))_x - q_7(x)w_7(x), \quad x \in (0,1), \\ w_7(0) = 0, \quad T_7(1)w_{7,x}(1,t) = 0. \end{cases}$$

The eigenvalues are gives by

$$\sigma_3 = \{\lambda \in \mathbb{C} \mid s_7'(1,\lambda) = 0\}.$$

Obviously, if $\cap_{j=1}^3 \sigma_j \neq \emptyset$, then the system is unstable. If $\cap_{j=1}^3 \sigma_j = \emptyset$, we consider the following cases.

1) If $\sigma_1 \cap \sigma_2 = \emptyset$, then one cut out edge w_1, and takes $w_{1,x}(1) = 0$. The boundary condition in (7.3.5) yields $w_{3,x}(0) = 0$. This together with $w_3(0) = 0$ leads to $w_3(x) \equiv 0$, so one can cut out edge w_3. For same reason one can cut out edge w_7.

2) If $\sigma_1 \cap \sigma_2 \neq \emptyset$, $\sigma_1 \cap \sigma_3 \neq \emptyset$ and $\sigma_2 \cap \sigma_3 \neq \emptyset$, we divide the graph G_1 into the subgraph sequence

$$G_1 \backslash G_{11}, \quad G_1 \backslash G_{12}, \quad G_1 \backslash G_{13}$$

and take the zero value of Neumann-Kirchhoff condition of G_{1j} at the intersection vertices.

Obviously, the eigenvalue problem on each subgraph $G_1 \backslash G_{1j}$ has no nonzero solution. We can cut out the subgraph $G_1 \backslash G_{11}$, $G_1 \backslash G_{12}$ and $G_1 \backslash G_{13}$ from the subgraph sequence. By now we cut out all edges from G. Therefore we have achieved the following result.

THEOREM 7.3.1 *If $\bigcap_{j=1}^{3} \sigma_j = \emptyset$, then the system (7.3.3) is asymptotically stable.*

EXAMPLE 7.3.2 *Let us consider a graph G with multi-circuit as shown in Fig. 7.3.3, with boundary controller at a_2 and interior controllers at a_4, a_6, a_8, a_{10} and a_{12}*

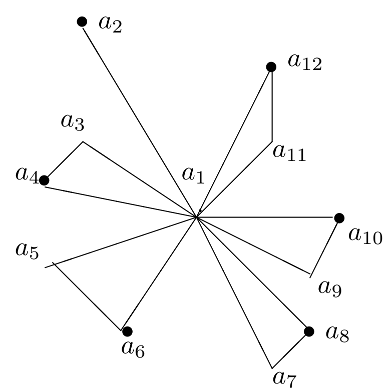

Fig. 7.3.3. A planar network with multi-circuit

The direct edges are defined by

$$\begin{array}{llll}
e_1 = (a_1, a_2), & e_2 = (a_1, a_3), & e_3 = (a_3, a_4), & e_4 = (a_4, a_1), \\
e_5 = (a_1, a_5), & e_6 = (a_6, a_1), & e_7 = (a_1, a_7), & e_8 = (a_8, a_1), \\
e_9 = (a_1, a_9), & e_{10} = (a_{10}, a_1), & e_{11} = (a_1, a_{11}), & e_{12} = (a_{12}, a_1) \\
e_{13} = (a_5, a_6), & e_{14} = (a_7, a_8), & e_{15} = (a_9, a_{10}), & e_{16} = (a_{11}, a_{12})
\end{array}$$

their parameterization x have same direction.

Let $w_j(x,t)$ be defined on e_j and satisfy the differential equations

$$m_j(x) w_{j,tt}(x,t) = (T_j(x) w_{j,x}(x,t))_x - q_j(x) w_j(x,t), \quad x \in (0,1)$$

and the boundary condition

$$T_1(1) w_{1,x}(1,t) + k_1 w(a_2,t) = -\alpha_1 w_t(a_2,t)$$

and the connection conditions: geometric conditions

$$w(a,t) = w_i(0,t) = w_j(1,t), j \in J^+(a), i \in J^-(a), a \neq a_2.$$

and dynamic conditions

$$\sum_{j\in J^+(a)} T_j(1)w_{j,x}(1,t) - \sum_{i\in J^-(a)} T_i(0)w_{i,x}(0,t) = -\alpha(a)w_t(a,t), \quad a \notin \{a_3, a_5, a_7, a_9, a_{11}\}$$

and

$$\sum_{j\in J^+(a)} T_j(1)w_{j,x}(1,t) - \sum_{i\in J^-(a)} T_i(0)w_{i,x}(0,t) = 0, \quad a \in \{a_3, a_5, a_7, a_9, a_{11}\}$$

Similar to discussion in Example 7.3.1, firstly we can delete the boundary edge e_1 from G. The resulted subgraph G_1 is shown as in Fig.7.3.4

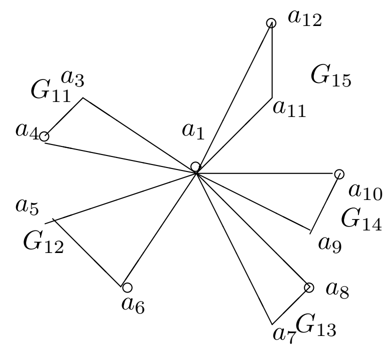

Fig. 7.3.4. The resulted subgraph G_1 with multi-subgraphs G_{1j}

Obviously, the Dirichlet vertex set is $V_{1d} = \{a_1, a_4, a_6, a_8, a_{10}, a_{12}\}$. G_1 has a subgraph sequence $\{G_{1j}\}_{j=1}^{5}$.

We are now in a position to solve the boundary eigenvalue problem on each subgraph G_{1j}. Here we reemphasize that we only take the Dirichlet boundary at the Dirichlet vertices. For example, G_{11}, corresponding Dirichlet boundary eigenvalue problem is

$$\begin{cases} \lambda^2 m_k(x)w_k(x) = (T_k(x)w_{k,x}(x))_x - q_k(x)w_k(x), \quad x \in (0,1), k = 2,3,4, \\ w_2(0) = w_4(1) = 0, \quad w_3(1) = w_4(0) = 0, \quad w_2(1) = w_3(0) \\ T_2(1)w_{2,x}(1) - T_3(0)w_{3,x}(0) = 0. \end{cases}$$

In fact, we can easily determine σ_j. We divide the subgraph G_{1j} into two subgraph $G_{1j,1}$, $G_{1j,2}$, where

$$G_{1j,1} = (a_1, a_{2j+2}), \quad G_{1j,2} = (a_1, a_{2j+1}) \cup (a_{2j+1}, a_{2j+2}).$$

Let $\sigma_{j,k}$ be eigenvalue set of the Dirichlet Problem corresponding to $G_{1j,k}$. Then we have $\sigma_j = \sigma_{j,1} \cap \sigma_{j,2}$, which is called the Dirichlet spectrum.

Let σ_j be the Dirichlet spectrum corresponding to subgraph G_{1j}, $j = 1,2,3,4,5$. If there is some $\sigma_k = \emptyset$, we can remove this subgraph from G_1. Further we compare the nonempty $\{\sigma_j\}$. According to the structure of subgraph G_1, we discuss the stability as follows:

1) If there are i and j such that $\sigma_i \cap \sigma_j \neq \emptyset$, then the system is unstable.

For example, $\sigma_1 \cap \sigma_5 \neq \emptyset$. We prove this assertion by three steps.

Step 1. For G_{11} and $\lambda \in \sigma_1 \cap \sigma_5$, one constructs a function $W_1(x, \lambda)$ satisfies boundary eigenvalue problem on G_{11} with Dirichlet boundary at a_1.

Suppose that $w(x, \lambda)$ is a nonzero solution on $(a_1, a_3) \cup (a_3, a_4)$ with Dirichlet boundary a_1 and a_4, $w_4(x, \lambda)$ is a nonzero solution on (a_4, a_1) with Dirichlet boundary at both ends. Define the function by

$$W_1(x, \lambda) = \begin{cases} w(x, \lambda), & x \in (a_1, a_3) \cup (a_3, a_4) \\ \gamma w_4(x, \lambda), & x \in (a_4, a_1) \end{cases}$$

where γ is a parameter. We can choose γ such that $W_1(x, \lambda)$ satisfy the dynamic condition at a_4

$$T_3(1) w_3'(1, \lambda) - T_4(0) w_4'(0, \lambda) = 0.$$

For simplicity, we define

$$\frac{\partial W_1(x, \lambda)}{\partial \vec{n}} = \gamma T_4(1) w_4'(1) - T_2(0) w_(' 0)$$

Similarly, for G_{15} one constructs the function $W_5(x, \lambda)$ and define $\frac{\partial W_2}{\partial \vec{n}}$.

Step 2. Choose a parameter β such that

$$\frac{\partial W_1(x, \lambda)}{\partial \vec{n}} + \beta \frac{\partial W_2(x, \lambda)}{\partial \vec{n}} = 0.$$

Step 3. For $\lambda \in \sigma_1 \cap \sigma_5$, one constructs a nonzero function $W(x, \lambda)$ defined on G_1 satisfying the boundary eigenvalue problem on G_1.

Set

$$W(x, \lambda) = \begin{cases} 0, & x \in G_{12} \cup G_{13} \cup G_{14} \\ W_1(x, \lambda), & x \in G_{11} \\ \beta W_2(x, \lambda), & x \in G_{15} \end{cases}$$

then $W(x, \lambda)$ satisfies all conditions. Clearly, the system is unstable

2) If $\sigma_j \cap \sigma_i = \emptyset$ for any i and j, we have subgraph sequence $G_{1j}, j = 1, 2, 3, 4, 5$. We solve the boundary eigenvalue problem on G_{1j}.

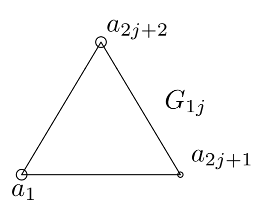

Fig 7.3.5. Resulted subgraph sequence with Dirichlet vertices a_1, a_{2j+2}

For example, on G_{11}, corresponding boundary eigenvalue problem is

$$\begin{cases} \lambda^2 m_k(x)w_k(x) = (T_k(x)w_{k,x}(x))_x - q_k(x)w_k(x), \quad x \in (0,1), k = 2,3,4, \\ w_2(0) = w_4(1) = 0, \quad w_3(1) = w_4(0) = 0, \quad w_2(1) = w_3(0) \\ T_2(1)w_{2,x}(1) - T_3(0)w_{3,x}(0) = 0, \\ T_3(1)w_{3,x}(1) - T_4(0)w_{4,x}(0) = 0, \\ T_4(1)w_{4,x}(1) - T_2(0)w_{2,x}(0) = 0. \end{cases}$$

If for any i, the boundary eigenvalue problem on G_{1j} has no nonzero solution, then the system is asymptotically stable. Otherwise, the system is unstable.

by now we have proved the following result.

THEOREM 7.3.2 *If $\sigma_i \cap \sigma_j = \emptyset, \forall i \neq j \in \{,1,2,3,4,5\}$ and each boundary eigenvalue problem on G_{1j} has no nonzero solution, then the system is asymptotically stable. Otherwise, the system is unstable.*

REMARK 7.3.1 *In example 7.3.2, we distinguish the boundary eigenvalue problem on G_{1j} from the Dirichlet boundary eigenvalue problem. Denote $\widehat{\sigma}_j$ the set of eigenvalue for the boundary eigenvalue problem on G_{1j}. Obviously, $\widehat{\sigma}_j \subset \sigma_j$. Usually, $\widehat{\sigma}_j = \emptyset$, but $\sigma_j \neq \emptyset$.*

Here we give an example to show that $\widehat{\sigma}_j = \emptyset$, but $\sigma_j \neq \emptyset$.

EXAMPLE 7.3.3 *We consider the boundary eigenvalue problem on a triangle-circuit shown as in Fig.7.3.6.*

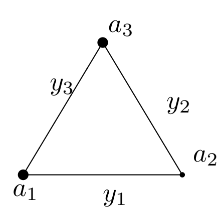

Fig 7.3.6. Triangle-circuit with Dirichlet vertices a_1 and a_3

$$\begin{cases} \lambda^2\pi^2 y_3(x) = y_3''(x) - \pi^2 y(x), \quad x \in (0,1), \\ y_3(0) = y_3(1) = 0 \\ \lambda^2 2\pi^2 y_k(x) = y_k''(x), \quad x \in (0,1), k = 1,2, \\ y_1(0) = y_2(1) = 0, \quad y_1(1) = y_2(0), \quad y_1'(1) = y_2'(0), \\ y_2'(1) - y_3'(0) = 0, \\ y_3'(1) - y_1'(0) = 0. \end{cases} \tag{7.3.6}$$

Firstly, we calculate the Dirichlet spectrum.

The Dirichlet spectrum determined by y_3 is

$$\sigma_3 = \{\lambda_n = \pm i\sqrt{n^2+1} \mid n \in \mathbb{N}\}$$

and the Dirichlet spectrum determined by y_1 and y_2 is

$$\sigma_{12} = \{\lambda = \pm\frac{ik\sqrt{2}}{2} \mid k \in \mathbb{N}\}.$$

Therefore, the Dirichlet spectrum of the system is

$$\sigma_3 \cap \sigma_{12} = \{\pm\sqrt{2}\}$$

and the functions corresponding $\lambda = \pm\sqrt{2}$ are

$$y_3(x) = \beta \sin \pi x, \quad y_1(x) = y_2(x) = \sin 2\pi x, \quad x \in (0,1).$$

But the boundary eigenvalue problem (7.3.6) has only zero solution. This is because the condition $y_2'(1) = y_3'(0)$ requires $\beta = 2$:

$$2\pi \cos 2\pi = \beta\pi$$

while $y_3'(1) = y_1'(0)$ requires $\beta = -2$:

$$\beta\pi \cos \pi = 2\pi.$$

Therefore, $\widehat{\sigma} = \emptyset$. □

7.4 Continuous networks and their equivalent forms

7.4.1 Variable coefficient equation and its equivalent form

Let us consider variable coefficients equation

$$m(x)w_{tt}(x,t) = (T(x)w_x(x,t))_x - q(x)w(x,t), \quad x \in (0,1) \tag{7.4.1}$$

whose energy function is

$$E(t) = \frac{1}{2}\int_0^1 [T(x)|w_x(x,t)|^2 + q(x)|w(x,t)|^2 + m(x)|w_t(x,t)|^2]dx$$

Set

$$\xi = \xi(x) = \int_0^x \sqrt{\frac{m(r)}{T(r)}}dr, \quad \ell = \int_0^1 \sqrt{\frac{m(r)}{T(r)}}dr.$$

Then

$$x'(\xi) = \frac{dx}{d\xi} = \sqrt{\frac{T(x)}{m(x)}}$$

and

$$\frac{d\ln x'(\xi)}{d\xi} = \frac{x''(\xi)}{x'(\xi)} = \frac{1}{2}\left[\frac{T'(x)}{T(x)} - \frac{m'(x)}{m(x)}\right]x'(\xi)$$

where $x(\xi)$ is the inverse function of $\xi(x)$.

We change the expression form of the energy function

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 [T(x)|w_x(x,t)|^2 + q(x)|w(x,t)|^2 + m(x)|w_t(x,t)|^2]dx \\ &= \frac{1}{2}\int_0^1 \left[|\sqrt{T(x)}w_x(x,t)|^2 + \frac{q(x)}{T(x)}|\sqrt{T(x)}w(x,t)|^2 + \frac{m(x)}{T(x)}|\sqrt{T(x)}w_t(x,t)|^2\right]dx. \end{aligned}$$

Introduce a function $\sqrt{T(x)}w(x,t) = y(x,t)$, then we have

$$\sqrt{T(x)}w_x(x,t) = y_x(x,t) - \frac{T'(x)}{2T(x)}y(x,t),$$

and

$$E(t) = \frac{1}{2}\int_0^1 \left[|y_x(x,t) - \frac{T'(x)}{2T(x)}y(x,t)|^2 + \frac{q(x)}{T(x)}|y(x,t)|^2 + \frac{m(x)}{T(x)}|y_t(x,t)|^2\right]dx$$

Define a new function

$$v(\xi,t) = \frac{1}{\sqrt{x'(\xi)}}y(x(\xi),t), \quad \xi \in (0,\ell)$$

then

$$y_x(x,t) = \frac{1}{\sqrt{x'(\xi)}}[v_\xi(\xi,t) + \frac{1}{4}(\frac{T'(x)}{T(x)} - \frac{m'(x)}{m(x)})x'(\xi)v(\xi,t)]$$

and

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^\ell \left|\frac{1}{\sqrt{x'(\xi)}}[v_\xi(\xi,t) + \frac{1}{4}(\frac{T'(x)}{T(x)} - \frac{m'(x)}{m(x)})x'(\xi)v(\xi,t)] - \frac{T'(x)}{2T(x)}\sqrt{x'(\xi)}v(\xi,t)\right|^2 x'(\xi)d\xi \\ &\quad + \frac{1}{2}\int_0^\ell \left[\frac{q(x)}{T(x)}|\sqrt{x'(\xi)}v(\xi,t)|^2 + \frac{m(x)}{T(x)}|\sqrt{x'(\xi)}v_t(\xi,t)|^2\right]x'(\xi)d\xi \\ &= \frac{1}{2}\int_0^\ell \left|v_\xi(\xi,t) - \frac{1}{4}[\frac{T'(x)}{T(x)} + \frac{m'(x)}{m(x)}]x'(\xi)v(\xi,t)\right|^2 d\xi \\ &\quad + \frac{1}{2}\int_0^\ell \left[\frac{q(x)}{T(x)}(x'(\xi))^2|v(\xi,t)|^2 + \frac{m(x)}{T(x)}(x'(\xi))^2|v_t(\xi,t)|^2\right]d\xi \\ &= \frac{1}{2}\int_0^\ell \left|v_\xi(\xi,t) - \frac{1}{4}[\frac{T'(x)}{T(x)} + \frac{m'(x)}{m(x)}]\sqrt{\frac{T(x)}{m(x)}}v(\xi,t)\right|^2 d\xi \\ &\quad + \frac{1}{2}\int_0^\ell \left[\frac{q(x)}{m(x)}|v(\xi,t)|^2 + |v_t(\xi,t)|^2\right]d\xi \end{aligned}$$

Set

$$\rho(\xi) = \sqrt[4]{T(x)m(x)}, \quad \widehat{q}(\xi) = \frac{q(x)}{m(x)}.$$

Then we have

$$\rho'(\xi) = \frac{1}{4}[\frac{T'(x)}{T(x)} + \frac{m'(x)}{m(x)}]\sqrt{\frac{T(x)}{m(x)}}$$

and the energy function

$$E(t) = \frac{1}{2}\int_0^\ell \left[|v_\xi(\xi,t) - \rho'(\xi)v(\xi,t)|^2 + \widehat{q}(\xi)|v(\xi,t)|^2 + |v_t(x,t)|^2\right]d\xi \qquad (7.4.2)$$

Therefore, $v(\xi,t)$ satisfies the following equation

$$\partial_t^2 v(\xi,t) + (\widehat{q}(\xi) + \rho''(\xi) + (\rho'(\xi))^2)v(\xi,t) = \partial_\xi^2 v(\xi,t), \quad \xi \in (0,\ell), t > 0 \tag{7.4.3}$$

the boundary conditions are of the form

$$v(\xi,t); \quad v_\xi(\xi,t) - \rho'(\xi)v(\xi,t), \quad \xi = 0, \ell. \tag{7.4.4}$$

The relation of both functions $w(x,t)$ and $v(\xi,t)$ is given by

$$v(\xi,t) = \rho(\xi)w(x(\xi),t), \quad \xi \in (0,\ell). \tag{7.4.5}$$

7.4.2 Continuous networks versus discontinuous networks

In this subsection, we consider the various forms of the continuous network of strings:

$$\begin{cases} m_j(s)w_{j,tt}(s,t) = (T_j(s)w_{j,s}(s,t))_s - q_j(s)w_j(s), \quad s \in (0,1) \\ w(a,t) = w_k(0,t) = w_i(1,t) = 0, \quad \forall k \in J^-(a), (or\ i \in J^+(a)), \quad a \in \partial G_D, \\ w(a,t) = w_k(0,t) = w_i(1,t), \quad \forall k \in J^-(a), i \in J^+(a), \quad a \in V\backslash\partial G_D \\ \sum\limits_{j\in J^+(a)} T_j(1)w_{j,s}(1,t) - \sum\limits_{k\in J^-(a)} T_k(0)w_{k,s}(0,t) + k(a)w(a,t) = -\alpha(a)s(a)w_t(a,t), a \in V\backslash\partial G_D \\ u_j(s,0) = u_{j0}(s), \quad u_{jt}(s,0) = u_{j1}(s), \quad s \in (0,1) \end{cases} \tag{7.4.6}$$

Let

$$\xi_j(s) = \int_0^s \sqrt{\frac{m_j(r)}{T_j(r)}}dr, \quad s \in (0,1), \quad \ell_j = \int_0^1 \sqrt{\frac{m_j(r)}{T_j(r)}}dr$$
$$\rho_j(\xi_j) = \sqrt[4]{T_j(s)m_j(s)}, \quad \widehat{q}_j(\xi_j) = \frac{q_j(s)}{m_j(s)}.$$

Under the transform

$$v_j(\xi_j,t) = \rho_j(\xi_j)w_j(s(\xi_j),t), \quad \xi_j \in (0,\ell_j)$$

the differential equations in (7.4.6) become

$$v_{j,tt}(\xi_j,t) + (\widehat{q}_j(\xi_j) + \rho_j''(\xi_j) + \rho_j'^2(\xi_j))v_j(\xi_j,t) = v_{j,\xi\xi}(\xi_j,t), \quad \xi \in (0,\ell_j), \tag{7.4.7}$$

the Dirichlet boundary vertices become

$$w(a,t) = \frac{v_j(0,t)}{\rho_j(0)} = \frac{v_i(\ell_i,t)}{\rho_i(\ell_i)} = 0, \quad j \in J^-(a), i \in J^+(a), \quad a \in \partial G_D.$$

At the other vertices, $a \in V\backslash\partial G_D$, the geometric continuity conditions become

$$w(a,t) = \frac{v_k(0,t)}{\rho_k(0)} = \frac{v_i(\ell_i,t)}{\rho_i(\ell_i)}, \quad \forall k \in J^-(a_j), i \in J^+(a_j), \tag{7.4.8}$$

and corresponding dynamic conditions become

$$\begin{cases} \sum\limits_{j\in J^+(a)} \rho_j(\ell_j)[v_{j,\xi}(\ell_j,t) - \rho_j'(\ell_j)v_j(\ell_j,t)] \\ \quad - \sum\limits_{k\in J^-(a)} \rho_k(0)[v_{k,\xi}(0,t) - \rho_k'(0)v_k(0,t)] \\ \quad\quad + k(a)w(a,t) = -\alpha(a)s(a)w_t(a,t), \quad a \in V\backslash\partial G_D \end{cases} \tag{7.4.9}$$

Using (7.4.8), we rewrite (7.4.9) into

$$\begin{cases} \sum_{j\in J^+(a)} \rho_j(\ell_j)v_{j,\xi}(\ell_j,t) - \sum_{k\in J^-(a)} \rho_k(0)v_{k,\xi}(0,t) \\ \quad + \left[k(a) - \sum_{j\in J^+(a)} \rho_j^2(\ell_j)\rho_j'(\ell_j) + \sum_{k\in J^-(a)} \rho_k^2(0)\rho_k'(0)\right] w(a,t) \\ \quad = -\alpha(a)w_t(a,t), \quad a\in V\backslash\partial G_D \end{cases} \tag{7.4.10}$$

Therefore, we get a new discontinuous network of strings

$$\begin{cases} v_{j,tt}(\xi_j,t) + (\widehat{q}_j(\xi_j) + \rho_j''(\xi_j) + {\rho'}_j^2(\xi_j))v_j(\xi_j,t) = v_{j,\xi\xi}(\xi_j,t), \quad \xi_j\in(0,\ell_j), t>0 \\ w(a,t) = \frac{v_j(0,t)}{\rho_j(0)} = \frac{v_i(\ell_i,t)}{\rho_i(\ell_i)} = 0, \quad j\in J^-(a), i\in J^+(a), \quad a\in\partial G_D \\ w(a,t) = \frac{v_k(0,t)}{\rho_k(0)} = \frac{v_i(\ell_i,t)}{\rho_i(\ell_i)}, \quad \forall k\in J^-(a), i\in J^+(a), a\in V\backslash\partial G_D \\ \sum_{j\in J^+(a)} \rho_j(\ell_j)v_{j,\xi}(\ell_j,t) - \sum_{k\in J^-(a)} \rho_k(0)v_{k,\xi}(0,t) \\ \quad + \left[k(a) - \sum_{j\in J^+(a)} \rho_j^2(\ell_j)\rho_j'(\ell_j) + \sum_{k\in J^-(a)} \rho_k^2(0)\rho_k'(0)\right] w(a,t) \\ \quad = -\alpha(a)s(a)w_t(a,t), \quad a\in V\backslash\partial G_D \\ v_j(\xi,0) = \rho_j(\xi_j)w_{j0}(s(\xi_j)), \quad v_{jt}(\xi_j,0) = \rho_j(\xi_j)w_{j,1}(s(\xi)), \quad \xi\in(0,\ell_j) \end{cases} \tag{7.4.11}$$

Since the transform $v_j(\xi_j,t) = \rho_j(\xi_j)w_j(s(\xi_j),t)$ is invertible, so system (7.4.11) is equivalent to system (7.4.6).

Let $E = \{e_j = (0,\ell_j), j = 1,2,\cdots,n\}$ and $V = \{a_1,a_2,\cdots,a_m\}$, we define a basic space by

$$H_E^1(E) = \left\{ v(x) = (v_j(\xi_j))_{j=1}^n \in H^1(E) \;\middle|\; \begin{array}{l} v_j\in H^1[0,\ell_j], \quad v(a)=0, a\in\partial G_D, \\ w_v(a) = \frac{v_k(0)}{\rho_k(0)} = \frac{v_j(\ell_j)}{\rho_j(\ell_j)}, k\in J^-(a), j\in J^+(a), a\in V\backslash\partial G_D \end{array} \right\}$$

equipped the inner product

$$(f,g)_{H_E} = \sum_{j=1}^n \int_0^{\ell_j} (f_j'(s)-\rho'(s)f(s))\overline{(g_j'(s)-\rho'(s)g(s))}ds + \widehat{q}_j(s)f(s)\overline{g(s)}ds + \sum_{j=1}^m k(a)w_f(a)\overline{w_g(a)}$$

where $w_f(a) = \frac{f_k(0)}{\rho_k(0)} = \frac{f_j(\ell_j)}{\rho_j(\ell_j)}$ indicates that $w_f(a)$ depends on f. Clearly, $H_E^1(E)$ is a Hilbert space.

Now let the state space corresponding to (7.4.11) be $\mathcal{H} = H_E^1 \times L^2(E)$ with the norm $||(f,g)||_{\mathcal{H}} = \sqrt{||f||_{H_E}^2 + ||g||_{L^2}^2}$, which also is a Hilbert space. Define an operator in $\mathcal{H}$ by

$$\mathcal{A}(f,g) = (g, f'' + (\rho'' + {\rho'}^2 + \widehat{q})f), \quad (f,g)\in D(\mathcal{A}) \tag{7.4.12}$$

with domain

$$D(\mathcal{A}) = \left\{ (f,g)\in\mathcal{H} \;\middle|\; \begin{array}{c} f=\{f_j\}\in H^2(E)\cap H_E^1, \quad g=\{g_j\}\in H_E^1(E), \\ \forall a\in V\backslash\partial G_D, \quad \sum_{j\in J^+(a)} \rho_j(\ell_j)f_j'(\ell_j) - \sum_{k\in J^-(a)} \rho_k(0)f_k'(0) \\ +[k(a)-\widehat{\rho}(a)]w_f(a) = -\alpha(a)s(a)w_g(a) \end{array} \right\} \tag{7.4.13}$$

where

$$\widehat{\rho}(a) = \sum_{j \in J^+(a)} \rho_j^2(\ell_j)\rho_j'(\ell_j) - \sum_{k \in J^-(a)} \rho_k^2(0)\rho_k'(0). \tag{7.4.14}$$

Then the system (7.4.11) can be rewritten into an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \frac{dV(t)}{dt} = \mathcal{A}V(t), \ t > 0 \\ V(0) = V_0 \end{cases} \tag{7.4.15}$$

where $V(t) = (v(x,t), v_t(x,t)) \in \mathcal{H}$ and $V_0 = (\{v_j(\xi,0)\}, \{v_{jt}(\xi,0)\}) \in \mathcal{H}$.

Then the energy function of the system (7.4.11) is

$$\mathcal{E}(t) = \frac{1}{2}||V(t)||_{\mathcal{H}}^2 = \frac{1}{2}||v(\xi,t)||_{H_E}^2 + \frac{1}{2}\|v_t(\xi,t)\|_{L^2}^2$$

in addition, it holds that

$$\frac{d\mathcal{E}(t)}{dt} = -\sum_{a \in V} \alpha(a)s(a)w_{v,t}^2(a,t).$$

Therefore, we have the following result.

THEOREM 7.4.1 *The system (7.4.11) is asymptotically stable if and only if the system (7.4.6) is.*

7.5 Comparison of systems

In this section we shall discuss properties of the system (7.4.11)(or (7.4.15)). Since (7.4.11) is still a system of variable coefficients, to treat it, we introduced a complex norm on $H_E^1(E)$ in preceding section. Note that the inner product on $H_E^1(E)$ is too complex to calculate in practice. For simplicity, let $M = \max_{1 \le j \le n}\{\max_{s \in [0,\ell_j]} q_j(s)\}$, we take an equivalent inner product in $H_E^1(E)$ defined by

$$(f,g)_{V_e} = \sum_{j=1}^n \int_0^{\ell_j} [f_j'(s))\overline{g_j'(s)}ds + Mf_j(s)\overline{g_j(s)}]ds + \sum_{a \in V} k(a)w_f(a)\overline{w_g(a)}, \quad f,g \in H_E^1(E).$$

Then an equivalent norm on $\mathcal{H}$ is given by

$$||(f,g)||_{\mathcal{H}}^2 = ||f||_{V_e}^2 + ||g||_{L^2}^2 = \sum_{j=1}^n \int_0^{\ell_j} [|f_j'(s)|^2 + M|f_j(s)|^2 + |g_j(s)|^2]ds + \sum_{a \in V} k(a)|w_f(a)|^2.$$

In what follows, we shall discuss the system (7.4.15) in $\mathcal{H}$ under this norm.

To discuss system (7.4.15), we firstly consider the following system

$$\begin{cases} u_{j,tt}(\xi,t) = u_{j,\xi\xi}(\xi,t) - Mu_j(\xi,t), \quad \xi \in (0,\ell_j), t > 0 \\ w_u(a,t) = \frac{u_j(0,t)}{\rho_j(0)} = \frac{u_i(\ell_i,t)}{\rho_i(\ell_i)} = 0, j \in J^-(a), i \in J^+(a), \quad a \in \partial G_D \\ w_u(a,t) = \frac{u_k(0,t)}{\rho_k(0)} = \frac{u_i(\ell_i,t)}{\rho_i(\ell_i)}, \quad \forall k \in J^-(a), i \in J^+(a), a \in V \backslash \partial G_D \\ \sum\limits_{j \in J^+(a)} \rho_j(\ell_j)u_{j,\xi}(\ell_j,t) - \sum\limits_{k \in J^-(a)} \rho_k(0)u_{k,\xi}(0,t) + k(a)w_u(a,t) \\ = -\alpha(a)s(a)w_{u,t}(a,t), \quad a \in V \backslash \partial G_D \\ u_j(\xi,0) = \rho_j(\xi_j)w_{j0}(s(\xi_j)), \quad u_{jt}(\xi_j,0) = \rho_j(\xi_j)w_{j,1}(s(\xi)), \quad \xi \in (0,\ell_j) \end{cases} \tag{7.5.1}$$

Define an operator in $\mathcal{H}$ by

$$\mathcal{A}_F(f,g) = (g, f'' - Mf), \quad (f,g) \in D(\mathcal{A}_F) \tag{7.5.2}$$

with domain

$$D(\mathcal{A}_F) = \left\{ (f,g) \in \mathcal{H} \;\middle|\; \begin{array}{l} f = \{f_j\} H^2(E) \cap H_E^1, g = \{g_j\} \in H_E^1(E), \text{ and for each } \; a \in V \backslash \partial G_D \\ \sum\limits_{j \in J^+(a)} \rho_j(\ell_j) f_j'(\ell_j) - \sum\limits_{k \in J^-(a)} \rho_k(0) f_k'(0) + k(a) w_f(a) = -\alpha(a) s(a) w_g(a) \end{array} \right\} \tag{7.5.3}$$

Then the system (7.5.1) can be rewritten into an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \frac{dU(t)}{dt} = \mathcal{A}_F U(t), \; t > 0 \\ U(0) = U_0 \end{cases} \tag{7.5.4}$$

where $U(t) = (u(x,t), u_t(x,t)) \in \mathcal{H}$ and $U_0 = (\{u_j(\xi, 0)\}, \{u_{jt}(\xi, 0)\}) \in \mathcal{H}$.

THEOREM 7.5.1 *Let $\mathcal{A}_F$ be defined by (7.5.2) and (7.5.3). Then the following statements are true.*

1) $\mathcal{A}_F$ is a densely defined and closed liner operator in $\mathcal{H}$;

2) $\mathcal{A}_F$ and $\mathcal{A}_F^*$ are dissipative in $\mathcal{H}$;

3) $\mathcal{A}_F$ generates a C_0 semigroup of contraction on $\mathcal{H}$;

4) The energy function of the system (7.5.1) (or (7.5.3) $\mathcal{E}(t)$ defined by $\mathcal{E}(t) = \frac{1}{2}||U(t)||^2_{\mathcal{H}}$ satisfies

$$\frac{d\mathcal{E}(t)}{dt} = -\sum_{a \in V} \alpha(a) s(a) |w_{u,t}(a,t)|^2$$

Proof Here we only prove that $\mathcal{A}_F$ is dissipative in $\mathcal{H}$, the other verifications are directly, we omit the detail.

For any real $(f,g) \in \mathcal{H}$, it holds that

$$\begin{aligned} \langle \mathcal{A}_F(f,g), (f,g) \rangle &= \sum_{j=1}^n \int_0^{\ell_j} [g_j'(\xi) f_j'(\xi) + f_j''(\xi) g_j(\xi)] d\xi + \sum_{a \in V} k(a) w_g(a) w_f(a) \\ &= \sum_{j=1}^n f_j'(\ell_j) g_j(\ell_j) - \sum_{j=1}^n f_j'(0) g_j(0) + \sum_{a \in V} k(a) w_g(a) w_f(a) \\ &= \sum_{a \in V} w_g(a) \left[\sum_{j \in J^+(a)} f_j'(\ell_j) \rho_j(\ell_j) - \sum_{k \in J^-(a)} f_k'(0) \rho_k(0) + k(a) w_f(a) \right] \\ &= -\sum_{a \in V} \alpha(a) s(a) w_g^2(a). \end{aligned}$$

So $\mathcal{A}_F$ is dissipative. $\square$

Now we define an operator $B_1 : H_E^1(E) \to \mathbb{C}$ by

$$B_1 f = \sum_{a \in V} \widehat{\rho}(a) w_f(a), \quad f \in H_E^1(E) \tag{7.5.5}$$

where $\widehat{\rho}(a)$ is defined by (7.4.14). Further we define an operator $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{B}(f,g) = (0, [\rho'' + {\rho'}^2 + \widehat{q} - M]f), \quad \forall (f,g) \in \mathcal{H}. \tag{7.5.6}$$

For these operators we have the following result.

THEOREM 7.5.2 *Let B_1 and $\mathcal{B}$ be defined by (7.5.5) and (7.5.6), respectively. Then the following statements are true.*

1) B_1 is a bounded linear operator on $H_E^1(E)$, and so is its dual operator $B_1^* : \mathbb{C} \to (H_E^1(E))'$.

2) $\mathcal{B}$ is a bounded and compact linear operator on $\mathcal{H}$.

Proof Obviously, B_1 and $\mathcal{B}$ are linear operators in $\mathcal{H}$. We at first prove the first assertion.

For any $f \in H_E^1(E)$, $f = \{f_j(\xi)\}$, it holds that

$$\ell_j f_j(\ell_j) = \int_0^{\ell_j} \xi f_j'(\xi) d\xi + \int_0^{\ell_j} f_j(\xi) d\xi$$

and

$$\ell_j f_j(0) = \int_0^{\ell_j} (\xi - \ell_j) f_j'(\xi) d\xi + \int_0^{\ell_j} f_j(\xi) d\xi.$$

So we have

$$|f_j(\ell_j)|^2, |f_j(0)|^2 \le 2\left(\frac{\ell_j}{3} + \frac{1}{M\ell_j}\right) \int_0^{\ell_j} [|f_j'(\xi)|^2 + M|f_j(\xi)|^2] d\xi$$

and hence

$$\begin{aligned}
\sum_{a\in V} |\widehat{\rho}(a) w_f(a)|^2 &\le \sum_{j=1}^n |\rho_j(\ell_j)\rho_j'(\ell_j)|^2 |f_j(\ell_j)|^2 + \sum_{j=1}^n |\rho_j(0)\rho_j'(0)|^2 |f_j(0)|^2 \\
&\le \max_{1\le j\le n} \left\{ \max_{\xi\in(0,\ell_j)} \{|\rho_j(\xi)\rho_j'(\xi)|^2\} \left(\frac{2\ell_j}{3} + \frac{2}{M\ell_j}\right) \right\} ||f||_{V_e}^2.
\end{aligned}$$

Therefore, B_1 is a bounded operator on $H_E^1(E)$. By the duality, B_1^* is bounded from $\mathbb{C}$ to $(H_E^1(E))'$.

In order to obtain an expression of B_1^*, for any $f \in H_E^1(E)$ and $c \in \mathbb{C}$, we calculate the dual product

$$\begin{aligned}
(B_1^* c, f)_{(H_E^1)', H_E^1} &= (c, B_1 f) = c \sum_{a\in V} \widehat{\rho}(a) w_f(a) \\
&= c \sum_{a\in V} \left[\sum_{j\in J^+(a)} \rho_j(\ell_j)\rho_j'(\ell_j) f_j(\ell_j) - \sum_{k\in J^-(a)} \rho_k(0)\rho_k'(0) f_k(0) \right] \\
&= c \sum_{j=1}^n [\rho_j(\ell_j)\rho_j'(\ell_j) f_j(\ell_j) - \rho_j(0)\rho_j'(0) f_j(0)] \\
&= c \sum_{j=1}^n \int_0^{\ell_j} [\rho_j(\xi)\rho_j'(\xi)\delta_j(\xi - \ell_j) - \rho_j(\xi)\rho_j'(\xi)\delta_j(\xi)] f_j(\xi) d\xi.
\end{aligned}$$

From above equality we get

$$[B_1^* c]_j(\xi) = [\rho_j(\xi)\rho_j'(\xi)\delta_j(\xi - \ell_j) - \rho_j(\xi)\rho_j'(\xi)\delta_j(\xi)]c, \quad \xi \in [0, \ell_j].$$

The first assertion is true.

Next, for any $(f,g)\in\mathcal{H}$,

$$\begin{aligned}
||\mathcal{B}(f,g)||^2_{\mathcal{H}} &= \sum_{j=1}^{n}\int_0^{\ell_j}|(\rho''_j(\xi)+(\rho'_j(\xi)^2+\widehat{q}_j(\xi)-M)f_j(\xi)|^2d\xi\\
&\le \frac{\max\limits_{1\le j\le n}\max\limits_{\xi\in(0,\ell_j)}\{|\rho''_j(\xi)+\rho'^2(\xi)+\widehat{q}(\xi)|\}+1}{M}\sum_{j=1}^{n}\int_0^{\ell_j}M|f_j(\xi)|^2d\xi\\
&\le \frac{\max\limits_{1\le j\le n}\max\limits_{\xi\in(0,\ell_j)}\{|\rho''_j(\xi)+\rho'^2(\xi)+\widehat{q}(\xi)|\}+1}{M}||(f,g)||^2_{\mathcal{H}}
\end{aligned}$$

So $\mathcal{B}$ is also bounded on $\mathcal{H}$. Note that $\mathcal{R}(\mathcal{B})\subset H^2(E)\times H^1_E(E)$, the Sobolev Embedding Theorem asserts that $\mathcal{B}$ is compact. The proof is then complete. $\square$

Now we regard the system (7.4.11)(or (7.4.15)) as the perturbation system (7.5.1)(or (7.5.4), respectively). Then we can rewrite the operator $\mathcal{A}$ defined by (7.4.13) into the following form

$$\mathcal{A}(f,g)=\mathcal{A}_F(f,g)-\mathcal{B}_1(f,g)+\mathcal{B}(f,g),\quad \mathcal{B}_1(f,g)=(0,B_1^*B_1f) \tag{7.5.7}$$

Let $T(t)$ and $S(t)$ be the C_0 semigroup generated by $\mathcal{A}$ and $\mathcal{A}_F$, respectively. Then the relation between $T(t)$ and $S(t)$ is given by

$$T(t)=S(t)-\int_0^t S(t-s)\mathcal{B}_1T(s)ds+\int_0^t S(t-s)\mathcal{B}T(s)dt. \tag{7.5.8}$$

The following theorem gives a comparison result.

THEOREM 7.5.3 *Let $T(t)$ and $S(t)$ be the C_0 semigroup generated by $\mathcal{A}$ and $\mathcal{A}_F$, then $T(t)-S(t)$ are compact operators on $\mathcal{H}$ for all $t\ge 0$. Hence $r_{ess}(T(t))=r_{ess}(S(t))$ for all $t>0$, where $r_{ess}(T)$ denotes the essential spectrum radius of operator T.*

Proof Thanks to the second assertion of Theorem 7.5.2, the third term at the right-hand side of equality (7.5.8) are compact operators on $\mathcal{H}$ for $\forall t\ge 0$. So we only need to prove the term

$$\int_0^t S(t-s)\mathcal{B}_1T(s)ds,\quad \forall t>0$$

are compact operators.

For any $(f,g)\in\mathcal{H}$, denote $(v(t),v_t(t))=T(t)(f,g)$, then

$$\mathcal{B}_1T(s)(f,g)=(0,B_1^*B_1v(s))=(0,B_1^*)B_1v(s),$$

and

$$S(t-s)\mathcal{B}_1T(s)(f,g)=(0,\beta(\cdot,t-s)B_1v(s))$$

where $(0,\beta(\cdot,t))=S(t)(0,B_1^*)$, $\beta(\cdot,t)\in(H^1_E(E))'$ and is continuous in t, and $B_1v(s)$ is a scalar continuous function in s. Therefore, we have

$$\int_0^t S(t-s)\mathcal{B}_1T(s)(f,g)ds=\left(0,\int_0^t\beta(\cdot,t-s)B_1v(s)ds\right),\quad \forall t>0$$

Thus, by using the admissibility of $\mathcal{B}_1^*$, for each $t \geq 0$, there exists a constants $K(t)$ such that

$$\begin{aligned}\left\|\int_0^t S(t-s)\mathcal{B}_1 T(s)(f,g)ds\right\|^2 &= \sum_{j=1}^n \int_0^{\ell_j} \left|\int_0^t \beta_j(\xi,t-s)B_1 v(s)ds\right|^2 d\xi \\ = & \int_0^t \int_0^t \left(\sum_{j=1}^n \int_0^{\ell_j} \beta_j(\xi,t-r)\overline{\beta_j(\xi,t-s)}d\xi\right) B_1 v(r)\overline{B_1 v(s)}drds \\ \leq & K^2(t)\int_0^t |B_1 v(s)|^2 ds.\end{aligned}$$

Let $\mathcal{O}$ be a bounded set in $\mathcal{H}$, whose bound denotes $M_{\mathcal{O}}$. Then for any $(f,g) \in \mathcal{O}$, we have

$$||(v(t), v_t(t)||_{\mathcal{H}} \leq ||T(t)||||(f,g)||_{\mathcal{H}} \leq M_{\mathcal{O}}, \quad t \geq 0$$

due to $T(t)$ being contraction. So, for each fixed $t > 0$, $\{B_1 v(s), (f,g) \in \mathcal{O}\}$ is a bounded set in $L^2[0,t]$. Note that $v(t)$ is differentiable in t and $v_t(t)$ is a square integrable function. Therefore, $\{B_1 v(s), (f,g) \in \mathcal{O}\}$ is a compact set in $L^2[0,t]$. Therefore, for each $t > 0$,

$$\int_0^t S(t-s)\mathcal{B}_1 T(s)ds, \quad \forall t > 0$$

is a compact operator on $\mathcal{H}$. The desired result follows. □

REMARK 7.5.1 *Let $\mathcal{K}(\mathcal{H})$ be the set consisting of all compact operator on Hilbert space $\mathcal{H}$, T be a bounded linear operator. The radius of essential spectrum of T is defined by*

$$r_{ess}(T) = \lim_{n\to\infty} \sqrt[n]{||[T]^n||}$$

where $||[T]|| = \inf\{||T-K||, K \in \mathcal{K}(\mathcal{H})\}$

As a consequence of Theorem 7.5.3, we have the following corollary.

COROLLARY 7.5.1 *Let $\mathcal{A}$ and $\mathcal{A}_F$ be defined as before. Then $\sigma(\mathcal{A})$ and $\sigma(\mathcal{A}_F)$ have same bound of the essential spectrum. Hence they have same the right-asymptote of the spectrum.*

As a direct result of above corollary, we have the stability result of system (7.4.11).

COROLLARY 7.5.2 *Let $T(t)$ and $S(t)$ be the C_0 semigroups generated by $\mathcal{A}$ and $\mathcal{A}_F$ respectively. Suppose that $T(t)$ and $S(t)$ are asymptotically stable. Then system (7.4.11) is exponentially stable if and only if the system (7.5.1) is.*

7.6 Conclusion remark

In this chapter, we discussed the continuous network of strings:

$$\begin{cases} m_j(s)u_{j,tt}(s,t) = (T_j(s)u_{j,s}(s,t))_s - q_j(s)u_j(s), \quad s \in (0,1) \\ u(a_j,t) = u_k(0,t) = u_i(1,t) = 0, \quad \forall k \in J^-(a_j), (or\ i \in J^+(a_j)), \quad a_j \in \partial G_D, \\ u(a_j,t) = u_k(0,t) = u_i(1,t), \quad \forall k \in J^-(a_j), i \in J^+(a_j), \quad a_j \in V\backslash\partial G_D \\ \sum\limits_{j\in J^+(a)} T_j(1)u_{j,s}(1,t) - \sum\limits_{k\in J^-(a)} T_k(0)u_{k,s}(0,t) + k(a)u(a,t) = -\alpha(a)s(a)u_t(a,t), a \in V\backslash\partial G_D \\ u_j(s,0) = u_{j0}(s), \quad u_{jt}(s,0) = u_{j1}(s), \quad s \in (0,1) \end{cases}$$

We presented the geometric approach for checking the asymptotic stability of the system. Further, we translated the system into the following form

$$\begin{cases} v_{j,tt}(\xi,t) + (\widehat{q}_j(\xi) + \rho''_j(\xi) + {\rho'}_j^2(\xi))v_j(\xi,t) = v_{j,\xi\xi}(\xi,t), \quad \xi \in (0,\ell_j), t > 0 \\ w_v(a,t) = \frac{v_j(0,t)}{\rho_j(0)} = \frac{v_i(\ell_i,t)}{\rho_i(\ell_i)} = 0, \quad j \in J^-(a), i \in J^+(a), \quad a \in \partial G_D \\ w_v(a,t) = \frac{v_k(0,t)}{\rho_k(0)} = \frac{v_i(\ell_i,t)}{\rho_i(\ell_i)}, \quad \forall k \in J^-(a), i \in J^+(a), a \in V \backslash \partial G_D \\ \sum\limits_{j\in J^+(a)} \rho_j(\ell_j) v_{j,\xi}(\ell_j,t) - \sum\limits_{k \in J^-(a)} \rho_k(0) v_{k,\xi}(0,t) \\ \quad + \left[k(a) - \sum\limits_{j\in J^+(a)} \rho_j^2(\ell_j)\rho'_j(\ell_j) + \sum\limits_{k\in J^-(a)} \rho_k^2(0)\rho'_k(0) \right] w_v(a,t) \\ \quad = -\alpha(a)s(a)w_{v,t}(a,t), \quad a \in V\backslash \partial G_D \\ v_j(\xi,0) = \rho_j(\xi_j) w_{j0}(s(\xi_j)), \quad v_{jt}(\xi_j,0) = \rho_j(\xi_j) w_{j,1}(s(\xi)), \quad \xi \in (0,\ell_j) \end{cases}$$

wherein the major terms are of the constants coefficient in the differential equations, although they are still of variable coefficients.

By comparing with a system with constant coefficients, we got that the exponential stability of the system can be determined via the following system

$$\begin{cases} u_{j,tt}(\xi,t) = u_{j,\xi\xi}(\xi,t) - M u_j(\xi,t), \quad \xi \in (0,\ell_j), t > 0 \\ w_u(a,t) = \frac{u_j(0,t)}{\rho_j(0)} = \frac{u_i(\ell_i,t)}{\rho_i(\ell_i)} = 0, j \in J^-(a), i \in J^+(a), \quad a \in \partial G_D \\ w_u(a,t) = \frac{u_k(0,t)}{\rho_k(0)} = \frac{u_i(\ell_i,t)}{\rho_i(\ell_i)}, \quad \forall k \in J^-(a), i \in J^+(a), a \in V\backslash \partial G_D \\ \sum\limits_{j\in J^+(a)} \rho_j(\ell_j) u_{j,\xi}(\ell_j,t) - \sum\limits_{k\in J^-(a)} \rho_k(0) u_{k,\xi}(0,t) + k(a) w_u(a,t) \\ = -\alpha(a)s(a)w_{u,t}(a,t), \quad a \in V\backslash \partial G_D \\ u_j(\xi,0) = \rho_j(\xi_j) w_{j0}(s(\xi_j)), \quad u_{jt}(\xi_j,0) = \rho_j(\xi_j) w_{j,1}(s(\xi)), \quad \xi \in (0,\ell_j). \end{cases}$$

Therefore, to assert the exponential stability of the continuous network of strings with variable coefficients, we need only to discuss the exponential stability of above system of the constant coefficients.

Chapter 8

Stabilization of Serially Connected Strings

In this chapter we study the stabilization problem of serially connected vibrating strings via joint feedback controls. Suppose that both ends of the strings are clamped, at the interior nodes, the shearing forces are continuous, but their displacements are discontinuous. We observe the shearing forces at interior nodes, and then design the compensators by the observation values. Finally we stabilize the system by using the feedback controllers at the interior nodes. We prove that the closed loop system is asymptotically stable under certain conditions. By a detail spectral analysis, we show that under certain conditions there exists a sequence of the generalized eigenvectors of the closed loop system that forms a Riesz basis with parentheses for the Hilbert state space. Hence we obtain the spectrum determined growth property.

8.1 Model and design of controllers

The modern large flexible space structures are often made of serially connected elastic elements; they are usually modeled as strings, beams, shell etc. Among them, the long chains flexible structure is the simplest one, for instance, the railway or aerial cable system in our real-life. Therefore, in the present chapter, we consider a network of strings defined on a long chains graph G as shown in Fig. 8.1.1. For such elastic structure, we need to control its vibration in practice.

Fig. 8.1.1. A long chain graph G

Suppose that each string in the system is homogeneous and inextensible, and denote by $w_j(x,t)$ the transversal deflection of the string departing from its equilibrium at position x between a_j and a_{j+1} at time t. The motion of strings in the vibrating system are governed by the partial differential equations

$$m_j \frac{\partial^2 w_j(x,t)}{\partial t^2} = T_j \frac{\partial^2 w_j(x,t)}{\partial x^2}, \quad j = 1, 2, \cdots n, \quad x \in (0,1), \quad t > 0, \tag{8.1.1}$$

where $m_j > 0$, $T_j > 0$, $j = 1, 2, \cdots, n$, are mass density and tension, respectively.

Suppose that the system is clamped at both ends, i.e.

$$w_1(0,t) = w_n(1,t) = 0, \quad t > 0. \tag{8.1.2}$$

At the interior nodes, the shearing forces of strings are assumed to be continuous

$$T_j w_{j,x}(1,t) = T_{j+1} w_{j+1,x}(0,t), \quad j = 1, 2, \cdots, n-1, \quad t > 0, \tag{8.1.3}$$

but the displacements are discontinuous. We act the control at each interior node, i.e.,

$$w_j(1,t) - w_{j+1}(0,t) = u_j(t), \quad j = 1, 2, \cdots, n-1, \quad t > 0, \tag{8.1.4}$$

where $u_j(t)$, $j = 1, 2, \cdots, n-1$, are external exciting forces.

We observe the shearing force $T_j w_{j,x}(1,t)$ at interior node a_{j+1}. For the system (8.1.1)–(8.1.4), if we choose simple feedback control law

$$w_{j,t}(1,t) - w_{j+1,t}(0,t) = -\alpha_j T_j w_{j,x}(1,t), \quad j = 1, 2, \cdots, n-1, \quad t > 0,$$

then 0 is still an eigenvalue of system operator, in fact, its multiplicity is $n-1$. With these feedback control law, corresponding closed loop system can not come back to its equilibrium position. Therefore, we design compensators as follows

$$\frac{dE_j(t)}{dt} = -\hat{\alpha}_j E_j(t) + T_j w_{j,x}(1,t), \quad j = 1, 2, \cdots, n-1. \tag{8.1.5}$$

Here the aim of the compensators is to remove the 0 eigenvalue of the system and to improve lower frequency of the system. Because 0 is an eigenvalue of multiplicity $(n-1)$, we need $(n-1)$ compensators. Finally, we take the feedback control law as

$$u_j(t) = -\alpha_j E_j(t), \quad \alpha_j > 0, \quad j = 1, 2, \cdots, n-1. \tag{8.1.6}$$

Thus the system (8.1.1)–(8.1.6) become a closed loop system:

$$\begin{cases} m_j w_{j,tt}(x,t) = T_j w_{j,xx}(x,t), \quad j = 1, 2, \cdots n, \quad j-1 < x < j, \quad t > 0, \\ w_1(0,t) = w_n(1,t) = 0, \quad t > 0, \\ T_j w_{j,x}(1,t) = T_{j+1} w_{j+1,x}(0,t), \quad j = 1, 2, \cdots, n-1, \quad t > 0, \\ w_j(1,t) - w_{j+1}(0,t) = -\alpha_j E_j(t), \quad j = 1, 2, \cdots, n-1, \quad t > 0, \\ \frac{dE_j(t)}{dt} = -\hat{\alpha}_j E_j(t) + T_j w_{j,x}(1,t), \quad j = 1, 2, \cdots, n-1, \ t > 0. \end{cases} \tag{8.1.7}$$

here and hereafter we shall use abbreviations $w_t = \frac{\partial w}{\partial t}$ and $w_x = \frac{\partial w}{\partial x}$.

Set

$$W(x,t) = (w_1(x,t), w_2(x,t), \cdots, w_n(x,t))^T.$$

Define $n \times n$ diagonal matrices:

$$\mathbb{M} = \text{diag}\,(m_1, m_2, \cdots, m_n)\,, \quad \mathbb{T} = \text{diag}\,(T_1, T_2, \cdots, T_n)\,,$$

and

$$\alpha = \text{diag}(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}), \quad \hat{\alpha} = \text{diag}(\hat{\alpha_1}, \hat{\alpha_2}, \cdots, \hat{\alpha_{n-1}}).$$

We define operators $P : \mathbb{C}^{n+1} \to \mathbb{C}^{n-1}$ by

$$P(v_1, v_2, \cdots, v_n, v_{n+1})^T = (v_2, \cdots, v_n)^T.$$

Let $\Phi^{\pm}$ be the incoming (outgoing) incidence matrix of the linear graph G. For simplification, we introduce operators $P\Phi^+ = P_{n-1}$ and $P\Phi^- = L_{n-1}$. It is easy to check that P_{n-1} and $L_{n-1} : \mathbb{C}^n \to \mathbb{C}^{n-1}$ with properties

$$P_{n-1}(y_1, y_2, y_3, \cdots, y_n)^T = (y_1, y_2, y_3, \cdots, y_{n-1})^T,$$

$$L_{n-1}(y_1, y_2, y_3, \cdots, y_n)^T = (y_2, y_3, y_4, \cdots, y_n)^T.$$

If we set matrix

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix}_{n \times n}, \tag{8.1.8}$$

then we have

$$L_{n-1}C = P_{n-1}, \quad P_{n-1}C^T = L_{n-1}$$

where C^T denotes the transpose of the matrix C. Moreover, we define operators E_1 and E_n from $\mathbb{C}^n$ to $\mathbb{C}$ by

$$E_1(y_1, y_2, y_3, \cdots, y_n)^T = y_1, \quad E_n(y_1, y_2, y_3, \cdots, y_n)^T = y_n.$$

With the help of these notations, the differential equations in (8.1.7) can be rewritten into an equation in $\mathbb{C}^n$:

$$\mathbb{M}W_{tt}(x,t) = \mathbb{T}W_{xx}(x,t), \quad x \in (0,1),\ t > 0. \tag{8.1.9}$$

The boundary conditions of the system become

$$E_1 W(0) = E_n W(1) = 0; \tag{8.1.10}$$

the dynamic continuous conditions at the interior nodes can be written into

$$P_{n-1}[\mathbb{T}W_x(1,t)] - L_{n-1}[\mathbb{T}W_x(0,t)] = 0; \tag{8.1.11}$$

the feedback control conditions at the interior nodes become

$$P_{n-1}W(1,t) - L_{n-1}W(0,t) = -\alpha E(t); \tag{8.1.12}$$

the compensators designed above become

$$\frac{dE(t)}{dt} = -\hat{\alpha}E(t) + L_{n-1}\mathbb{T}W_x(0,t). \tag{8.1.13}$$

Therefore, the equations (8.1.7) are equivalent to the following differential equation in $\mathbb{C}^n$:

$$\begin{cases} \mathbb{M}W_{tt}(x,t) = \mathbb{T}W_{xx}(x,t), \quad x \in (0,1),\ t>0, \\ \frac{dE(t)}{dt} = -\hat{\alpha}E(t) + L_{n-1}\mathbb{T}W_x(0,t),\ t>0, \\ E_1W(0,t) = E_nW(1,t) = 0,\ t>0, \\ P_{n-1}[\mathbb{T}W_x(1,t)] - L_{n-1}[\mathbb{T}W_x(0,t)] = 0,\ t>0, \\ P_{n-1}W(1,t) - L_{n-1}W(0,t) = -\alpha E(t),\ t>0 \end{cases} \tag{8.1.14}$$

with appropriate initial data.

8.2 Well-posed-ness and asymptotic stability of the system

In this section we study the well-posed-ness and asymptotic stability of the closed loop system (8.1.7). To this end, we formulate the system into a Hilbert state space. Let us begin with introducing some notations.

Set

$$\mathcal{W}^k = H^k([0,1],\mathbb{C}^n) \times H^{k-1}([0,1],\mathbb{C}^n) \times \mathbb{C}^{n-1}.$$

Let the state space be

$$\mathcal{H} = \left\{ (f,g,p) \in \mathcal{W}^1 \;\middle|\; \begin{array}{c} E_1f(0) = E_nf(1) = 0 \\ P_{n-1}f(1) - L_{n-1}f(0) = -\alpha p \end{array} \right\}$$

endowed with the inner product, for $(f_j,g_j,p_j) \in \mathcal{H}, j=1,2,$

$$((f_1,g_1,p_1),(f_2,g_2,p_2))_{\mathcal{H}} := \int_0^1 (\mathbb{T}f_{1,x}(x), f_{2,x}(x))_c\,dx + \int_0^1 (\mathbb{M}g_1(x), g_2(x))_c\,dx + (\hat{\alpha}\alpha p_1, p_2)_{c-1}, \tag{8.2.1}$$

where $(\cdot,\cdot)_c$ and $(\cdot,\cdot)_{c-1}$ denote the inner product in $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$, respectively. Since $\mathbb{T}$ and $\mathbb{M}$ are positive definite matrices, a direct verification shows that $\|(f,g,p)\| = \sqrt{((f,g,p),(f,g,p))_{\mathcal{H}}}$ induces a norm on $\mathcal{H}$ and $(\mathcal{H}, ||\cdot||)$ is a Hilbert space.

We define an operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}\begin{bmatrix} W \\ Z \\ Q \end{bmatrix} = \begin{bmatrix} Z \\ \mathbb{M}^{-1}\mathbb{T}W_{xx} \\ -\hat{\alpha}Q + L_{n-1}\mathbb{T}W_x(0) \end{bmatrix} \tag{8.2.2}$$

and

$$\mathcal{D}(\mathcal{A}) = \left\{(W,Z,Q) \in \mathcal{W}^2 \;\middle|\; \mathcal{A}(W,Z,Q)^T \in \mathcal{H}; P_{n-1}\mathbb{T}W_x(1) = L_{n-1}\mathbb{T}W_x(0)\right\}. \tag{8.2.3}$$

Then, we can rewrite (8.1.14) into an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \frac{dU(t)}{dt} = \mathcal{A}U(t), \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{8.2.4}$$

where $U(t) = (W(x,t), W_t(x,t), Q(t))^T$ and $U(0) = (W^0(x), Z^0(x), Q^0)^T \in \mathcal{H}$ is given.

THEOREM 8.2.1 *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. Then $\mathcal{A}$ is dissipative, $\mathcal{A}^{-1}$ is compact, and hence $\mathcal{A}$ generates a C_0 semigroup of contraction on $\mathcal{H}$.*

Proof We prove firstly that $\mathcal{A}$ is a dissipative operator. For any $(W,Z,Q) \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}(W,Z,Q) \in \mathcal{H}$ implies $Z \in H^1([0,1],\mathbb{C}^n)$ and

$$P_{n-1}Z(1) - L_{n-1}Z(0) = -\alpha(-\widehat{\alpha}Q + L_{n-1}\mathbb{T}W_x(0))$$

and $P_{n-1}W(1) - L_{n-1}W(0) = -\alpha Q$. So we have

$$\begin{aligned} & \left(\mathcal{A}(W,Z,Q)^T, (W,Z,Q)^T\right)_{\mathcal{H}} \\ = & \int_0^1 (\mathbb{T}Z_x(x), W_x(x))_c dx + \int_0^1 (\mathbb{M}\mathbb{M}^{-1}\mathbb{T}W_{xx}(x), Z(x))_c dx + (\hat{\alpha}\alpha(-\hat{\alpha}Q + L_{n-1}\mathbb{T}W_x(0)), Q)_{c-1} \\ = & \int_0^1 (\mathbb{T}Z_x(x), W_x(x))_c dx + (\mathbb{T}W_x, Z)_c|_0^1 - \int_0^1 (\mathbb{T}W_x, Z_x)_c dx \\ & +(\hat{\alpha}\alpha^{-1}(P_{n-1}Z(1) - L_{n-1}Z(0)), P_{n-1}W(1) - L_{n-1}W(0))_{c-1} \end{aligned}$$

and

$$\begin{aligned} & \left((W,Z,Q)^T, \mathcal{A}(W,Z,Q)^T\right)_{\mathcal{H}} \\ = & \int_0^1 (W_x(x), \mathbb{T}Z_x(x))_c dx + \int_0^1 (Z(x), \mathbb{M}\mathbb{M}^{-1}\mathbb{T}W_{xx}(x))_c dx + (\hat{\alpha}\alpha Q, -\hat{\alpha}Q + L_{n-1}\mathbb{T}W_x(0))_{c-1} \\ = & \int_0^1 (W_x(x), \mathbb{T}Z_x(x))_c dx + (Z, \mathbb{T}W_x)_c|_0^1 - \int_0^1 (Z_x, \mathbb{T}W_x)_c dx \\ & +(\hat{\alpha}\alpha^{-1}(P_{n-1}W(1) - L_{n-1}W(0)), (P_{n-1}Z(1) - L_{n-1}Z(0)))_{c-1}. \end{aligned}$$

Since $\mathbb{T}$ and $\mathbb{M}$ are positive definite matrices, and $(W,Z,Q) \in \mathcal{D}(\mathcal{A})$, so we have

$$\begin{aligned} & 2\Re\left((W,Z,Q)^T, \mathcal{A}(W,Z,Q)^T\right)_{\mathcal{H}} \\ = & (\mathbb{T}W_x, Z)_c|_0^1 + (\hat{\alpha}\alpha^{-1}(P_{n-1}Z(1) - L_{n-1}Z(0)), P_{n-1}W(1) - L_{n-1}W(0))_{c-1} \\ & +(Z, \mathbb{T}W_x)_c|_0^1 + (\hat{\alpha}\alpha^{-1}(P_{n-1}W(1) - L_{n-1}W(0)), P_{n-1}Z(1) - L_{n-1}Z(0))_{c-1} \\ = & (\mathbb{T}W_x(1), Z(1))_c - (\mathbb{T}W_x(0), Z(0))_c + (Z(1), \mathbb{T}W_x(1))_c - (Z(0), \mathbb{T}W_x(0))_c \\ & +(\hat{\alpha}\alpha^{-1}(P_{n-1}Z(1) - L_{n-1}Z(0)), P_{n-1}W(1) - L_{n-1}W(0))_{c-1} \\ & +(\hat{\alpha}\alpha^{-1}(P_{n-1}W(1) - L_{n-1}W(0)), P_{n-1}Z(1) - L_{n-1}Z(0))_{c-1} \\ = & (C^T\mathbb{T}W_x(0), Z(1))_c - (\mathbb{T}W_x(0), Z(0))_c + (Z(1), C^T\mathbb{T}W_x(0))_c - (Z(0), \mathbb{T}W_x(0))_c \end{aligned}$$

$$
\begin{aligned}
&+(\hat\alpha\alpha^{-1}(P_{n-1}Z(1)-L_{n-1}Z(0)),P_{n-1}W(1)-L_{n-1}W(0))_{c\text{-}1}\\
&+(\hat\alpha\alpha^{-1}(P_{n-1}W(1)-L_{n-1}W(0)),P_{n-1}Z(1)-L_{n-1}Z(0))_{c\text{-}1}\\
=\;&(\mathbb{T}W_x(0),CZ(1)-Z(0))_c+(CZ(1)-Z(0),\mathbb{T}W_x(0))_c\\
&+(\hat\alpha\alpha^{-1}(P_{n-1}Z(1)-L_{n-1}Z(0)),P_{n-1}W(1)-L_{n-1}W(0))_{c\text{-}1}\\
&+(\hat\alpha\alpha^{-1}(P_{n-1}W(1)-L_{n-1}W(0)),P_{n-1}Z(1)-L_{n-1}Z(0))_{c\text{-}1}\\
=\;&(L_{n-1}\mathbb{T}W_x(0),L_{n-1}(CZ(1)-Z(0)))_{c\text{-}1}+(L_{n-1}(CZ(1)-Z(0)),L_{n-1}\mathbb{T}W_x(0))_{c\text{-}1}\\
&+(\hat\alpha\alpha^{-1}(P_{n-1}Z(1)-L_{n-1}Z(0)),P_{n-1}W(1)-L_{n-1}W(0))_{c\text{-}1}\\
&+(\hat\alpha\alpha^{-1}(P_{n-1}W(1)-L_{n-1}W(0)),P_{n-1}Z(1)-L_{n-1}Z(0))_{c\text{-}1}\\
=\;&(L_{n-1}\mathbb{T}W_x(0),P_{n-1}Z(1)-L_{n-1}Z(0))_{c\text{-}1}+(P_{n-1}Z(1)-L_{n-1}Z(0),L_{n-1}\mathbb{T}W_x(0))_{c\text{-}1}\\
&+(\hat\alpha\alpha^{-1}(P_{n-1}Z(1)-L_{n-1}Z(0)),P_{n-1}W(1)-L_{n-1}W(0))_{c\text{-}1}\\
&+(\hat\alpha\alpha^{-1}(P_{n-1}W(1)-L_{n-1}W(0)),P_{n-1}Z(1)-L_{n-1}Z(0))_{c\text{-}1}\\
=\;&-((-\hat\alpha Q+L_{n-1}\mathbb{T}W_x(0)),\alpha(-\hat\alpha Q+L_{n-1}\mathbb{T}W_x(0)))_{c\text{-}1}\\
&-(\alpha(-\hat\alpha Q+L_{n-1}\mathbb{T}W_x(0)),(-\hat\alpha Q+L_{n-1}\mathbb{T}W_x(0)))_{c\text{-}1}\\
=\;&-2\left\|\alpha^{\frac12}(-\hat\alpha Q+L_{n-1}\mathbb{T}W_x(0))\right\|^2_{c-1}\le 0.
\end{aligned}
$$

Therefore, $\mathcal{A}$ is dissipative in $\mathcal{H}$.

Next, we prove that $\mathcal{A}^{-1}$ exists and is compact. Clearly, $\mathcal{A}$ is densely defined and closed operator in $\mathcal{H}$. For any fixed $F=(F_1,F_2,F_3)\in\mathcal{H}$, we consider the solvability of the equation $\mathcal{A}Y=F$, $Y=(W,Z,Q)\in\mathcal{D}(\mathcal{A})$, i.e.,

$$
\begin{cases}
Z(x)=F_1(x),\\
\mathbb{T}W_{xx}(x)=\mathbb{M}F_2(x),\\
-\hat\alpha Q+L_{n-1}\mathbb{T}W_x(0)=F_3
\end{cases}
\tag{8.2.5}
$$

with boundary conditions

$$
\begin{cases}
E_1W(0)=E_nW(1)=0,\\
P_{n-1}[\mathbb{T}W_x(1)]-L_{n-1}[\mathbb{T}W_x(0)]=0,\\
P_{n-1}W(1)-L_{n-1}W(0)=-\alpha Q.
\end{cases}
\tag{8.2.6}
$$

Solving the differential equation in (8.2.5), we get

$$
\begin{aligned}
\mathbb{T}W_x(x)-\mathbb{T}W_x(0)&=\int_0^x\mathbb{M}F_2(s)ds,\\
\mathbb{T}W_x(1)-\mathbb{T}W_x(0)&=\int_0^1\mathbb{M}F_2(s)ds.
\end{aligned}
$$

So we have

$$
P_{n-1}(C^T-I)\mathbb{T}W_x(0)=P_{n-1}\int_0^1\mathbb{M}F_2(s)ds,
$$

i.e.,

$$P_{n-1}\begin{pmatrix} -T_1 w_{1,x}(0)+T_2 w_{2,x}(0)\\ -T_2 w_{2,x}(0)+T_3 w_{3,x}(0)\\ -T_3 w_{3,x}(0)+T_4 w_{4,x}(0)\\ \vdots \\ -T_{n-1} w_{n-1,x}(0)+T_n w_{n,x}(0)\end{pmatrix} = P_{n-1}\int_0^1 MF_2(s)ds.$$

A direct calculation leads to

$$\begin{aligned} T_2 w_{2,x}(0) &= \int_0^1 m_1 F_{21}(s)ds + T_1 w_{1,x}(0),\\ &\cdots \\ T_j w_{j,x}(0) &= \sum_{i=1}^{j-1}\int_0^1 m_i F_{2i}(s)ds + T_1 w_{1,x}(0),\\ &\cdots \\ T_n w_{n,x}(0) &= \sum_{i=1}^{n-1}\int_0^1 m_i F_{2i}(s)ds + T_1 w_{1,x}(0). \end{aligned}$$

Now we define the vectors $I_{n\times 1}$ and G_1 by

$$I_{n\times 1} := \begin{pmatrix} 1\\ 1\\ \vdots\\ 1\end{pmatrix}_{n\times 1}, \quad G_1 := \begin{pmatrix} 0\\ \int_0^1 m_1 F_{21}(s)ds\\ \sum_{i=1}^{2}\int_0^1 m_i F_{2i}(s)ds\\ \sum_{i=1}^{3}\int_0^1 m_i F_{2i}(s)ds\\ \vdots\\ \sum_{i=1}^{n-1}\int_0^1 m_i F_{2i}(s)ds\end{pmatrix}.$$

So we have

$$\mathbb{T}W_x(0) = T_1 w_{1,x}(0) I_{n\times 1} + G_1.$$

Thus

$$\begin{aligned} \mathbb{T}W_x(x) &= \mathbb{T}W_x(0) + \int_0^x \mathbb{M}F_2(s)ds\\ &= T_1 w_{1,x}(0) I_{n\times 1} + G_1 + \int_0^x \mathbb{M}F_2(s)ds. \end{aligned} \tag{8.2.7}$$

From above we get

$$\begin{aligned} W(1)-W(0) &= \int_0^1 W_x(s)ds = \int_0^1 \mathbb{T}^{-1}\left[T_1 w_{1,x}(0) I_{n\times 1} + G_1 + \int_0^x \mathbb{M}F_2(s)ds\right]\\ &= \mathbb{T}^{-1}\left[T_1 w_{1,x}(0) I_{n\times 1} x + G_1 x + \int_0^x ds\int_0^s \mathbb{M}F_2(r)dr\right]. \end{aligned}$$

From the third equation in (8.2.5), we get

$$\begin{aligned} Q &= \hat{\alpha}^{-1}(L_{n-1}\mathbb{T}W_x(0) - F_3) \\ &= \hat{\alpha}^{-1}\left(T_1 w_{1,x}(0) I_{(n-1)\times 1} + L_{n-1}G_1 - F_3\right). \end{aligned}$$

Because of $w_n(1) = 0$, we have $W(1) = \begin{pmatrix} P_{n-1}W(1) \\ 0 \end{pmatrix}$, and hence

$$W(1) = C^T W(0) - \alpha\hat{\alpha}^{-1}\begin{pmatrix} T_1 w_{1,x}(0) I_{(n-1)\times 1} + L_{n-1}G_1 - F_3 \\ 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned} & W(1) - W(0) \\ = \;& C^T W(0) - \alpha\hat{\alpha}^{-1}\begin{pmatrix} T_1 w_{1,x}(0) I_{(n-1)\times 1} + L_{n-1}G_1 - F_3 \\ 0 \end{pmatrix} - W(0) \\ = \;& \mathbb{T}^{-1}\left[T_1 w_{1,x}(0) I_{n\times 1} + G_1 + \int_0^1 ds \int_0^s \mathbb{M}F_2(r)dr\right]. \end{aligned}$$

Therefore, we get

$$\begin{aligned} W(0) &= (C^T - I)^{-1}\left\{\alpha\hat{\alpha}^{-1}\begin{pmatrix} T_1 w_{1,x}(0) I_{(n-1)\times 1} + L_{n-1}G_1 - F_3 \\ 0 \end{pmatrix}\right. \\ &\quad \left. + \mathbb{T}^{-1}\left[T_1 w_{1,x}(0) I_{n\times 1} + G_1 + \int_0^1 ds \int_0^s \mathbb{M}F_2(r)dr\right]\right\}. \end{aligned}$$

Since $w_1(0) = 0$ and

$$\begin{aligned} -w_1(0) &= \sum_{i=1}^{n-1} \alpha_i \hat{\alpha_i}^{-1}\left(T_1 w_{1,x}(0) + \sum_{j=1}^{i}\int_0^1 m_j F_{2j}(s)ds - F_{3i}\right) \\ &\quad + \sum_{i=1}^{n} T_i^{-1}\left(T_1 w_{1,x}(0) + \int_0^1 ds \int_0^s m_i F_{2i}(r)dr\right) \\ &\quad + \sum_{i=2}^{n} T_i^{-1}\sum_{j=1}^{i-1}\int_0^1 m_j F_{2j} ds, \end{aligned}$$

a direct calculation yields

$$\begin{aligned} w_{1,x}(0) &= \frac{-\sum\limits_{i=1}^{n-1}\frac{\alpha_i}{\hat{\alpha_i}}\left(\sum\limits_{j=1}^{i}\int_0^1 m_j F_{2j}(s)ds - F_{3i}\right) - \sum\limits_{i=1}^{n}\frac{1}{T_i}\int_0^1 ds\int_0^s m_i F_{2i}(r)dr - \sum\limits_{i=2}^{n}\frac{1}{T_i}\sum\limits_{j=1}^{i-1}\int_0^1 m_j F_{2j} ds}{\sum\limits_{i=1}^{n-1}\frac{\alpha_i}{\hat{\alpha_i}}T_1 + \sum\limits_{i=1}^{n}\frac{1}{T_i}T_1} \\ &:= G_2. \end{aligned}$$

So we have

$$Q = \hat{\alpha}^{-1}\left(a_1 G_2 I_{(n-1)\times 1} + L_{n-1}G_1 - F_3\right),$$

$$
\begin{aligned}
W(0) &= (C^T - I)^{-1}\left\{\alpha\hat{\alpha}^{-1}\begin{pmatrix} T_1G_2I_{(n-1)\times 1} + L_{n-1}G_1 - F_3 \\ 0 \end{pmatrix}\right. \\
&\quad \left. +\mathbb{T}^{-1}\left[T_1G_2I_{n\times 1} + G_1 + \int_0^1 ds\int_0^s \mathbb{M}F_2(r)dr\right]\right\}
\end{aligned}
$$

and hence

$$
\begin{aligned}
W(x) &= \mathbb{T}^{-1}\left[T_1w_{1,x}(0)I_{n\times 1}x + G_1x + \int_0^x ds\int_0^s \mathbb{M}F_2(r)dr\right] + W(0) \\
&= \mathbb{T}^{-1}\left[T_1G_2I_{n\times 1}x + G_1x + \int_0^x ds\int_0^s \mathbb{M}F_2(r)dr\right] \\
&\quad +(C^T - I)^{-1}\left\{\alpha\hat{\alpha}^{-1}\begin{pmatrix} T_1G_2I_{(n-1)\times 1} + L_{n-1}G_1 - F_3 \\ 0 \end{pmatrix}\right. \\
&\quad \left. +\mathbb{T}^{-1}\left[T_1G_2I_{n\times 1} + G_1 + \int_0^1 ds\int_0^s \mathbb{M}F_2(r)dr\right]\right\}.
\end{aligned}
$$

Let $W(x)$ and Q be given by above, then $(W, F_1, Q)) \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}(W, F_1, Q) = F$. So $\mathcal{A}^{-1}$ exists and $\mathcal{A}^{-1}F = (W, F_1, Q)$. Note that

$$(W, F_1, Q) \in \mathcal{D}(\mathcal{A}) \subset H^2 \times H^1 \times \mathbb{C}^{n-1} \subset H^1 \times L^2 \times \mathbb{C}^{n-1}.$$

The Sobolev's Embedding Theorem asserts that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. Therefore, $\mathcal{A}$ generates a C_0 semigroup of contraction by the Lumer-Phillips Theorem (cf. [92]). □

As a consequence of Theorem 8.2.1, we have the following result.

COROLLARY 8.2.1 *The spectrum of $\mathcal{A}$ consists of all isolated eigenvalue, i.e., $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$.*

In order to get the asymptotic stability of the closed loop system (8.2.4), thank to a result in [70], we only need to check that there is no eigenvalue on the imaginary axis.

THEOREM 8.2.2 *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before, $S(t)$ be the C_0 semigroup generated by $\mathcal{A}$. If function equations in σ*

$$
\left\{\begin{array}{lll}
\cos\sqrt{\frac{m_1}{T_1}}\sigma = 0, & \cos\sqrt{\frac{m_n}{T_n}}\sigma = 0, & \sin\sqrt{\frac{m_2}{T_2}}\sigma = 0, \\
\sin\sqrt{\frac{m_3}{T_3}}\sigma = 0, & \cdots \quad \cdots \quad , & \sin\sqrt{\frac{m_{n-1}}{T_{n-1}}}\sigma = 0
\end{array}\right. \tag{8.2.8}
$$

have no solution on the real axis, then $S(t)$ is asymptotically stable.

Proof We see from Theorem 8.2.1 that $0 \notin \sigma(\mathcal{A})$. It only needs to check that $\lambda = i\sigma$ is not an eigenvalue of $\mathcal{A}$ provided that $\sigma \in \mathbb{R},\ \sigma \neq 0$.

By contradiction, if $\lambda = i\sigma$ is an eigenvalue of $\mathcal{A}$ and (W, Z, Q) is corresponding an eigenvector, then we have

$$Z(x) = \lambda W(x), \quad \mathbb{T}W_{xx}(x) = \lambda\mathbb{M}Z(x), \quad x \in (0, 1), \quad -\hat{\alpha}Q + L_{n-1}\mathbb{T}W_x(0) = \lambda Q,$$

and hence

$$0 = \Re\lambda\left((W, Z, Q), (W, Z, Q)\right)_{\mathcal{H}}$$

$$
\begin{aligned}
&= \Re(\mathcal{A}(W,Z,Q),(W,Z,Q))_{\mathcal{H}} \\
&= \lambda^2(Q,-\alpha^{-1}Q)_{c-1} + \lambda^2(-\alpha^{-1}Q,Q)_{c-1}
\end{aligned}
$$

which implies that $Q = 0$. Thus W satisfies the equations:

$$
\begin{cases}
\lambda^2 \mathbb{M} W(x) = \mathbb{T} W_{xx}(x), \\
E_1 W(0) = E_n W(1) = 0, \\
P_{n-1}\mathbb{T}W_x(1) - L_{n-1}\mathbb{T}W_x(0) = 0, \\
P_{n-1}W(1) - L_{n-1}W(0) = 0, \\
L_{n-1}\mathbb{T}W_x(0) = 0.
\end{cases}
\tag{8.2.9}
$$

Let us consider the first string. Due to $E_1 W(0) = 0$, we have

$$
w_1(x) = c_{11}\sinh\sqrt{\frac{m_1}{T_1}}\lambda x = ic_{11}\sin\sqrt{\frac{m_1}{T_1}}\sigma x.
$$

Since $P_{n-1}\mathbb{T}W_x(1) = L_{n-1}\mathbb{T}W_x(0) = 0$, we have

$$
w_{1,x}(1) = c_{11}\sqrt{\frac{m_1}{T_1}}\lambda\cosh\sqrt{\frac{m_1}{T_1}}\lambda = c_{11}\sqrt{\frac{m_1}{T_1}}\lambda\cos\sqrt{\frac{m_1}{T_1}}\sigma = 0.
$$

Thus we have either $c_{11} = 0$ or $\cos\sqrt{\frac{m_1}{T_1}}\sigma = 0$.

If $c_{11} = 0$, then $w_1(x) = 0$. From this we can easily deduce that $w_2(x) = w_3(x) = \cdots = w_n(x) = 0$, i.e, $W(x) = 0$, and hence $(W,Z,Q) = 0$, which contradicts that (W,Z,Q) is an eigenvector of $\mathcal{A}$. Therefore, $\cos\sqrt{\frac{m_1}{T_1}}\sigma = 0$.

When $\cos\sqrt{\frac{m_1}{T_1}}\sigma = 0$, from $P_{n-1}\mathbb{T}W_x(1) = L_{n-1}\mathbb{T}W_x(0) = 0$ and $E_n W(1) = 0$ we can obtain that

$$
w_j(x) = c_j\cosh\sqrt{\frac{m_j}{T_j}}\lambda x = c_j\cos\sqrt{\frac{m_j}{T_j}}\sigma x, \quad j = 2,3,\cdots,n,
$$

$$
w_{j,x}(1) = -c_j\sqrt{\frac{m_j}{T_j}}\sigma\sin\sqrt{\frac{m_j}{T_j}}\sigma = 0, \quad j = 2,3,\cdots,n-1.
$$

Therefore, $\sigma \in \mathbb{R}$ satisfies the following function equations:

$$
\begin{cases}
\cos\sqrt{\frac{m_1}{T_1}}\sigma = 0, \quad \cos\sqrt{\frac{m_n}{T_n}}\sigma = 0, \quad \sin\sqrt{\frac{m_2}{T_2}}\sigma = 0, \\
\sin\sqrt{\frac{m_3}{T_3}}\sigma = 0, \qquad \cdots \quad \cdots, \qquad \sin\sqrt{\frac{m_{n-1}}{T_{n-1}}}\sigma = 0.
\end{cases}
$$

This also contradicts the assumption. Therefore, there is no an eigenvalue of $\mathcal{A}$ on the imaginary axis. The stability theorem [70] asserts that the closed loop system (8.2.4) is asymptotically stable. $\square$

8.3 Asymptotic analysis of spectrum of $\mathcal{A}$

In this section we discuss the asymptotic distribution of the spectrum of $\mathcal{A}$. Theorem 8.2.1 shows that $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$, so we need only to discuss the eigenvalues of $\mathcal{A}$.

Let $\lambda \in \sigma(\mathcal{A})$ and $(W, Z, Q) \in \mathcal{D}(\mathcal{A})$ be a corresponding eigenvector. Then $Z(x) = \lambda W(x)$, and $W(x)$ satisfies the following differential equations

$$\begin{cases} \lambda^2 \mathbb{M} W(x) - \mathbb{T} W_{xx}(x) = 0, \\ E_1 W(0) = E_n W(1) = 0, \\ P_{n-1} \mathbb{T} W_x(1) - L_{n-1} \mathbb{T} W_x(0) = 0, \\ P_{n-1} W(1) - L_{n-1} W(0) = -\alpha Q, \\ \lambda Q = -\hat{\alpha} Q + L_{n-1} \mathbb{T} W_x(0). \end{cases} \tag{8.3.1}$$

For the sake of convenience, we denote

$$B = \operatorname{diag}\left(\sqrt{\frac{m_1}{T_1}}, \sqrt{\frac{m_2}{T_2}}, \cdots, \sqrt{\frac{m_n}{T_n}} \right) = \sqrt{\mathbb{T}}^{-1} \sqrt{\mathbb{M}}. \tag{8.3.2}$$

The general solution of the differential equations in (8.3.1) is of the form

$$W = e^{B\lambda x} \eta_1 + e^{-B\lambda x} \eta_2$$

where η_1 and η_2 are vectors in $\mathbb{C}^n$ that will be determined later. Using the boundary conditions in (8.3.1), we get

$$\begin{aligned} & E_1(\eta_1 + \eta_2) = 0, \\ & E_n(e^{B\lambda} \eta_1 + e^{-B\lambda} \eta_2) = 0, \\ & P_{n-1} \mathbb{T}(\lambda B e^{B\lambda} \eta_1 - \lambda B e^{-B\lambda} \eta_2) - L_{n-1} \mathbb{T}(\lambda B \eta_1 - \lambda B \eta_2) = 0, \\ & P_{n-1} \left(e^{B\lambda} \eta_1 + e^{-B\lambda} \eta_2 \right) - L_{n-1}(\eta_1 + \eta_2) = -\alpha(\lambda + \hat{\alpha})^{-1} L_{n-1} \mathbb{T}(\lambda B \eta_1 - \lambda B \eta_2). \end{aligned}$$

Since $\lambda \neq 0$, we can write the above into the matrix form

$$\begin{pmatrix} S_{11}(\lambda) & S_{12}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0, \tag{8.3.3}$$

where

$$S_{11}(\lambda) = \begin{bmatrix} E_1 \\ P_{n-1} \sqrt{\mathbb{T}\mathbb{M}} e^{B\lambda} - L_{n-1} \sqrt{\mathbb{T}\mathbb{M}} \end{bmatrix}$$

$$S_{12}(\lambda) = \begin{bmatrix} E_1 \\ -P_{n-1} \sqrt{\mathbb{T}\mathbb{M}} e^{-B\lambda} + L_{n-1} \sqrt{\mathbb{T}\mathbb{M}} \end{bmatrix}$$

$$S_{21}(\lambda) = \begin{bmatrix} P_{n-1} e^{B\lambda} - L_{n-1} + \lambda\alpha(\lambda + \hat{\alpha})^{-1} L_{n-1} \sqrt{\mathbb{T}\mathbb{M}} \\ 0 \quad 0 \quad 0 \cdots 0 \quad e^{\sqrt{\frac{m_n}{a_n}}\lambda} \end{bmatrix}$$

$$S_{22}(\lambda) = \begin{bmatrix} P_{n-1} e^{-B\lambda} - L_{n-1} - \lambda\alpha(\lambda + \hat{\alpha})^{-1} L_{n-1} \sqrt{\mathbb{T}\mathbb{M}} \\ 0 \quad 0 \quad 0 \cdots 0 \quad e^{-\sqrt{\frac{m_n}{T_n}}\lambda} \end{bmatrix}$$

where we have used $\mathbb{T} B = \sqrt{\mathbb{T}\mathbb{M}}$. Set matrices

$$\widehat{S}_{11} = \begin{bmatrix} 1 \quad 0 \quad 0 \cdots 0 \quad 0 \\ P_{n-1} \sqrt{\mathbb{T}\mathbb{M}} e^{B\lambda} - L_{n-1} \sqrt{\mathbb{T}\mathbb{M}} \end{bmatrix}_{n \times n}$$

$$\widehat{S}_{12} = \begin{bmatrix} 1 \quad 0 \quad 0\cdots 0 \quad 0 \\ -P_{n-1}\sqrt{\mathbb{TM}}e^{-B\lambda} + L_{n-1}\sqrt{\mathbb{TM}} \end{bmatrix}_{n\times n}$$

$$\widehat{S}_{21} = \begin{bmatrix} P_{n-1}e^{B\lambda} - L_{n-1} + \lambda\alpha(\lambda+\hat{\alpha})^{-1}L_{n-1}\sqrt{\mathbb{TM}} \\ 0 \quad 0 \quad 0\cdots 0 \quad e^{\sqrt{\frac{m_n}{T_n}}\lambda} \end{bmatrix}_{n\times n}$$

$$\widehat{S}_{22} = \begin{bmatrix} P_{n-1}e^{-B\lambda} - L_{n-1} - \lambda\alpha(\lambda+\hat{\alpha})^{-1}L_{n-1}\sqrt{\mathbb{TM}} \\ 0 \quad 0 \quad 0\cdots 0 \quad e^{-\sqrt{\frac{m_n}{T_n}}\lambda} \end{bmatrix}_{n\times n}$$

Set

$$\Delta(\lambda) = \det\begin{pmatrix} \widehat{S}_{11} & \widehat{S}_{12} \\ \widehat{S}_{21} & \widehat{S}_{22} \end{pmatrix}.$$

Then, $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ if and only if $\Delta(\lambda) = 0$. Therefore we only need to discuss the zeros of $\Delta(\lambda)$.

When $\Re\lambda \to +\infty$, we have

$$\lim_{\Re\lambda\to+\infty} \frac{\Delta(\lambda)}{e^{\sum\limits_{i=1}^{n}\sqrt{\frac{m_i}{T_i}}\lambda}}$$

$$= \left| \begin{array}{cccccc} 0 & 0 & 0 & \cdots & \cdots & 0 \\ \sqrt{m_1T_1} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \sqrt{m_2T_2} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \sqrt{m_3T_3} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \sqrt{m_{n-1}T_{n-1}} & 0 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{array} \right.$$

$$
\begin{array}{cccccc|}
1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & \sqrt{m_2T_2} & 0 & \cdots & \cdots & 0 \\
0 & 0 & \sqrt{m_3T_3} & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \sqrt{m_{n-1}T_{n-1}} & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \sqrt{m_nT_n} \\
0 & -1-\alpha_1\sqrt{m_2T_2} & 0 & \cdots & \cdots & 0 \\
0 & 0 & -1-\alpha_2\sqrt{m_3T_3} & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & -1-\alpha_{n-2}\sqrt{m_{n-1}T_{n-1}} & 0 \\
0 & 0 & \cdots & \cdots & \cdots & -1-\alpha_{n-1}\sqrt{m_nT_n} \\
0 & 0 & 0 & \cdots & \cdots & 0
\end{array}
$$

$$
\begin{aligned}
= \quad & (-1)^{n^2}\left\{\sqrt{T_1m_1}(1+\alpha_1\sqrt{T_2m_2})+\sqrt{T_2m_2}\right\}\times\left\{\sqrt{T_2m_2}(1+\alpha_2\sqrt{T_3m_3})+\sqrt{T_3m_3}\right\} \\
& \times\cdots\times\left\{\sqrt{T_{n-2}m_{n-2}}(1+\alpha_{n-2}\sqrt{T_{n-1}m_{n-1}})+\sqrt{T_{n-1}m_{n-1}}\right\} \\
& \times\left\{\sqrt{T_{n-1}m_{n-1}}(1+\alpha_{n-1}\sqrt{T_nm_n})+\sqrt{T_nm_n}\right\}\neq 0
\end{aligned}
$$

and when $\Re\lambda\to-\infty$,

$$
\lim_{\Re\lambda\to-\infty}\frac{\Delta(\lambda)}{e^{-\sum\limits_{i=1}^{n}\sqrt{\frac{m_i}{T_i}}\lambda}}
$$

$$
= \left|\begin{array}{cccccc}
1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & -\sqrt{m_2T_2} & 0 & \cdots & \cdots & 0 \\
0 & 0 & -\sqrt{m_3T_3} & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & -\sqrt{m_{n-1}T_{n-1}} & 0 \\
0 & 0 & \cdots & \cdots & \cdots & -\sqrt{m_nT_n} \\
0 & -1+\alpha_1\sqrt{m_2T_2} & 0 & \cdots & \cdots & 0 \\
0 & 0 & -1+\alpha_2\sqrt{m_3T_3} & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & -1+\alpha_{n-2}\sqrt{m_{n-1}T_{n-1}} & 0 \\
0 & 0 & \cdots & \cdots & \cdots & -1+\alpha_{n-1}\sqrt{m_nT_n} \\
0 & 0 & 0 & \cdots & \cdots & 0
\end{array}\right.
$$

$$
\left.\begin{array}{cccccc}
0 & 0 & 0 & \cdots & \cdots & 0 \\
-\sqrt{m_1T_1} & 0 & 0 & \cdots & \cdots & 0 \\
0 & -\sqrt{m_2T_2} & 0 & \cdots & \cdots & 0 \\
0 & 0 & -\sqrt{m_3T_3} & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & -\sqrt{m_{n-1}T_{n-1}} & 0 \\
1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 1 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 1 & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 1
\end{array}\right|
$$

$$
\begin{aligned}
= \;& \left\{\sqrt{T_1m_1}(-1+\alpha_1\sqrt{T_2m_2})-\sqrt{T_2m_2}\right\} \times \left\{\sqrt{T_2m_2}(-1+\alpha_2\sqrt{T_3m_3})-\sqrt{T_3m_3}\right\} \\
& \times \cdots \times \left\{\sqrt{T_{n-2}m_{n-2}}(-1+\alpha_{n-2}\sqrt{T_{n-1}m_{n-1}})-\sqrt{T_{n-1}m_{n-1}}\right\} \\
& \times \left\{\sqrt{T_{n-1}m_{n-1}}(-1+\alpha_{n-1}\sqrt{T_nm_n})-\sqrt{T_nm_n}\right\}.
\end{aligned}
$$

So we have

$$\lim_{\Re\lambda\to-\infty}\left|\frac{\Delta(\lambda)}{e^{-\sum\limits_{i=1}^{n}\sqrt{\frac{m_i}{T_i}}\lambda}}\right|>0$$

provided that α satisfies

$$\alpha_i \neq \frac{1}{\sqrt{m_iT_i}}+\frac{1}{\sqrt{m_{i+1}T_{i+1}}}, \quad i=1,2,\cdots,n-1.$$

When the above conditions are fulfilled, there exist positive constants d_1, d_2 and h such that

$$d_1e^{|\Re\lambda|\sum\limits_{i=1}^{n}\sqrt{\frac{m_i}{a_i}}} \leq |\Delta(\lambda)| \leq d_2e^{|\Re\lambda|\sum\limits_{i=1}^{n}\sqrt{\frac{m_i}{a_i}}}, \quad |\Re\lambda|>h, \tag{8.3.4}$$

which implies that the zeros of $\Delta(\lambda)$ lie in the strip of $|\Re\lambda|\leq h$.

By now we have proved the following result.

THEOREM 8.3.1 *Let $\mathcal{A}$ be defined as before. If the gain constants satisfy the conditions*

$$\alpha_i \neq \frac{1}{\sqrt{m_ia_i}}+\frac{1}{\sqrt{m_{i+1}a_{i+1}}}, \quad i=1,2,\cdots,n-1, \tag{8.3.5}$$

then there is a positive constant h such that

$$\sigma(\mathcal{A})=\{\lambda\in\mathbb{C}\mid\Delta(\lambda)=0\}\subset\{\lambda\in\mathbb{C}\mid|\Re\lambda|\leq h\}. \tag{8.3.6}$$

Let us recall some notions before we proceed. A set σ is said to be *separable* if

$$\inf_{\lambda,\mu\in\sigma,\lambda\neq\mu}|\lambda-\mu|>0.$$

An entire function f of exponential type is said to be of sine type if
(a). the zeros of f lie in a strip $\{\lambda \in \mathbb{C} \mid |y| \le h, \lambda = y + ix\}$ for some $h > 0$;
(b). there are positive constants c_1, c_2 and $y_0 \in \mathbb{R}$ such that

$$c_1 e^{|y|} \le |f(y+ix)| \le c_2 e^{|y|}, \quad |y| \ge |y_0|.$$

The following result for the sine type functions is due to Levin (cf. see [9]).

LEMMA 8.3.1 *If f is an entire function of sine type, then the set of its zeros is a union of finitely many separated sets.*

As a direct consequence of Lemma 8.3.1 together (8.3.5), we have the following result.

COROLLARY 8.3.1 *Let $\mathcal{A}$ be defined as before. Suppose that conditions in (8.3.5) are fulfilled. Then $\sigma(\mathcal{A})$ is a union of finitely many separated sets.*

8.4 Completeness and Riesz basis property of root vectors of $\mathcal{A}$

In this section we discuss the completeness and Riesz basis property of eigenvectors and generalized eigenvectors of $\mathcal{A}$. Firstly, we establish the completeness of the eigenvectors and generalized eigenvectors of $\mathcal{A}$ and then use the spectral distribution of $\mathcal{A}$ to obtain the Riesz basis property.

THEOREM 8.4.1 *Suppose that the conditions in (8.3.5) are fulfilled, then the system of eigenvectors and generalized eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$.*

Proof We can assume that $\sigma(\mathcal{A}) = \{\lambda_k,\ k \in \mathbb{N}\}$ due to Theorem 8.2.1. Set

$$Sp(\mathcal{A}) = \overline{\left\{\sum_k y_k, y_k \in E(\lambda_k, \mathcal{A})\mathcal{H}, \lambda_k \in \sigma(\mathcal{A})\right\}}$$

where $E(\lambda_k, \mathcal{A})$ is the Riesz projection corresponding to λ_k. Then the completeness of eigenvectors and generalized eigenvectors of $\mathcal{A}$ is equivalent to $Sp(\mathcal{A}) = \mathcal{H}$. We shall finish the proof of completeness by showing that for any $F = (f_1, f_2, f_3) \perp Sp(\mathcal{A})$ implies $F = 0$.

Let $F = (f_1, f_2, f_3) \perp Sp(\mathcal{A})$, then $R^*(\lambda, \mathcal{A})F$ is an $\mathcal{H}$-valued entire function on $\mathbb{C}$. For any $G = (g_1, g_2, g_3) \in \mathcal{H}$, the scalar function

$$U(\lambda) = (G, R^*(\lambda, \mathcal{A})F)_{\mathcal{H}}, \quad \forall \lambda \in \mathbb{C} \tag{8.4.1}$$

also is an entire function. In particular, it holds that $\lim\limits_{\Re\lambda \to +\infty} U(\lambda) = 0$ because $\mathcal{A}$ is the generator of a C_0 semigroup. For $\lambda \in \rho(\mathcal{A})$, we have

$$U(\lambda) = (R(\lambda, \mathcal{A})G, F)_{\mathcal{H}}.$$

In what follows, we prove that $U(\lambda)$ is bounded on the negative real axis. To this end, denote by $Y_1 = R(\lambda, \mathcal{A})G,\ \lambda \in \rho(\mathcal{A}) \cap \mathbb{R}^-$ where $Y_1 = (y_1, z_1, Q_1) \in \mathcal{D}(\mathcal{A})$. Then $(\lambda I - \mathcal{A})Y_1 = G$ is equivalent to the equations

$$\begin{cases} \lambda y_1 - z_1 = g_1, \\ \lambda z_1 - \mathbb{M}^{-1}\mathbb{T} y_{1,xx} = g_2, \\ \lambda Q_1 + \hat{\alpha} Q_1 - L_{n-1}\mathbb{T} y_{1,x}(0) = g_3, \\ P_{n-1}\mathbb{T} y_{1,x}(1) = L_{n-1}\mathbb{T} y_{1,x}(0), \\ E_1 y_1(0) = E_n y_1(1) = 0, \\ P_{n-1} y_1(1) - L_{n-1} y_1(0) = -\alpha Q_1. \end{cases} \tag{8.4.2}$$

We introduce $Y_2 = (y_2, z_2, Q_2) \in \mathcal{H}$, which satisfies auxiliary equations:

$$\begin{cases} \lambda y_2 - z_2 = g_1, \\ \lambda z_2 - \mathbb{M}^{-1}\mathbb{T} y_{2,xx} = g_2, \\ \lambda Q_2 + \hat{\alpha} Q_2 - L_{n-1}\mathbb{T} y_{2,x}(0) = g_3, \\ P_{n-1}\mathbb{T} y_{2,x}(1) = L_{n-1}\mathbb{T} y_{2,x}(0), \\ E_1 y_2(0) = E_n y_2(1) = 0, \\ P_{n-1} y_2(1) - L_{n-1} y_2(0) = 0, \\ \hat{\alpha} Q_2 - L_{n-1}\mathbb{T} y_{2,x}(0) = 0, \end{cases} \tag{8.4.3}$$

where λ is the same as that in (8.4.2). Then we have

$$\begin{aligned}
&\|G\|\|Y_2\| \geq |\Re(G, Y_2)| \\
=\ & \left|\Re\left(\int_0^1 [(\mathbb{T} g_{1,x}, y_{2,x})_c + (\mathbb{M} g_2, z_2)_c]\, dx + (\hat{\alpha}\alpha g_3, Q_2)_{c-1}\right)\right| \\
=\ & \left|\Re\left(\int_0^1 [(\mathbb{T}(\lambda y_2 - z_2)_x, y_{2,x})_c + (\mathbb{M}(\lambda z_2 - \mathbb{M}^{-1}\mathbb{T} y_{2,xx}), z_2)_c] dx + (\hat{\alpha}\alpha g_3, Q_2)_{c-1}\right)\right| \\
=\ & \left|\Re\left(\int_0^1 ((\lambda \mathbb{T} y_{2,x}, y_{2,x})_c dx - \int_0^1 (\mathbb{T} z_{2,x}, y_{2,x})_c dx + \int_0^1 \lambda(\mathbb{M} z_2, z_2)_c dx\right.\right. \\
& \left.\left. - \int_0^1 (\mathbb{T} y_{2,xx}, z_2)_c dx + (\hat{\alpha}\alpha g_3, Q_2)_{c-1}\right)\right| \\
=\ & \left|\Re\left(\int_0^1 (\lambda \mathbb{T} y_{2,x}, y_{2,x})_c dx + \int_0^1 \lambda(\mathbb{M} z_2, z_2)_c dx - (\mathbb{T} y_{2,x}, z_2)_c|_0^1 + (\hat{\alpha}\alpha\lambda Q_2, Q_2)_{c-1}\right)\right| \\
=\ & \left|\Re\lambda\left(\int_0^1 (\mathbb{T} y_{2,x}, y_{2,x})_c dx + \int_0^1 (\mathbb{M} z_2, z_2)_c dx + (\hat{\alpha}\alpha Q_2, Q_2)_{c-1}\right) - \Re(\mathbb{T} y_{2,x}, z_2)_c|_0^1\right| \\
=\ & \left|\Re\lambda\|Y_2\|^2 + \Re\left[(\mathbb{T} y_{2,x}(1), z_2(1))_c - (\mathbb{T} y_{2,x}(0), z_2(0))_c\right]\right| \\
=\ & \left|\Re\lambda\|Y_2\|^2 - Re[(P_{n-1}\mathbb{T} y_{2,x}(1), P_{n-1} z_2(1))_{c-1} - (L_{n-1}\mathbb{T} y_{2,x}(0), L_{n-1} z_2(0))_{c-1}]\right| \\
=\ & \left|\Re\lambda\|Y_2\|^2 + \Re[(L_{n-1}\mathbb{T} y_{2,x}(0), P_{n-1} z_2(1) - L_{n-1} z_2(0))_{c-1}]\right| \\
=\ & |\Re\lambda|\|Y_2\|^2,
\end{aligned}$$

i.e.,

$$\|G\|\|Y_2\| \geq |\Re\lambda|\, \|Y_2\|^2.$$

Therefore, we have

$$\|Y_2\| \le \frac{1}{|\Re\lambda|}\|G\|, \quad \lambda \in \rho(\mathcal{A}) \cap \mathbb{R}^-. \tag{8.4.4}$$

Now set $Y_3(\lambda) = Y_1 - Y_2 = (y_3, z_3, Q_3)$, then the components of $Y_3(\lambda)$ satisfy the following equations

$$\begin{cases} \lambda y_3 - z_3 = 0, \\ \lambda z_3 - \mathbb{M}^{-1}\mathbb{T}y_{3,xx} = 0, \\ \lambda Q_3 + \hat{\alpha}Q_3 - L_{n-1}\mathbb{T}y_{3,x}(0) = 0, \\ P_{n-1}\mathbb{T}y_{3,x}(1) = L_{n-1}\mathbb{T}y_{3,x}(0), \\ E_1 y_3(0) = E_n y_3(1) = 0, \\ P_{n-1}y_3(1) - L_{n-1}y_3(0) = -\alpha(Q_3 + Q_2). \end{cases} \tag{8.4.5}$$

Clearly, $y_3(x)$ is of the form

$$y_3(x) = e^{B\lambda x}\widehat{\eta_1} + e^{-B\lambda x}\widehat{\eta_2}, \tag{8.4.6}$$

where $\widehat{\eta}_1, \widehat{\eta}_2$ are the vectors in $\mathbb{C}^n$, they are determined later. Substituting (8.4.6) into the boundary conditions in (8.4.5), we get

$$D(\lambda)\begin{pmatrix} \widehat{\eta_1} \\ \widehat{\eta_2} \end{pmatrix} = \begin{pmatrix} 0 \\ Q_2 \end{pmatrix} \tag{8.4.7}$$

where

$$D(\lambda) = \begin{pmatrix} \widehat{S}_{11} & \widehat{S}_{12} \\ \widehat{S}_{21} & \widehat{S}_{22} \end{pmatrix}.$$

According to a result in Kato's book [66, Formula (4.12),pp. 28], there is a constant $\mu > 0$ such that

$$\|D^{-1}\| \le \mu\frac{\|D\|^{n-1}}{\det D},$$

where D is a matrix and $\mu > 0$ is a constant independent of D. By a complicated calculation we can show that there exists $\mu_1 > 0$ such that

$$\|D(\lambda)^{-1}\| \le \mu_1\frac{1}{e^{\rho_1|\lambda|}}, \tag{8.4.8}$$

where $\rho_1 = \min\sqrt{\frac{m_j}{T_j}}$, $j = 1, 2, \cdots, n$, $\|\cdot\|$ is the norm in $\mathbb{C}^n$. So we have

$$\left\|\begin{pmatrix} \widehat{\eta_1} \\ \widehat{\eta_2} \end{pmatrix}\right\| \le \|D(\lambda)^{-1}\|\|\begin{pmatrix} 0 \\ Q_2 \end{pmatrix}\| = \|D(\lambda)^{-1}\|\|Q_2\|.$$

Therefore, we have estimate

$$\left\|\begin{pmatrix} \widehat{\eta_1} \\ \widehat{\eta_2} \end{pmatrix}\right\| \le \mu_1\frac{1}{e^{\eta_1|\lambda|}}\|Q_2\|. \tag{8.4.9}$$

Since $Q_3 = (\lambda + \hat{\alpha})^{-1}L_{n-1}\mathbb{T}y_{3,x}(0)$, a direct calculation leads to

$$Q_3 = (\lambda + \hat{\alpha})^{-1}L_{n-1}\sqrt{\mathbb{T}\mathbb{M}}\lambda(\widehat{\eta_1} - \widehat{\eta_2}).$$

Consequently, we have

$$\|Q_3\| \leq \frac{|\Re\lambda|}{\big|\,|\Re\lambda| - \|\hat{\alpha}\|\,\big|}\|\sqrt{\mathbb{TM}}(\widehat{\eta_1} - \widehat{\eta_2})\| \leq \frac{\|\sqrt{\mathbb{TM}}\||\Re\lambda|}{\big|\,|\Re\lambda| - \|\hat{\alpha}\|\,\big|}\left\|\begin{pmatrix} \widehat{\eta_1} \\ \widehat{\eta_2} \end{pmatrix}\right\|.$$

Combining (8.4.9) we have

$$\|Q_3\| \leq \upsilon_1 \frac{|\Re\lambda|}{e^{\eta_1|\lambda|}} \frac{\|\sqrt{\mathbb{TM}}\|}{\big|\,|\Re\lambda| - \|\hat{\alpha}\|\,\big|}\|Q_2\|. \tag{8.4.10}$$

From (8.4.5) we can get

$$\begin{aligned} \int_0^1 (\mathbb{T}y_{3,xx}, y_3)_c dx &= \int_0^1 \lambda^2 (\mathbb{M}y_3(x), y_3(x))_c dx \\ &= (\mathbb{T}y_{3,x}, y_3)_c|_0^1 - \int_0^1 (\mathbb{T}y_{3,x}, y_{3,x})_c dx. \end{aligned}$$

Therefore, we can calculate the norm of $||Y_3||$ as follows

$$\begin{aligned} \|Y_3\|^2 &= \int_0^1 (\mathbb{T}y_{3,x}, y_{3,x})_c dx + \int_0^1 (\mathbb{M}z_3, z_3)_c dx + (\hat{\alpha}\alpha Q_3, Q_3)_{c-1} \\ &= (\mathbb{T}y_{3,x}, y_3)_c|_0^1 + (\hat{\alpha}\alpha Q_3, Q_3)_{c-1} \\ &= (\mathbb{T}y_{3,x}(1), y_3(1))_c - (\mathbb{T}y_{3,x}(0), y_3(0))_c + (\hat{\alpha}\alpha Q_3, Q_3)_{c-1} \\ &= (P_{n-1}\mathbb{T}y_{3,x}(1), P_{n-1}y_3(1))_{c-1} - (L_{n-1}\mathbb{T}y_{3,x}(0), L_{n-1}y_3(0))_{c-1} + (\hat{\alpha}\alpha Q_3, Q_3)_{c-1} \\ &= (L_{n-1}\mathbb{T}y_{3,x}(0), P_{n-1}y_3(1) - L_{n-1}y_3(0))_{c-1} + (\hat{\alpha}\alpha Q_3, Q_3)_{c-1} \\ &= (L_{n-1}\mathbb{T}y_{3,x}(0), -\alpha(Q_2 + Q_3))_{c-1} + (\hat{\alpha}\alpha Q_3, Q_3)_{c-1} \\ &= (-\lambda Q_3, \alpha Q_3)_{c-1} + (-\lambda Q_3 - \hat{\alpha}Q_3, \alpha Q_2)_{c-1} \\ &\leq |\Re\lambda|\|\alpha\|\|Q_3\|^2 + |\Re\lambda|\|\alpha\|\|Q_3\|\|Q_2\| + \|\hat{\alpha}\alpha\|\|Q_3\|\|Q_2\| \\ &\leq \left(\frac{\mu_1^2|Re\lambda|^3\|\alpha\|\|\mathbb{TM}\|}{e^{2\eta_1|\lambda|}(|\Re\lambda| - \|\hat{\alpha}\|)^2} + \frac{\upsilon_1\|\alpha\|\|\sqrt{AM}\||\Re\lambda|^2}{e^{\eta_1|\lambda|}\big|(|\Re\lambda| - \|\hat{\alpha}\|)\big|} + \frac{\mu_1\|\hat{\alpha}\alpha\|\|\sqrt{\mathbb{TM}}\||Re\lambda|}{e^{\eta_1|\lambda|}\big|(|\Re\lambda| - \|\hat{\alpha}\|)\big|}\right)\|Q_2\|^2 \\ &\leq \left(\frac{\mu_1^2|\Re\lambda|^3\|\alpha\|\|\mathbb{TM}\|}{e^{2\eta_1|\lambda|}(|\Re\lambda| - \|\hat{\alpha}\|)^2} + \frac{\upsilon_1\|\alpha\|\|\sqrt{AM}\||\Re\lambda|^2}{e^{\eta_1|\lambda|}\big|(|\Re\lambda| - \|\hat{\alpha}\|)\big|} + \frac{\mu_1\|\hat{\alpha}\alpha\|\|\sqrt{\mathbb{TM}}\||\Re\lambda|}{e^{\eta_1|\lambda|}\big|(|\Re\lambda| - \|\hat{\alpha}\|)\big|}\right)\|Y_2\|^2 \\ &\leq \left(\frac{\mu_1^2|\Re\lambda|^3\|\alpha\|\|\mathbb{TM}\|}{e^{2\eta_1|\lambda|}(|\Re\lambda| - \|\hat{\alpha}\|)^2} + \frac{\mu_1\|\alpha\|\|\sqrt{\mathbb{TM}}\||\Re\lambda|^2}{e^{\eta_1|\lambda|}\big|(|\Re\lambda| - \|\hat{\alpha}\|)\big|} + \frac{\mu_1\|\hat{\alpha}\alpha\|\|\sqrt{\mathbb{TM}}\||\Re\lambda|}{e^{\eta_1|\lambda|}\big|(|\Re\lambda| - \|\hat{\alpha}\|)\big|}\right)\frac{\|G\|^2}{|\Re\lambda|^2}, \end{aligned}$$

that is, for $\lambda \in \rho(\mathcal{A}) \cap \mathbb{R}^-$,

$$\|Y_3\|^2 \leq \left(\frac{\mu_1^2|\Re\lambda|^3\|\alpha\|\|\mathbb{TM}\|}{e^{2\eta_1|\lambda|}(|\Re\lambda| - \|\hat{\alpha}\|)^2} + \frac{\mu_1\|\alpha\|\|\sqrt{\mathbb{TM}}\||\Re\lambda|^2}{e^{\eta_1|\lambda|}\big|(|\Re\lambda| - \|\hat{\alpha}\|)\big|} + \frac{\mu_1\|\hat{\alpha}\alpha\|\|\sqrt{\mathbb{TM}}\||\Re\lambda|}{e^{\eta_1|\lambda|}\big|(|\Re\lambda| - \|\hat{\alpha}\|)\big|}\right)\frac{\|G\|^2}{|\Re\lambda|^2}. \tag{8.4.11}$$

We know from Theorem 8.3.1 that $\lambda \in \rho(\mathcal{A}) \cap \mathbb{R}^-$ for $|\Re\lambda|$ sufficiently large. (8.4.4) together with (8.4.11) lead to

$$\lim_{\Re\lambda \to -\infty} \|R(\lambda, \mathcal{A})G\| = \lim_{\Re\lambda \to -\infty} \|Y_2 + Y_3\| = 0. \tag{8.4.12}$$

That leads to $U(\lambda)$ is bounded on the real axis.

Since $U(\lambda)$ is an entire function of finite exponential type, $U(\lambda)$ is uniformly bounded on the line $\Re\lambda = \alpha > 0$, the Phragmén-Linderöf Theorem (cf. [127]) asserts that

$$|U(\lambda)| \leq M, \quad \forall \lambda \in \mathbb{C}.$$

The Liouville's Theorem further says that $U(\lambda) \equiv 0$. Note that $U(\lambda) = (G, R^*(\lambda, \mathcal{A})F)_{\mathcal{H}}$ for any given $G \in \mathcal{H}$. It must be $R^*(\lambda, \mathcal{A})F = 0$. This means that $F = 0$. Therefore $Sp(\mathcal{A}) = \mathcal{H}$, the desired result follows. $\square$

Next we discuss the generation of Riesz basis of the generalized eigenvectors of $\mathcal{A}$. Firstly, we introduce a result, which comes from [119] and is an extension of the result in [117].

THEOREM 8.4.2 *Let $\mathcal{A}$ be the generator of a C_0 semigroup $T(t)$ on a separable Hilbert space $\mathcal{H}$. Suppose that the following conditions are satisfied:*

1) The spectrum of $\mathcal{A}$ has a decomposition

$$\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A}) \tag{8.4.13}$$

where $\sigma_2(\mathcal{A})$ consists of the isolated eigenvalues of $\mathcal{A}$ of finite multiplicity (repeated many times according to its algebraic multiplicity).

2) There exists a real number $\alpha \in \mathbb{R}$ such that

$$\sup\{\Re\lambda, \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf\{\Re\lambda, \lambda \in \sigma_2(\mathcal{A})\} \tag{8.4.14}$$

3) The set $\sigma_2(\mathcal{A})$ is an union of finite many separated sets.

Then the following statements are true:

i). There exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ such that $\sigma(\mathcal{A}\big|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$ and $\sigma(\mathcal{A}\big|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$; and there exists a finite combination $E(\Omega_k, \mathcal{A})$ of some $\{E(\lambda_k, \mathcal{A})\}_{k=1}^{\infty}$:

$$E(\Omega_k, \mathcal{A}) = \sum_{\lambda \in \Omega_k \cap \sigma_2(\mathcal{A})} E(\lambda, \mathcal{A}) \tag{8.4.15}$$

such that $\{E(\Omega_k, \mathcal{A})\mathcal{H}_2\}_{k \in \mathbb{N}}$ forms a Riesz basis of subspaces for $\mathcal{H}_2$. Furthermore,

$$\mathcal{H} = \overline{\mathcal{H}_1 \oplus \mathcal{H}_2}.$$

ii). If $\sup_{k \geq 1} ||E(\lambda_k, \mathcal{A})|| < \infty$, then

$$\mathcal{D}(\mathcal{A}) \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}. \tag{8.4.16}$$

iii). $\mathcal{H}$ has a decomposition of the topological direct sum, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, if and only if

$$\sup_{n \geq 1} \left\| \sum_{k=1}^{n} E(\Omega_k, \mathcal{A}) \right\| < \infty. \tag{8.4.17}$$

Now we can prove the following result.

THEOREM 8.4.3 *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. If conditions in (8.3.5) are fulfilled, then there is a sequence of eigenvectors and generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$. Indeed, in this case, $\mathcal{A}$ generates a C_0 group on $\mathcal{H}$. In particular, the system associated with $\mathcal{A}$ will satisfy the spectrum determined growth condition.*

Proof Set $\sigma_1(\mathcal{A}) = \{-\infty\}$, $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$. Theorem 8.3.1 shows that all conditions in Theorem 8.4.2 are fulfilled. So the results of Theorem 8.4.2 are true. Hence there is a sequence of eigenvectors and generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}_2$. Theorem 8.4.1 says that the eigenvectors and generalized eigenvectors is complete in $\mathcal{H}$, that is $\mathcal{H}_2 = \mathcal{H}$. Therefore the sequence is also a Riesz basis with parentheses for $\mathcal{H}$. The Riesz basis property of the eigenvectors and generalized eigenvectors together with distribution of spectrum of $\mathcal{A}$ implies that $\mathcal{A}$ generates a C_0 group on $\mathcal{H}$. At the same time, the Riesz basis property together with the uniform boundedness of multiplicities of eigenvalues of $\mathcal{A}$ ensure that the system associated with $\mathcal{A}$ satisfies the spectrum determined growth condition. The proof is then complete. $\square$

We have the following remark about the exponential stability of this system.

REMARK 8.4.1 *When both endpoints are clamped, if one applies the controllers only at the interior nodes, the closed loop system usually is not exponentially stable, even though the system of two-connected strings it also needs to satisfy much more strict conditions, see ([114])*

8.5 Conclusion remark

In this chapter, we studied the property of serially connected strings with discontinuous displacements. One wants to stabilize the system by the tension feedback of the interior nodes. The main results are as follows:

1) Since 0 is an eigenvalue of geometrical multiplicity $n-1$ for the uncontrolled system, one needs at least $n-1$ controllers (the number of controllers is not less than the geometrical multiplicity of the eigenvalue);

2). If the function equations in σ

$$\left\{\begin{array}{lll} \cos\sqrt{\frac{m_1}{T_1}}\sigma = 0, & \cos\sqrt{\frac{m_n}{T_n}}\sigma = 0, & \sin\sqrt{\frac{m_2}{T_2}}\sigma = 0, \\ \sin\sqrt{\frac{m_3}{T_3}}\sigma = 0, & \cdots \quad \cdots \quad , & \sin\sqrt{\frac{m_{n-1}}{T_{n-1}}}\sigma = 0 \end{array}\right. \tag{8.5.1}$$

has no solution on the real axis, then the closed loop system is asymptotically stable. Otherwise, the system is unstable;

3). If the feedback gains α_j satisfy conditions

$$\alpha_i \neq \frac{1}{\sqrt{m_i T_i}} + \frac{1}{\sqrt{m_{i+1}T_{i+1}}}, \quad i = 1, 2, \cdots, n-1, \tag{8.5.2}$$

then the frequencies of the closed loop system are in a strip parallel to the imaginary axis. In this case, there is a sequence of eigenvectors and generalized eigenvectors of the system that forms a Riesz basis with parenthesis for the Hilbert state space.

4). The system satisfies the spectrum determined growth condition.

Let us revisit the controlled system

$$\begin{cases} m_j \frac{\partial^2 w_j(x,t)}{\partial t^2} = T_j \frac{\partial^2 w_j(x,t)}{\partial x^2}, \quad j=1,2,\cdots n, \quad x \in (0,1), \quad t>0, \\ w_1(0,t) = w_n(1,t) = 0, \quad t>0 \\ T_j w_{j,x}(1,t) = T_{j+1} w_{j+1,x}(0,t), \quad j=1,2,\cdots,n-1, \quad t>0, \\ w_j(1,t) - w_{j+1}(0,t) = u_j(t), \quad j=1,2,\cdots,n-1, \quad t>0, \end{cases} \tag{8.5.3}$$

where $u_j(t),\ j=1,2,\cdots,n-1$, are external exciting forces.

The observable nodal values of the system are

$$W'(v) = (w'(a_1,t), w'(a_2,t), w'(a_3,t), \cdots, w'(a_n,t), w'(a_{n+1},t)).$$

Set $y_j(x,t) = T_j w_{j,x}(x,t)$, then $y_j(x,t)$ is a continuous function on $(0,1)$ and satisfies

$$m_j y_{j,tt}(x,t) = T_j y_{j,xx}(x,t).$$

With this transform, the first boundary condition in (8.5.3) becomes

$$y_{1,x}(0,t) = 0, \quad y_{n,x}(1,t) = 0.$$

the second is

$$y_j(1,t) = y_{j+1}(0,t),$$

and the third becomes

$$\frac{1}{m_j} y_{j,x}(1,t) - \frac{1}{m_{j+1}} y_{j+1,x}(0,t) = u_{j,tt}(t) = \widehat{u}_j(t), \quad j=1,2,\cdots,n-1.$$

Thus we deduce the system satisfied displacement continuity,

$$\begin{cases} m_j \frac{\partial^2 y_j(x,t)}{\partial t^2} = T_j \frac{\partial^2 y_j(x,t)}{\partial x^2}, \quad j=1,2,\cdots n, \quad x \in (0,1), \quad t>0, \\ y_{1,x}(0,t) = y_{n,x}(1,t) = 0, \quad t>0 \\ y_j(1,t) = y_{j+1}(0,t), \quad j=1,2,\cdots,n-1, \quad t>0, \\ \frac{1}{m_j} y_{j,x}(1,t) - \frac{1}{m_{j+1}} y_{j+1,x}(0,t) = \widehat{u}_j(t), \quad j=1,2,\cdots,n-1, \quad t>0, \end{cases} \tag{8.5.4}$$

This is a system both ends free and is acted the controllers on all the interior nodes.

REMARK 8.5.1 *For $n=1$, the system of string with both ends free, literature [120] gives a complete controller design and stability analysis. The authors in [79] presented the controller design for serially connected strings.*

Chapter 9

Network of Strings with A Triangle-Shape Circuit

In this chapter we consider a continuous network of strings with a triangle-shaped circuit. Suppose that the network of strings at all internal nodes are continuously joined, and its boundary (the exterior vertices) are free. The velocity feedback controllers are placed on all vertices of the network. This system is a particular case of the general continuous network of strings. For the sake of completeness, we firstly discuss in section 2 the well-posed-ness of the closed loop system by the semigroup theory. In section 3, by spectral analysis of the system operator, we show that the spectra of the system are located in the left half complex plane and are distributed in a strip parallel to the imaginary axis under certain conditions. Further we prove in section 4 that there is a sequence of the generalized eigenvectors of the system that forms a Riesz basis with parentheses for the Hilbert state space, and hence the spectrum determined growth condition holds. In section 5, we analyze conditions of asymptotical stability of the network. The result shows that if there is one of ratio of the wave speeds of strings in triangle-shape circuit being a irrational number, then the system is asymptotically stable. Finally, in section 6, we give a conclusion remark.

9.1 Introduction

Networks of vibrating strings are often used as models in large flexible space structures, satellite antenna, and information transmission and so on. In last two decades, the control problem of elastic networks had been one of hot topics in control engineering and mathematical control field, involving controllability, observability and stabilization ([5], [6], [34], [72], [71]). Riesz basis approach, as one of the powerful tools in control theory of distribute parameter system ([26],[43],[45],[79],[119], [111], [117]), was used successfully in study of control of vibration of flexible system. In particular, Xu et al in [125] obtained Riesz basis property for a class of abstract differential equations, which has applied to study of tree-shaped network of strings.

But network systems with circuit were seldom studied. Here we consider a planar network of elastic strings with a triangle circuit, whose structure is shown as Fig. 9.1.1. Suppose that every string has unit length. The network is linked as follows (see, Fig 9.1.1)

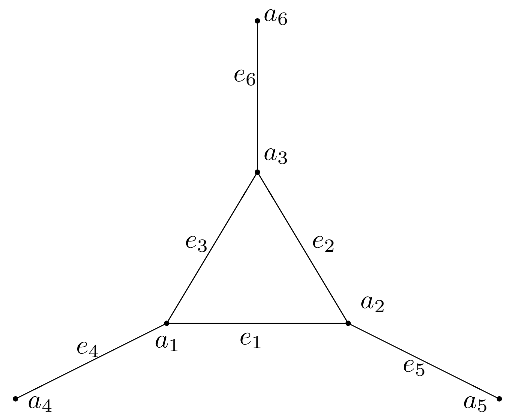

Fig 9.1.1. A planar network of strings with a triangle circuit

where e_k represents the k-th string, and $a_k,\ k = 1, 2, \ldots, 6$ represent the nodes of the network. The parameterization directions of strings are as follows

$$\begin{aligned} &e_1 = (a_1, a_2), \quad e_2 = (a_2, a_3), \quad e_3 = (a_3, a_1) \\ &e_4 = (a_1, a_4), \quad e_5 = (a_2, a_5), \quad e_3 = (a_3, a_6). \end{aligned}$$

We place controllers on all vertices. Then the motion of the controlled network system is governed by the partial differential equations

$$\begin{cases} T_k \dfrac{\partial^2 y_k(x,t)}{\partial x^2} = m_k \dfrac{\partial^2 y_k(x,t)}{\partial t^2}, \ x \in (0,\, 1), t > 0, \\ y_1(0,\, t) = y_3(1,\, t) = y_4(0,\, t), \\ y_2(0,\, t) = y_1(1,\, t) = y_5(0,\, t), \\ y_3(0,\, t) = y_2(1,\, t) = y_6(0,\, t), \\ T_1 \dfrac{\partial y_1(1,\, t)}{\partial x} - T_2 \dfrac{\partial y_2(0,\, t)}{\partial x} - T_5 \dfrac{\partial y_5(0,\, t)}{\partial x} = u_1(t), \\ T_2 \dfrac{\partial y_2(1,\, t)}{\partial x} - T_3 \dfrac{\partial y_3(0,\, t)}{\partial x} - T_6 \dfrac{\partial y_6(0,\, t)}{\partial x} = u_2(t), \\ T_3 \dfrac{\partial y_3(1,\, t)}{\partial x} - T_1 \dfrac{\partial y_1(0,\, t)}{\partial x} - T_4 \dfrac{\partial y_4(0,\, t)}{\partial x} = u_3(t), \\ T_4 \dfrac{\partial y_4(1,\, t)}{\partial x} = u_4(t), \quad T_5 \dfrac{\partial y_5(1,\, t)}{\partial x} = u_5(t), \quad T_6 \dfrac{\partial y_6(1,\, t)}{\partial x} = u_6(t), \\ y_k(x,\, 0) = y_{k,0}(x), \ \dfrac{\partial y_k(x,\, 0)}{\partial t} = y_{k,1}(x), \ k = 1, 2, \ldots, 6. \end{cases} \tag{9.1.1}$$

where $y_k(x, t)$ describes the transversal displacement of the k-th string on position x at time t, the constant coefficients m_k and T_k are the mass density and the tension of the k-th string of the network, respectively.

We design the feedback controllers $u_k(t),\ k = 1, 2, \ldots, 6$ as follows

$$u_k(t) = -\beta_k \frac{\partial y_k(1,\, t)}{\partial t} - \gamma_k y_k(1, t), \quad \beta_k, \gamma_k > 0, \tag{9.1.2}$$

Let

$$Y(x,t)=[y_1(x,t),y_2(x,t),y_3(x,t),y_4(x,t),y_5(x,t),y_6(x,t)]^T=[Y_{lp}(x,t)^T,Y_{nl}(x,t)^T]^T, \tag{9.1.3}$$

where $Y_{lp}(x,t)=[y_1(x,t),y_2(x,t),y_3(x,t)]^T$ denotes the part of interior circuit and $Y_{nl}(x,t)=[y_4(x,t),y_5(x,t),y_6(x,t)]^T$ denotes the exterior of the circuit, where the superscript T denotes the transpose of matrix.

Define the diagonal matrices

$$\mathbb{M}=\mathrm{diag}\{m_1,\, m_2,\, m_3, m_4,\, m_5,\, m_6\},\quad \mathbb{T}=\mathrm{diag}\{T_1,\, T_2,\, T_3, T_4,\, T_5,\, T_6\}$$

$$\beta=\mathrm{diag}\{\beta_1,\dots,\beta_6\},\quad \gamma=\mathrm{diag}\{\gamma_1,\dots,\gamma_6\}$$

Then the closed loop system (9.1.1)–(9.1.2) can be rewritten as follows:

$$\begin{cases} \mathbb{M}Y_{tt}(x,t)=\mathbb{T}Y_{xx}(x,t),\ x\in(0,\ 1),\ t>0,\\ Y(0,t)=CY(1,t),\\ \mathbb{T}Y_x(1,t)-C^T\mathbb{T}Y_x(0,t)=-\beta Y_t(1,t)-\gamma Y(1,t),\\ Y(x,0)=Y_0(x),\ Y_t(x,0)=Y_1(x), \end{cases} \tag{9.1.4}$$

where

$$C=\begin{pmatrix} C_{lp} & 0\\ C_{lp} & 0\end{pmatrix},\quad C_{lp}=\begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&1&0\end{pmatrix},$$

and

$$Y_0=[y_{1,0}(x),\dots,y_{6,0}(x)]^T,\quad Y_1=[y_{1,1}(x),\dots,y_{6,1}(x)]^T.$$

In this chapter, we mainly analyze the stability of the closed loop system (9.1.4).

9.2 Well-pose-ness of the system

To discuss the well-posed-ness of the system (9.1.4), firstly we formulated it into an appropriate Hilbert state space.

Let $H^k[(0,1),\mathbb{C}^6]\,(k=1,2)$ be the usual vector-valued Sobolev space and $L^2[(0,1),\mathbb{C}^6]$ be the usual vector-valued square integrable function space, which also a Hilbert space.

Set

$$V_E^k(0,1)=\{f\in H^k[(0,1),\mathbb{C}^6]\ \big|\ f(0)=Cf(1)\},$$

and define the state space by

$$\mathcal{H}=V_E^1(0,1)\times L^2[(0,1),\mathbb{C}^6]$$

equipped with an inner product, $\forall (f,g)^T,\ (u,v)^T\in\mathcal{H}$

$$\langle (f,g),(u,v)\rangle_{\mathcal{H}}=\int_0^1[(\mathbb{T}f'(x),u'(x))_{\mathbb{C}^6}+(\mathbb{M}g(x),v(x))_{\mathbb{C}^6}]dx+(\gamma f(1),u(1))_{\mathbb{C}^6}$$

where $(\cdot,\cdot)_{\mathbb{C}^6}$ denotes the inner product of the complex space $\mathbb{C}^6$. Clearly, $\mathcal{H}$ is a Hilbert space.

In the space $\mathcal{H}$, we define an operator $\mathcal{A}$ by

$$\mathcal{A}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g(x) \\ \mathbb{M}^{-1}\mathbb{T}f''(x) \end{pmatrix} \tag{9.2.1}$$

with the domain

$$D(\mathcal{A}) = \left\{ (f,g) \in \mathcal{H} \mid \begin{array}{l} f \in V_E^2(0,1),\ g \in V_E^1(0,1), \\ \mathbb{T}f'(1) - C^T\mathbb{T}f'(0) = -\beta g(1) - \gamma f(1), \end{array} \right\}. \tag{9.2.2}$$

Thus we can rewrite (9.1.4) into an abstract evolutionary equation in $\mathcal{H}$

$$\begin{cases} \frac{dZ(t)}{dt} = \mathcal{A}Z(t), \quad t > 0 \\ Z(0) = Z_0. \end{cases} \tag{9.2.3}$$

where $Z(t) = (Y(x,t), Y_t(x,t))^T$, $Z_0 = (Y_0, Y_1)^T$.

Firstly, we have the following conclusion.

THEOREM 9.2.1 *Let $\mathcal{H}$ be defined as before and $\mathcal{A}$ be defined by in (9.2.1) and (9.2.2). Then the following assertions hold*

1) $\mathcal{A}$ is dissipative and $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$;

2) the spectrum of $\mathcal{A}$ consists of all isolated eigenvalues of finite multiplicity, i.e. $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$;

3) $\mathcal{A}$ generates a C_0 semigroup of contraction $S(t)$ on $\mathcal{H}$. Hence the closed loop system (9.1.4) or the evolution system (9.2.3) is well-posed.

Proof Firstly, for any $(f,g)^T \in D(\mathcal{A})$, we have

$$\begin{aligned} \Re(\mathcal{A}(f,g)^T, (f,g)^T)_{\mathcal{H}} &= \Re\left\{ (\mathbb{T}f'(x), g(x))_{\mathbb{C}^6}\Big|_0^1 + (\gamma g(1),\ f(1))_{\mathbb{C}^6} \right\} \\ &= \Re\left\{ (\mathbb{T}f'(1), g(1))_{\mathbb{C}^6} - (\mathbb{T}f'(0), g(0))_{\mathbb{C}^6} + (\gamma g(1),\ f(1))_{\mathbb{C}^6} \right\} \\ &= \Re\left\{ (\mathbb{T}f'(1) - C^T\mathbb{T}f'(0), g(1))_{\mathbb{C}^6} + (\gamma g(1),\ f(1))_{\mathbb{C}^6} \right\} \end{aligned}$$

where we have used the conditions $g(0) = Cg(1)$ and $f(0) = Cf(1)$. Again using the condition $\mathbb{T}f'(1) - C^T\mathbb{T}f'(0) = -\beta g(1) - \gamma f(1)$, we get that

$$\Re(\mathcal{A}(f,g)^T, (f,g)^T)_{\mathcal{H}} = -(\beta g(1), g(1))_{\mathbb{C}^6} \le 0 \tag{9.2.4}$$

i.e. $\mathcal{A}$ is a dissipative operator.

Next, we prove $0 \in \rho(\mathcal{A})$, i.e. $\mathcal{A}^{-1}$ exists and is bounded. For each fixed $(\zeta,\nu)^T \in \mathcal{H}$, we solve the resolvent equation

$$\mathcal{A}(f,g)^T = (\zeta,\nu)^T, \quad (f,g)^T \in D(\mathcal{A}), \tag{9.2.5}$$

i.e.,

$$\begin{cases} g(x) = \zeta(x), \\ \mathbb{T}f''(x) = \mathbb{M}\nu(x). \\ f(0) = Cf(1), \\ \mathbb{T}f'(1) - C^T\mathbb{T}f'(0) = -\beta g(1) - \gamma f(1) \end{cases}$$

Integrating the second equation from x to 1 leads to

$$\mathbb{T}f'(x) \quad = \quad \mathbb{T}f'(1) - \int_x^1 \mathbb{M}\nu(x)dx. \tag{9.2.6}$$

So,

$$\mathbb{T}f(x) = \mathbb{T}f(1) - (1-x)\mathbb{T}f'(1) + \int_x^1 \varrho(s)ds, \tag{9.2.7}$$

where $\varrho(x) = \int_x^1 \mathbb{M}\nu(s)ds$, $f(1)$ and $\mathbb{T}f'(1)$ are unknown.

From (9.2.6) we get that $\mathbb{T}f'(1) = \mathbb{T}f'(0) + \varrho(0)$. Substituting it into the dynamic boundary condition yields

$$\gamma f(1) + (I - C^T)\mathbb{T}f'(1) = -\beta\zeta(1) - C^T\varrho(0).$$

From (9.2.7) we get

$$\mathbb{T}f'(1) - \mathbb{T}(I - C)f(1) = \kappa, \tag{9.2.8}$$

where $\kappa = \int_0^1 \varrho(s)ds$. Therefore we get the following linear equations with unknown vectors $f(1)$ and $\mathbb{T}f'(1)$

$$\begin{pmatrix} \gamma & (I - C^T) \\ -\mathbb{T}(I - C) & I \end{pmatrix} \begin{pmatrix} f(1) \\ \mathbb{T}f'(1) \end{pmatrix} = \begin{pmatrix} -\beta\zeta(1) - C^T\varrho(0) \\ \kappa \end{pmatrix} \tag{9.2.9}$$

Denote by $\gamma_{lp} = \operatorname{diag}\{\gamma_1, \gamma_2, \gamma_3\}$, $\gamma_{nl} = \operatorname{diag}\{\gamma_4, \gamma_5, \gamma_6\}$, $\mathbb{T}_{lp} = \operatorname{diag}\{T_1, T_2, T_3\}$ and $\mathbb{T}_{nl} = \operatorname{diag}\{T_4, T_5, T_6\}$. A direct calculation of the determinant gives

$$\begin{aligned} & \det \begin{pmatrix} \gamma & (I - C^T) \\ -\mathbb{T}(I - C) & I \end{pmatrix} = \det \begin{pmatrix} \gamma + (I - C^T)\mathbb{T}(I - C) & 0 \\ -\mathbb{T}(I - C) & I \end{pmatrix} \\ = \ & \det(\gamma + (I - C^T)T(I - C)) \\ = \ & \det\left(\gamma_{lp} + (I - C_{lp}^T)T_{lp}(I - C_{lp}) + C_{lp}^T\mathbb{T}_{nl}C_{lp} - C_{lp}^T\mathbb{T}_{nl}\left[\gamma_{nl} + \mathbb{T}_{nl}\right]^{-1}\mathbb{T}_{nl}C_{lp}\right) \\ = \ & \det \begin{pmatrix} d_1 & -T_2 & -T_1 \\ -T_2 & d_2 & -T_3 \\ -T_1 & -T_3 & d_3 \end{pmatrix} \neq 0 \end{aligned}$$

where $d_1 = \gamma_1 + T_1 + T_2 + \frac{\gamma_5 T_5}{\gamma_5 + T_5}$, $d_2 = \gamma_2 + T_2 + T_3 + \frac{\gamma_6 T_6}{\gamma_6 + T_6}$ and $d_3 = \gamma_3 + T_1 + T_3 + \frac{\gamma_4 T_4}{\gamma_4 + T_4}$. Thus, $f(1)$ and $\mathbb{T}f'(1)$ are uniquely solvable via (9.2.9). Hence $f(x)$ and $g(x)$ can be determined by ζ, ν uniquely, which implies that $\mathcal{A}^{-1}$ exists. By the arbitrariness of $(\zeta, \nu) \in \mathcal{H}$, so $\mathcal{A}^{-1}$ is bounded, i.e., $0 \in \rho(\mathcal{A})$. Note that $D(\mathcal{A}) \subset H^2 \times H^1$, the Sobolev's Embedding Theorem asserts that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$.

As a result of resolvent compact operator, we know the assertion 2) holds. Finally, according to Lumer-Phillips Theorem ([92]), $\mathcal{A}$ generates a C_0 semigroup of contraction. □

REMARK 9.2.1 *From the proof of Theorem 9.2.1 we see that $\mathcal{A}^{-1}$ exists and is bounded provided that γ satisfies $\sum_{k=1}^6 \gamma_k^2 \neq 0, \gamma_k \geq 0$. So $\sum_{k=1}^6 \gamma_k^2 \neq 0$ is a sufficient and necessary condition for $0 \in \rho(\mathcal{A})$*

9.3 Spectral analysis

In this section, we consider the eigenvalue problem of the system (9.2.3)(i.e., the closed loop system (9.1.4)).

Let $\lambda \in \sigma(\mathcal{A})$ and $(f,g)^T \in D(\mathcal{A})$ be corresponding an eigenvector, then $(\lambda I - \mathcal{A})(f,g)^T = 0$, i.e.,

$$\begin{cases} \mathbb{T}f''(x) = \lambda^2 \mathbb{M} f(x), \\ g(x) = \lambda f(x), \\ f(0) = \mathbb{C} f(1), \\ \mathbb{T}f'(1) - C^T \mathbb{T} f'(0) = -(\lambda\beta + \gamma) f(1). \end{cases} \tag{9.3.1}$$

Set $\eta(x) = (f(x), \lambda^{-1}\mathbb{T}f'(x))^T$, then $\eta(x)$ satisfies

$$\frac{d\eta}{dx} = \lambda \begin{bmatrix} 0 & \mathbb{T}^{-1} \\ \mathbb{M} & 0 \end{bmatrix} \eta \tag{9.3.2}$$

with boundary condition

$$\begin{pmatrix} I & 0 \\ 0 & -C^T \end{pmatrix} \eta(0) + \begin{pmatrix} -C & 0 \\ \widetilde{\beta} & I \end{pmatrix} \eta(1) = 0, \tag{9.3.3}$$

where $\widetilde{\beta} = \beta + \lambda^{-1}\gamma$. The theory of ordinary differential equations shows that the fundamental matrix of (9.3.2) is given by

$$W(x,\lambda) = \widehat{Q} \begin{pmatrix} \exp(-\lambda x B) & 0 \\ 0 & \exp(\lambda x B) \end{pmatrix} \widehat{Q}^{-1} \tag{9.3.4}$$

where

$$\widehat{Q} = \begin{pmatrix} -Q_{MT}^{-1} & Q_{MT}^{-1} \\ I & I \end{pmatrix}, \quad \widehat{Q}^{-1} = \frac{1}{2} \begin{pmatrix} -Q_{MT} & I \\ Q_{MT} & I \end{pmatrix}$$

where $Q_{MT} = \mathbb{M}^{1/2}\mathbb{T}^{1/2} = \operatorname{diag}\{c_1, \ldots, c_6\}$, $B = \mathbb{M}^{1/2}\mathbb{T}^{-1/2} = \operatorname{diag}\{b_1, \ldots, b_6\}$, $b_k = m_k^{1/2} T_k^{-1/2}$, $c_k = \sqrt{(m_k T_k)}$. Substituting it into boundary condition (9.3.3) leads to

$$D(\lambda)\eta(0) = 0,$$

where

$$D(\lambda) = \left[\begin{pmatrix} I & 0 \\ 0 & -C^T \end{pmatrix} + \begin{pmatrix} -C & 0 \\ \widetilde{\beta} & I \end{pmatrix} W(1,\lambda) \right]. \tag{9.3.5}$$

Therefore, the problem (9.3.2) and (9.3.3) have a nonzero solution if and only if

$$\Delta(\lambda) = \det(D(\lambda)) = 0. \tag{9.3.6}$$

Since the eigenvalue problem (9.3.1) is equivalent to the problem (9.3.2) and (9.3.3), so the zeros of $\Delta(\lambda)$ are the eigenvalues of the operator $\mathcal{A}$.

Since

$$D(\lambda)\widehat{Q} = \left[D_- + D_+ \begin{pmatrix} e^{\lambda B} & 0 \\ 0 & e^{\lambda B} \end{pmatrix}\right] \begin{pmatrix} e^{-\lambda B} & 0 \\ 0 & I \end{pmatrix},$$

where

$$D_- = \begin{pmatrix} CQ_{MT}^{-1} & Q_{MT}^{-1} \\ I - \widetilde{\beta} Q_{MT}^{-1} & -C^T \end{pmatrix}, \tag{9.3.7}$$

$$D_+ = \begin{pmatrix} -Q_{MT}^{-1} & -CQ_{MT}^{-1} \\ -C^T & I + \widetilde{\beta} Q_{MT}^{-1} \end{pmatrix}, \tag{9.3.8}$$

in addition,

$$\begin{aligned} \det(D_-) &= \det \begin{pmatrix} CQ_{MT}^{-1} & Q_{MT}^{-1} \\ I - \beta Q_{MT}^{-1} & -C^T \end{pmatrix} \\ &= \det(Q_{MT}^{-1}) \det\left(I + C^T Q_{MT} C Q_{MT}^{-1} - \beta Q_{MT}^{-1}\right), \end{aligned}$$

we conclude that

$$\begin{aligned} \Delta_- &= \lim_{\Re\lambda \to -\infty} \frac{\Delta(\lambda)}{\det(\exp(-\lambda B))} \\ &= \det\left(I + C^T Q_{MT} C Q_{MT}^{-1} - \beta Q_{MT}^{-1}\right) \\ &= (1 + (c_2 + c_5 - \beta_1)/c_1)(1 + (c_3 + c_6 - \beta_2)/c_2) \\ &\quad (1 + (c_1 + c_4 - \beta_3)/c_3)(1 - \beta_4/c_4)(1 - \beta_5/c_5)(1 - \beta_6/c_6) \end{aligned}$$

and

$$\begin{aligned} \Delta_+ &= \lim_{\Re\lambda \to +\infty} \frac{\Delta(\lambda)}{\det(\exp(\lambda B))} \\ &= \det\left(I + C^T Q_{MT} C Q_{MT}^{-1} + \beta Q_{MT}^{-1}\right) \\ &= (1 + (c_2 + c_5 + \beta_1)/c_1)(1 + (c_3 + c_6 + \beta_2)/c_2) \\ &\quad (1 + (c_1 + c_4 + \beta_3)/c_3)(1 + \beta_4/c_4)(1 + \beta_5/c_5)(1 + \beta_6/c_6). \end{aligned}$$

Therefore, when $\Delta_- \neq 0$, i.e., β satisfies the following conditions

$$\begin{cases} \beta_1 \neq c_1 + c_2 + c_5 = \sqrt{m_1 T_1} + \sqrt{m_2 T_2} + \sqrt{m_5 T_5}, \\ \beta_2 \neq c_2 + c_3 + c_6 = \sqrt{m_2 T_2} + \sqrt{m_3 T_3} + \sqrt{m_6 T_6}, \\ \beta_3 \neq c_1 + c_3 + c_4 = \sqrt{m_1 T_1} + \sqrt{m_3 T_3} + \sqrt{m_4 T_4}, \\ \beta_k \neq c_k = \sqrt{m_k T_k}, k = 4, 5, 6, \end{cases} \tag{9.3.9}$$

there exist positive constants $\widetilde{c}_1$, $\widetilde{c}_2$ and δ such that for $|\Re(\lambda)| > \delta$,

$$\widetilde{c}_1 \exp\left(\Re(\lambda) tr(B)\right) \leq |\Delta(\lambda)| \leq \widetilde{c}_2 \exp\left(\Re(\lambda) tr(B)\right), \tag{9.3.10}$$

which shows that the zeros of $\Delta(\lambda)$ are located in the region $\{\lambda \in \mathbb{C} \mid |\Re\lambda| \leq \delta\}$. By Theorem 9.2.1, we conclude

$$\sigma(\mathcal{A}) \subset \left\{\lambda \in \mathbb{C} \mid -\delta \leq \Re\lambda \leq 0\right\}. \tag{9.3.11}$$

In addition, the inequality (9.3.10) shows that $\Delta(i\lambda)$ is an entire function of sine type on $\mathbb{C}$ (see, [9, Definition II, 1.27, pp61]). Levin theorem (see, [9, Proposition II, 1.28]) asserts that the set of zeros of $\Delta(\lambda)$ is a union of finitely many separable sets. So is $\sigma(\mathcal{A})$.

By above analysis, we have achieved the following result.

THEOREM 9.3.1 *Let $\mathcal{A}$ be defined by (9.2.1)–(9.2.2). Then we have*

$$\sigma(\mathcal{A}) = \left\{\lambda \in \mathbb{C} \;\middle|\; \Delta(\lambda) = 0\right\}. \tag{9.3.12}$$

In particular, when β satisfies (9.3.9), $\sigma(\mathcal{A})$ is a union of finite many separated sets, and there exists a positive constant δ such that (9.3.11) holds.

9.4 Completeness and Riesz basis of root vectors of $\mathcal{A}$

In this section, we discuss the completeness and basis property of root vectors of $\mathcal{A}$. We begin with the following lemma.

LEMMA 9.4.1 *Let $\mathcal{H}$ be defined as before, and $\mathcal{A}_0$ be the uncontrolled operator in $\mathcal{H}$ defined by*

$$\mathcal{A}_0\,(f,g) = \left(g(x), \mathbb{M}^{-1}\mathbb{T}f''(x)\right), \tag{9.4.1}$$

with

$$D(\mathcal{A}_0) = \left\{ (f,g) \in \mathcal{H} \;\middle|\; \begin{array}{l} f \in V_E^2(0,1), g \in V_E^1(0,1), \\ \mathbb{T}f'(1) - C^T\mathbb{T}f'(0) = -\gamma f(1) \end{array} \right\} \tag{9.4.2}$$

and $\sum_{k=1}^{6}\gamma_k^2 \neq 0, \gamma_k \geq 0$. Then the following assertions hold

1) $\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$;

2) for any $(\zeta,\nu)^T \in \mathcal{H}$ and $\lambda \in \mathbb{R}$, the solution of the resolvent equation

$$(\lambda I - \mathcal{A}_0)\,(f,\, g)^T = (\zeta,\nu)^T \tag{9.4.3}$$

satisfies

$$\|g(1)\|_{\mathbb{C}^6}^2 \leq 2c^2\|(\zeta,\nu)^T\|_{\mathcal{H}}^2 \tag{9.4.4}$$

where $c > 0$ is a constant.

Proof It is easy to check that $\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$, which implies that

$$\|\lambda R(\lambda, \mathcal{A}_0)\| \leq 1, \quad \forall \Re\lambda \neq 0. \tag{9.4.5}$$

Without loss of generality, we assume that $\gamma_1 \neq 0$. The resolvent equation (9.4.3) implies that $g(x) = \lambda f(x) - \zeta(x)$ and $g(1) = \lambda f(1) - \zeta(1)$. So, we have

$$\begin{aligned} g(1) &= g(0) + \int_0^1 g'(x)dx = Cg(1) + \int_0^1 (\lambda f'(x) - \zeta'(x))dx \\ &= (g_3(1), g_1(1), g_2(1), g_3(1), g_1(1), g_2(1)) + \int_0^1 (\lambda f'(x) - \zeta'(x))dx. \end{aligned}$$

Note that the following two inequalities,

$$\left\|\int_0^1 T^{1/2} f'(x)dx\right\|^2 \le \int_0^1 (\mathbb{T}f'(x), f'(x))_{\mathbb{C}^6} dx,$$

$$\int_0^1 (\mathbb{T}f'(x), f'(x))_{\mathbb{C}^6} dx + \gamma_1 |f_1(1)|^2 \le \|(f,\, g)^T\|^2.$$

Thus we have

$$\begin{aligned}
\|g(1)\|^2_{c^6} &\le 3\left\|\int_0^1 \lambda f'(x)dx\right\|^2 + 3\left\|\int_0^1 \zeta'(x)dx\right\|^2 + 6(|\lambda f_1(1)|^2 + |\zeta_1(1)|^2)\\
&\le c^2|\lambda|^2\left(\int_0^1 (\mathbb{T}f'(x), f'(x))_{\mathbb{C}^6} dx + \gamma_1|f_1(1)|^2\right)\\
&\quad + c^2\left(\int_0^1 (\mathbb{T}\zeta'(x), \zeta'(x))_{\mathbb{C}^6} dx + \gamma_1|\zeta_1(1)|^2\right)\\
&\le c^2\left(|\lambda|^2\|R(\lambda, \mathcal{A}_0)(\zeta,\nu)^T\|^2_{\mathcal{H}} + \|(\zeta,\nu)^T\|^2_{\mathcal{H}}\right)\\
&\le 2c^2\|(\zeta,\nu)^T\|^2_{\mathcal{H}}
\end{aligned}$$

where $c^2 = \max\{3\|T^{-1/2}\|^2, 6(\gamma_1)^{-1}\}$. The proof is then complete. □

Using above lemma, we can prove the main result in this section.

THEOREM 9.4.1 *Let $\mathcal{A}$ be defined by (9.2.1) –(9.2.2). If β satisfies (9.3.9), then the system of eigenvectors and generalized eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$.*

Proof Denote by

$$Sp(\mathcal{A}) = \overline{\text{span}\left\{\sum_k y_k \;\middle|\; y_k \in E(\lambda_k, \mathcal{A})\mathcal{H}, \forall \lambda_k \in \sigma(\mathcal{A})\right\}},$$

where $E(\lambda_k, \mathcal{A})$ is the Riesz projector corresponding to λ_k. We shall prove $Sp(\mathcal{A}) = \mathcal{H}$.

Let $(\widetilde{\zeta}, \widetilde{\nu})^T \in \mathcal{H}$ such that $(\widetilde{\zeta}, \widetilde{\nu})^T \perp Sp(\mathcal{A})$. For any $(\zeta,\nu)^T \in \mathcal{H}$ given, we define a function on complex plane $\mathbb{C}$ by

$$F(\lambda) = \langle (\zeta,\nu)^T,\, R^*(\lambda,\, \mathcal{A})(\widetilde{\zeta}, \widetilde{\nu})^T \rangle_{\mathcal{H}}. \tag{9.4.6}$$

Clearly, $F(\lambda)$ is an entire function of finite exponential type and

$$|F(\lambda)| \le (\Re\lambda)^{-1}\, \|(\zeta,\nu)\|\, \left\|(\widetilde{\zeta}, \widetilde{\nu})\right\|, \text{ for } \Re\lambda > 0.$$

Hence $\lim\limits_{\Re\lambda\to+\infty} |F(\lambda)| = 0$.

Now we consider the following equations

$$\begin{cases} (\lambda I - \mathcal{A})\,(f,g)^T = (\zeta,\nu)^T \\ (\lambda I - \mathcal{A}_0)\left(\widehat{f}, \widehat{g}\right)^T = (\zeta,\nu)^T, \end{cases} \tag{9.4.7}$$

where $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0)$ and $\lambda < 0$. Let $u(x) = f(x) - \widehat{f}(x)$, $v(x) = g(x) - \widehat{g}(x)$, then

$$R\,(\lambda, \mathcal{A})\,(\zeta,\nu)^T = R\,(\lambda, \mathcal{A}_0)\,(\zeta,\nu)^T + (u,v)^T \tag{9.4.8}$$

and $(u,v)^T$ satisfies equation

$$\begin{cases} \mathbb{T}u''(x) = \lambda^2 \mathbb{M}u(x), \\ v(x) = \lambda u(x), \\ u(0) = Cu(1), \\ \mathbb{T}u'(1) - C^T\mathbb{T}u'(0) + (\gamma + \lambda\beta)u(1) = -\beta\widehat{g}(1). \end{cases} \tag{9.4.9}$$

According to (9.4.6) and (9.4.8), it holds that

$$F(\lambda) = \langle R(\lambda, \mathcal{A}_0)(\zeta, \nu)^T, (\widetilde{\zeta}, \widetilde{\nu})^T\rangle_{\mathcal{H}} + \langle (u,v)^T, (\widetilde{\zeta}, \widetilde{\nu})^T\rangle_{\mathcal{H}}. \tag{9.4.10}$$

Set $\eta(x) = (u(x), \lambda^{-1}\mathbb{T}u'(x))^T$. Similar to (9.3.2), the solution of (9.4.9) satisfies

$$\eta(x) = W(x,\lambda)\eta_0 \tag{9.4.11}$$

and

$$D(\lambda)\eta_0 = \lambda^{-1}(0, \beta\widehat{g}(1))^T, \tag{9.4.12}$$

where $W(x,\lambda)$ and $D(\lambda)$ are defined as (9.3.4) and (9.3.5), respectively.

Taking transform

$$\widetilde{\eta} = \begin{pmatrix} e^{-\lambda A} & 0 \\ 0 & I \end{pmatrix} \widehat{Q}^{-1}\eta_0, \tag{9.4.13}$$

then from (9.4.12) we deduce that

$$\widetilde{\eta} = \lambda^{-1}\left[D_- + D_+ \begin{pmatrix} e^{\lambda A} & 0 \\ 0 & e^{\lambda A} \end{pmatrix}\right]^{-1} \begin{pmatrix} 0 \\ \beta\hat{g}(1) \end{pmatrix},$$

where D_- and D_+ are defined by (9.3.7) and (9.3.8), respectively. Obviously, $\|\widetilde{\eta}\| = O(\lambda^{-1})\|\hat{g}(1)\|$ for sufficiently large $-\lambda > 0$. Therefore, from (9.4.13), (9.4.11) and (9.3.4) we obtain

$$\eta(x) = \widehat{Q}\begin{pmatrix} e^{\lambda(1-x)A} & 0 \\ 0 & e^{\lambda x A} \end{pmatrix}\widetilde{\eta}.$$

So, we have

$$u(x) = -Q_{MT}^{-1}e^{\lambda(1-x)A}\widetilde{\eta}_1 + Q_{MT}^{-1}e^{\lambda x A}\widetilde{\eta}_2,$$

and

$$u(1) = -Q_{MT}^{-1}\widetilde{\eta}_1 + Q_{MT}^{-1}e^{\lambda A}\widetilde{\eta}_2.$$

When $-\lambda$ is large enough, we have

$$\|u(1)\|^2 = O(|\lambda|^{-2})\,\|\hat{g}(1)\|^2. \tag{9.4.14}$$

From (9.4.9) it follows that

$$\left\|(u,v)^T\right\|^2 = -\langle u(1), \beta\widehat{g}(1)\rangle_{\mathbb{C}^6} - \langle\, u(1), \lambda\beta u(1)\,\rangle_{\mathbb{C}^6}.$$

The equality (9.4.10) combining with (9.4.4) in Lemma 9.4.1 yields that

$$\left\|(u,v)^T\right\| \le O(|\lambda|^{-1/2}) \left\|(\zeta,\nu)^T\right\|_{\mathcal{H}}.$$

Therefore, when $-\lambda > 0$ is sufficiently large, we have

$$\begin{aligned} |F(\lambda)| &\le |\lambda^{-1}| \left\|(\zeta,\nu)^T\right\|_{\mathcal{H}} \cdot \left\|(\widetilde{\zeta},\widetilde{\nu})^T\right\|_{\mathcal{H}} + \left\|(u,v)^T\right\|_{\mathcal{H}} \cdot \left\|(\widetilde{\zeta},\widetilde{\nu})^T\right\|_{\mathcal{H}} \\ &\le \left(|\lambda^{-1}| + O(|\lambda|^{-1/2})\right) \left\|(\zeta,\nu)^T\right\|_{\mathcal{H}} \cdot \left\|(\widetilde{\zeta},\widetilde{\nu})^T\right\|_{\mathcal{H}} \\ &\le O(|\lambda|^{-1/2}) \left\|(\zeta,\nu)^T\right\|_{\mathcal{H}} \cdot \left\|(\widetilde{\zeta},\widetilde{\nu})^T\right\|_{\mathcal{H}}. \end{aligned}$$

Since $F(\lambda)$ is an entire function of finite exponential type, the Phrángmen-Lindelöf theorem (see [127]) and the above inequality show that $F(\lambda) \equiv 0$. From (9.4.6) we get that $R^*(\lambda, \mathcal{A})(\widetilde{\zeta},\widetilde{\nu})^T \equiv 0$, which leads to $(\widetilde{\zeta},\widetilde{\nu})^T \equiv 0$. Therefore, $Sp\,(\mathcal{A}) = \mathcal{H}$, the desire result follows. □

Combining Theorem 9.2.1, Theorem 9.3.1 and the result in[125, Theorem 3.4], we have the following basis property of root vectors of $\mathcal{A}$.

THEOREM 9.4.2 *Let $\mathcal{H}$ be defined as before and $\mathcal{A}$ be defined by (9.2.1)–(9.2.2). If β satisfies 9.3.9, then there is a sequence of eigenvectors and generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$. Hence the C_0 semigroup $S(t)$ generated by $\mathcal{A}$ satisfies the spectrum determined growth assumption.*

Proof Set $\sigma_1(\mathcal{A}) = \{-\infty\}$, $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$. Theorem 9.2.1 and Theorem 9.3.1 ensure all the conditions in [125, Theorem 3.4] are fulfilled. Thus there is a sequence of eigenvectors and generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}_2 = Sp(\mathcal{A})$. Theorem 9.4.1 says that the root vectors are complete in $\mathcal{H}$, the sequence is also a Riesz basis with parentheses for $\mathcal{H}$. The Riesz basis property together with the uniform boundedness of the multiplicities of eigenvalues of $\mathcal{A}$ implies that $S(t)$ satisfies the spectrum determined growth assumption. The proof is then complete. □

9.5 Asymptotical stability of the closed loop system

In this section we analyze stability of the system (9.2.3). Firstly, as a consequence of the Riesz basis property, we have the following stability result of the system.

THEOREM 9.5.1 *Let $\mathcal{A}$ be defined by (9.2.1) and (9.2.2), and β satisfy condition (9.3.9). Then the following statements are true:*

1) If $\inf\limits_{\lambda \in i\mathbb{R}} |\Delta(\lambda)| \neq 0$, then the system is exponentially stable.

2) If $\inf\limits_{\lambda \in i\mathbb{R}} |\Delta(\lambda)| = 0$, then the system at most is asymptotically stable but not exponentially stable.

Proof Under assumption in proposition, Theorem 9.4.2 shows that the system (9.2.3) satisfies the spectrum determined growth condition. Note that

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \Delta(\lambda) = 0\} \subset \{\lambda \in \mathbb{C} \mid \Re\lambda \le 0\}.$$

If $\inf\limits_{\lambda\in i\mathbb{R}} |\Delta(\lambda)| \neq 0$, then the imaginary axis is not an asymptote of $\sigma(\mathcal{A})$, which implies the system is exponentially stable. If there exists a $\lambda \in i\mathbb{R}$ such that $\Delta(\lambda) = 0$, then the system is unstable. If the imaginary axis is an asymptote of $\sigma(\mathcal{A})$, then there exists a sequence $\lambda_n = \alpha_n + i\beta_n$ such that $\alpha_n \to 0$ as $n \to \infty$. Since $\Delta(\lambda)$ is sine-type function, so is $\Delta'(\lambda)$. Thus we have

$$\sup_{|\Re\lambda|\leq\delta} |\Delta'(\lambda)| < \infty.$$

Note that

$$\Delta(\alpha_n + i\beta_n) - \Delta(i\beta_n) = \Delta'(\alpha_n\theta_n + i\beta_n)\alpha_n, \quad \theta_n \in (0,1).$$

So we have $\inf\limits_{\lambda\in i\mathbb{R}} |\Delta(\lambda)| = 0$. In this case, the system is asymptotically stable but not exponentially stable. The proof is then complete. □

REMARK 9.5.1 *In the proof of Theorem 9.5.1, we only show if* $\inf_{\lambda\in i\mathbb{R}} |\Delta(\lambda)| > 0$, *the imaginary axis is not an asymptote of* $\sigma(\mathcal{A})$. *If the imaginary axis is an asymptote of* $\sigma(\mathcal{A})$, *then* $\inf_{\lambda\in i\mathbb{R}} |\Delta(\lambda)| = 0$. *A question is if* $\inf_{\lambda\in i\mathbb{R}} |\Delta(\lambda)| = 0$, *whether the imaginary axis is an asymptote of* $\sigma(\mathcal{A})$. *In fact, using the property of sine-type function we can prove that if the imaginary axis is not an asymptote of* $\sigma(\mathcal{A})$, *then* $\inf_{\lambda\in i\mathbb{R}} |\Delta(\lambda)| > 0$.

According to Theorem 9.5.1 and the spectrum determined growth assumption, the closed loop system (9.2.3) is asymptotically stable if and only if

$$\sigma(\mathcal{A}) \subset \{\, \lambda \in \mathbb{C} \mid -\delta \leq \Re(\lambda) < 0 \,\}. \tag{9.5.1}$$

To guarantee the asymptotic stability, it is necessary that there is no spectral points of $\mathcal{A}$ on the imaginary axis. The following theorem gives a sufficient and necessary condition for no spectral point of $\mathcal{A}$ on the imaginary axis.

THEOREM 9.5.2 *Let* $\mathcal{A}$ *be defined as (9.2.1) and (9.2.2). If* $\beta_k > 0$, $k = 1, \ldots, 6$ *and at least one of all ratios* $\dfrac{\sqrt{m_1/T_1}}{\sqrt{m_2/T_2}}$, $\dfrac{\sqrt{m_2/T_2}}{\sqrt{m_3/T_3}}$ *and* $\dfrac{\sqrt{m_3/T_3}}{\sqrt{m_1/T_1}}$ *is irrational number, then there is no spectral points of* $\mathcal{A}$ *on the imaginary axis.*

Proof Assume that $\lambda \in \sigma_p(\mathcal{A})$ and $\lambda = i\theta (0 \neq \theta \in \mathbb{R})$, $(f,g)^T \in D(\mathcal{A})$ is corresponding an eigenvector, then (9.2.4) implies that

$$\Re\lambda\langle (f,g)^T, (f,g)^T\rangle_{\mathcal{H}} = \Re(\mathcal{A}(f,g)^T, (f,g)^T)_{\mathcal{H}} = -\langle \beta g(1), g(1)\rangle_{\mathbb{C}^6} = 0 \tag{9.5.2}$$

so, $g(1) = 0$. This together with (9.3.1) yields that

$$\begin{cases} \mathbb{T}f''(x) = -\theta^2\mathbb{M}f(x), \quad g(x) = \theta f(x) \\ f(0) = f(1) = 0 \\ \mathbb{T}f'(1) - C^T\mathbb{T}f'(0) = 0. \end{cases} \tag{9.5.3}$$

i.e.,

$$\begin{cases} T_j f_j''(x) = -\theta^2 m_j f_j(x), j = 1,2,3,4,5,6 \\ f_j(0) = f_j(1) = 0, = 1,2,3,4,5,6, \\ T_1 f_1'(1) - T_2 f_2'(0) - T_5 f_5'(0) = 0 \\ T_2 f_2'(1) - T_3 f_3'(0) - T_6 f_6'(0) = 0 \\ T_3 f_3'(1) - T_1 f_1'(0) - T_4 f_5'(0) = 0 \\ f_4'(1) = f_5'(1) = f_6'(1) = 0. \end{cases} \tag{9.5.4}$$

Set $b_k = \sqrt{m_k/T_k}$,$k = 1, \dots, 6$, the general solution to (9.5.4) are of the form

$$f_j(x) = a_j \sin(\theta b_j x), \tag{9.5.5}$$

From the boundary conditions (9.5.4), one gets that $a_j = 0, j = 4,5,6$. If $a_j \neq 0, j = 1,2,3$, then

$$f_j(1) = 0, \quad \text{if and only if} \quad \sin\theta b_j = 0, \quad j = 1,2,3. \tag{9.5.6}$$

Since

$$\sin\theta b_j = 0 \Longleftrightarrow \theta b_j = k_j\pi, \quad k_j \in \mathbb{N}, j = 1,2,3,$$

we have

$$\frac{b_1}{b_2} = \frac{k_1}{k_2}, \quad \frac{b_1}{b_3} = \frac{k_1}{k_3} \tag{9.5.7}$$

and hence $\frac{b_i}{b_j} = \frac{\sqrt{m_i/T_i}}{\sqrt{m_j/T_j}}$ are rational numbers. This contradicts to the assumption of Theorem 9.5.1. Therefore, at least one of $a_j, j = 1,2,3$ vanishes. This together with (9.5.3) and (9.5.4) leads to $f(x) = g(x) = 0$. Therefore there is no spectral point of $\mathcal{A}$ on the imaginary axis. □

As a consequence of Theorems 9.5.1 and 9.5.2, the following conclusion is immediately.

COROLLARY 9.5.1 *Suppose that all conditions of Theorem 9.5.2 hold, then the closed loop system (9.1.4) (or the system (9.2.3)) is asymptotically stable at least.*

From above discussions we see that the stability of the system is closely relative to shape of the graph. Note that the conditions in Theorem 9.5.2 are the necessary and sufficient conditions for the system being asymptotically stable. If those conditions are fulfilled, whether or not the system might achieve the exponential stability? This is an unsolve problem.

9.6 Conclusions

In this chapter we discussed the stability of a network of strings with a triangle circuit. In particular, we obtained the Riesz basis property of the system. By the elasticity theory, the wave speed of the constant coefficient string is $\sqrt{\frac{m}{T}}$. Using the ratio $\frac{\sqrt{m_k/T_k}}{\sqrt{m_p/T_p}}$ of wave speeds of the p-th string and k-th string in Theorem 9.5.1, we give the sufficient and necessary condition for the closed loop system (9.1.4) being asymptotically stable.

If $\ell_k \neq 1$, we can take a change of variable: $x = \ell_k s$, under which the system is changed into we obtain that

$$T_k \frac{\partial^2 y_k(s,t)}{\partial s^2} = \ell_k^2 m_k \frac{\partial^2 y_k(s,t)}{\partial t^2}, \quad 0 < s < 1, t > 0, \tag{9.6.1}$$

where $y_k(s,t) = y_k(\ell_k s, t)$. Therefore, the result can be applied to the general network of the form (9.1.1).

In the system (9.1.1), all vertices are controlled. This control method is improvable. From the stability analysis we see that if we setup only controllers on the exterior vertices, then the corresponding system has same property. From this point of view, the interior controllers are unavailing (relative to exterior controller). This motivates us to study the location problem of controllers for more complex network.

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