A Mathematical Model for the Fire-extinguishing Rocket Flight in a Turbulent Atmosphere

CRISTINA MIHAILESCU
Electromecanica –Ploiesti SA
Soseaua Ploiesti-Tirgoviste, Km 8
ROMANIA
crismihailescu@yahoo.com http://www.elmec.ro

Abstract: The paper analyses the evolution of rockets used to extinguish fires developed in open space in conditions of turbulent atmosphere. After the general definitions of the turbulent atmosphere and working hypotheses, the model of uniform and linear turbulent atmosphere obtained from analytical relations is presented in detail. Starting from this model, the components of velocity of turbulence as functions of time are deduced. Introducing the components of velocity into the model of simulation of the rocket flight, a fascicle of the trajectories is obtained. The work novelty consists in the new technique used to model the turbulent atmosphere influence on unguided rocket flight.

Key-Words: turbulent atmosphere, rocket trajectory, statistical function, dynamics of flight, turbulence models

1 Introduction

Fire-extinguishing rocket trajectory must be evaluated in the early project design stage, when the extreme atmospheric turbulences, the vertical air flows and the no uniform wind must be considered as possible perturbations. The effect of large scale perturbation is important for establishing the average trajectory. A supplementary task would be to determine the trajectory dispersion due to perturbation gradients produced by the atmospheric gusts. The importance of the study lies in the results it provides: the probable trajectory, its dispersion and the rocket response to dynamic loading.
This paper tries to provide an answer to the problem of turbulent atmosphere influence on fire-extinguishing rocket flight.

2 Definition of Atmospheric Turbulence. Hypotheses

Since turbulence is a random process that cannot be described by explicit functions of time, only a statistical, probabilistic approach can be taken.
The velocity field, variable in time and space, may be considered as result of an average value and a perturbation around it. In the case of long time flights, the average value may be modelled as referencing the flight with respect to a mobile rectangular frame which moves with the wind's average velocity. In a point located by $\bar{r}(x_1, x_2, x_3)$ the air average velocity is:

$$\bar{u}(\bar{r},t) = [u_1 \quad u_2 \quad u_3]^T, \quad (1)$$

It may be described properly by a correlation matrix composed of elements like this:

$$R_{i,j}(\bar{\xi},\tau) = < u_i(\bar{r},t) u_j(\bar{r}+\bar{\xi}, t+\tau) >, \quad (2)$$

which represents the temporal and spatial average of the product between a component u_i located at a point defined by $\bar{r}$ at the time *t*, and the component u_j from the point $\bar{r}+\bar{\xi}$ at the time $t+\tau$.
To this matrix we can associate a spectral function matrix whose elements are defined by the Fourier transform function:

$$\Theta_{i,j}(\bar{\Omega},\omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{i,j}(\bar{\xi},\tau) e^{-i(\bar{\Omega}\bar{\xi}+\omega\tau)} \, d\xi_1 \, d\xi_2 \, d\xi_3 \, d\tau \quad (3)$$

The inverse of these elements belong to the correlation matrix:

$$R_{i,j}(\bar{\xi},\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Theta_{i,j}(\bar{\Omega},\omega) e^{-i(\bar{\Omega}\bar{\xi}+\omega\tau)} \, d\Omega_1 \, d\Omega_2 \, d\Omega_3 \, d\omega \quad (4)$$

Generally, the atmosphere turbulences are not Gaussian. However, for practical purposes, the normal repartition may be used, producing significant simplifications. On the other hand, the statistical parameters $R_{i,j}$ and $\Theta_{i,j}$ of the turbulence are defined in each point in space, $\bar{r}$ and variable with time *t*.
A particular aspect of the fire-extinguishing rocket is that it flies a long time, with a slight change in velocity magnitude, due to slow burning of rocket propellant. This is a significant theoretical advantage because it allows admitting the hypothesis that the turbulence is a stationary process. Another hypothesis is that the turbulence could be considered homogenous on small intervals, which means that $R_{i,j}$ and $\Theta_{i,j}$ do not depend on $\bar{r}$, alongside of at least a trajectory segment. At high altitude, the turbulence is identical in all points belonging to the same layer, which means it might be considered homogenous. Close to the ground often changes with altitude occur. However, in the case of small launch angles, homogenous turbulence might be considered along the trajectories close to ground.

$R_{i,j}$ and $\Theta_{i,j}$ usually depend on wind referenced frame axes orientation. If the statistical functions do not depend on space, the turbulence is considered isotropic, which makes the velocity components square means to be equal:

$$\sigma^2 = \overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}\ , \qquad (5)$$

where the standard deviation σ will be called turbulence intensity.

Due to the fact that rocket velocity is much higher than their variation velocity, the turbulence may be regarded to as „a frozen model”. This allows us to neglect time in the function $\overline{u}(\overline{r},t)$, which corresponds to Taylor's hypothesis. Consequently, the correlation and spectral functions become:

$$R_{i,j}(\overline{\xi},\tau) \to R_{i,j}(\overline{\xi})\ ;\ \ \Theta_{i,j}(\overline{\Omega},\omega) \to \Theta_{i,j}(\overline{\Omega})\ , \quad (6)$$

Consequently, the simplest model is the one of homogenous and isotropic turbulences, Gaussian and „frozen”, a model which is used for the flight outside the ground proximity layer. Inside of this layer is necessary to consider anisotropic turbulence.

For the isotropic turbulence, using Batchelor's relation:

$$\frac{R_{ij}(\overline{\xi})}{\sigma^2} = [f(\xi) - g(\xi)]\frac{\xi_i \xi_j}{\xi^2} + g(\xi)\delta_{i,j}\ , \qquad (7)$$

where $\xi = \left|\overline{\xi}\right|$; $\delta_{i,j}$ - Kroneker's symbol;

The longitudinal correlation function $f(\xi)$ represents the correlation between velocity components along an axis, given in two points of that axis (Figure 1.):

$$f(\xi) = \frac{R_{11}(\xi)}{\sigma^2} = \lim_{X\to\infty}\frac{1}{2X\sigma^2}\int_{-X}^{X} u_1(x_1)u_1(x_1+\xi)\,\mathrm{d}x_1 \qquad (8)$$

Because of the hypothesis regarding isotropy:

$$f(\xi_1) = f(\xi_2) = f(\xi_3) = f(\xi)\ , \qquad (9)$$

and:

$$f(\xi) = \frac{R_{11}(\xi_1)}{\sigma^2} = \lim_{X\to\infty}\frac{1}{2X\sigma^2}\int_{-X}^{X} u_1(x_1)u_1(x_1+\xi_1)\,\mathrm{d}x_1 \qquad (10)$$

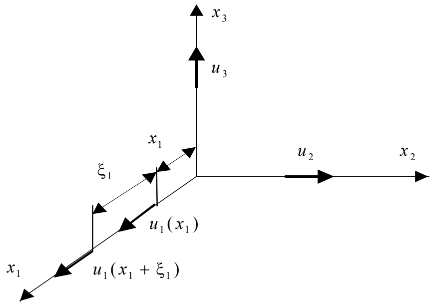

Fig. 1 Longitudinal correlation of velocity components

The other function $g(\xi)$, is the lateral correlation function and represents the correlation between the transversal velocity components with respect to an axis (Figure 2), defined in two points:

$$g(\xi) = \frac{R_{33}(\xi)}{\sigma^2} = \lim_{X\to\infty}\frac{1}{2X\sigma^2}\int_{-X}^{X} u_3(x_1)u_3(x_1+\xi)\,\mathrm{d}x_1 \qquad (11)$$

Based on the hypothesis regarding isotropy, it results:

$$\begin{aligned}&< u_3(x_1)u_3(x_1+\xi_1) > = < u_2(x_1)u_2(x_1+\xi_1) > = \\ &= < u_3(x_2)u_3(x_2+\xi_2) > = \ldots\end{aligned}\ , \qquad (12)$$

and:

$$g(\xi) = \frac{R_{33}(\xi_1)}{\sigma^2} = \lim_{X\to\infty}\frac{1}{2X\sigma^2}\int_{-X}^{X} u_3(x_1)u_3(x_1+\xi_1)\,\mathrm{d}x_1 \qquad (13)$$

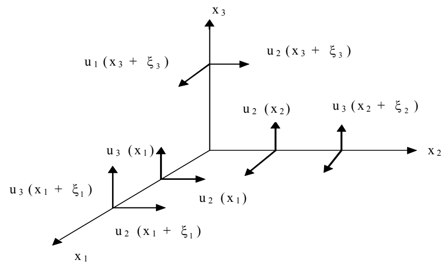

Fig. 2 Transversal correlation of velocity components

3 The Uniform Turbulence Model

For the characteristic correlation functions, the following relations from work [1] may be used:

$$f(\xi) = \frac{2^{2/3}}{\Gamma(1/3)}\zeta^{1/3}K_{1/3}(\zeta)\ ;$$

$$g(\xi) = f(\xi) - \frac{2^{-1/3}}{\Gamma(1/3)}\zeta^{4/3}K_{2/3}(\zeta) \qquad (14)$$

where: $\zeta = \dfrac{\xi}{aL}$, $a = 1{,}339$ and L representing the characteristic length. Γ is the second degree Euler function and K_υ is the second degree Bessel modified function. The Fourier transforms of the functions $f(\xi)$ and $g(\xi)$ are spectral one-dimensional functions:

$$\Phi_{11}(\Omega_1) = \frac{\sigma^2}{\pi}\int_0^\infty f(\xi)e^{-i\Omega_1\xi}\,\mathrm{d}\xi\ ;$$

$$\Phi_{33}(\Omega_1) = \frac{\sigma^2}{\pi}\int_0^\infty g(\xi)e^{-i\Omega_1\xi}\,\mathrm{d}\xi \qquad (15)$$

Figures 3 and 4 illustrate graphical representation of functions defined by (14) and (15).

For these spectral functions, through experimental measurements, several models have been established, among which, the most commonly used are the von Karman model:

$$\Phi_{11}(\Omega_1) = \frac{\sigma^2 L}{\pi}\frac{1}{[1+(aL\Omega_1)^2]^{5/6}}\ ;\ \Phi_{33}(\Omega_1) = \frac{\sigma^2 L}{2\pi}\frac{1+(8/3)(aL\Omega_1)^2}{[1+(aL\Omega_1)^2]^{11/6}} \qquad (16)$$

and the Dryden model:

$$\Phi_{11}(\Omega_1) = \frac{\sigma^2 L}{\pi}\frac{1}{1+(L\Omega_1)^2}\ ;\ \Phi_{33}(\Omega_1) = \frac{\sigma^2 L}{2\pi}\frac{1+3(L\Omega_1)^2}{[1+(L\Omega_1)^2]^2} \qquad (17)$$

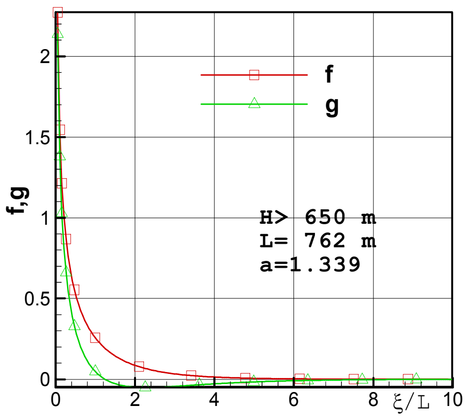

Fig. 3 Characteristic correlation functions

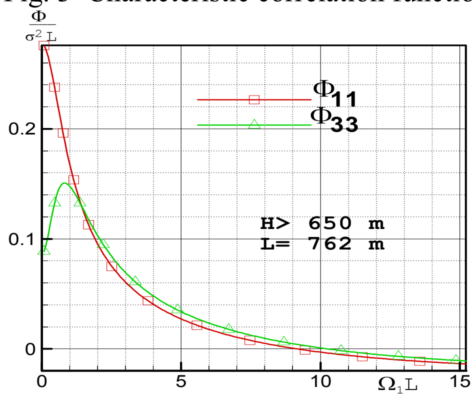

Fig. 4 Characteristic spectral functions

The established experimental relations [1] are in good accordance with relations (15) and may be successfully used in the case of the study models with uniform turbulence. However, in the case of linear turbulence method, due to necessity to define cross correlation functions which show the mutual dependence between velocity components, the characteristic functions defined by (14) will be used.

In case of altitudes over 650m the boundary layer influence is diminished and the atmosphere may be considered, a unique value for the characteristic length *L* being recommended, both to longitudinal and lateral spectral functions. For the model considered, according to [1] and [2] *L*=762*m.* The standard deviation is the same along all the three directions, depending only on the turbulence intensity: $\sigma = 1{,}53\,[m/s]$ - low turbulence; $\sigma = 3{,}05\,[m/s]$ - mean turbulence; $\sigma = 6{,}4\,[m/s]$ - height turbulence (storm).

The turbulence model obtained in this way is the simplest possible, being known as the uniform field model, in which only the linear components of velocity along the three axes, reduced in the gravity centre of the rocket, are considered: u_g, v_g, w_g.

4 The Linear Turbulence Method

A higher order approximation is the one of a linear field, the turbulence velocities being considered as linear functions depending on position.

Authors of [1], [2] and [3] developed in case of airplane linear turbulence models in four points. These models also allow calculation of rotation velocity components p_g, q_g, r_g due to turbulence. Starting from the method proposed in [1], we developed for rockets a linear model of turbulence in two points, which allow deducing of angular velocity of pitch and yaw motion.

In the linear turbulence case, due to the slender axi-symmetrical shape of the rocket, some gradients may be neglected. We shall consider a model of the rocket in two points as shown in Figure 5.

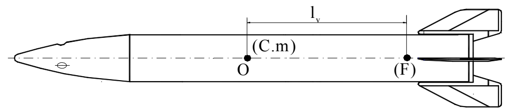

Fig. 5 Model of the rocket in two points

The length l_v, called aerodynamic length, represents the distance from the mass centre to the focal point. Based on this model, the pitch and yaw velocities, equivalent to the velocity gradients, are:

$$q_g = \frac{1}{l_v}(w_1 - w_0)\,;\quad r_g = \frac{1}{l_v}(v_0 - v_1)\,, \qquad (18)$$

to which there are added the linear velocities due translation for the uniform turbulence:

$$w_g = w_0\,;\ u_g = u_0\,;\ v_g = v_0 \qquad (19)$$

To calculate the velocities produced by turbulence, we expressed their correlation functions through the characteristic functions *f* and *g*, proceeding as in the case of uniform distribution, we calculated the spectral functions corresponding to the correlation functions and, from these, we found some possible forms of the translation and rotation velocities generated by turbulence. For the case of the pitch turbulence velocity the following initial relation was used:

$$R_{qq}(\tau) = \langle q_g(t) q_g(t+\tau) \rangle\,, \qquad (20)$$

obtaining:

$$\frac{(l_v)^2}{\sigma^2} R_{qq}(\tau) = 2g(\xi_1) - g(\xi_2) - g(\xi_3)\,, \qquad (21)$$

where:

$$\xi_1 = V\tau\,;\quad \xi_2 = \xi_1 + l_v \quad \xi_3 = \xi_1 - l_v$$

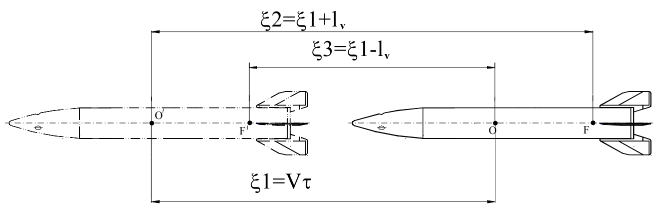

Fig. 6 Correlation functions calculus schema

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Similarly the angular yaw velocity:

$$R_{rr}(\tau)=<r_g(t)r_g(t+\tau)>=\frac{1}{(l_v)^2}<(v_0-v_1)(v'_0-v'_1)>= \\ =\frac{1}{(l_v)^2}[<v_0v'_0>+<v_1v'_1>-<v_1v'_0>-<v_0v'_1>] \quad (22)$$

from which we obtained:

$$R_{rr}(\tau)=\frac{1}{(l_v)^2}[2R_{22}(V\tau,0,0)-R_{22}(V\tau+l_v 0,0)-R_{22}(V\tau-l_v 0,0)]$$

$$\frac{(l_v)^2}{\sigma^2}R_{rr}(\tau)=2g(\xi_1)-g(\xi_2)-g(\xi_3)\,, \quad (23)$$

From the previous relations and (13), the correlation functions for the rotation velocities were determined. The two rotation velocity components are equal, $R_{qq}=R_{rr}$, due to axial symmetry (Figure 7). The spectral function for the rotation velocity is represented in Figure 8. We compare the velocity components calculated with the linear turbulence field theory to those obtained using the uniform field theory. These proved to be identical.

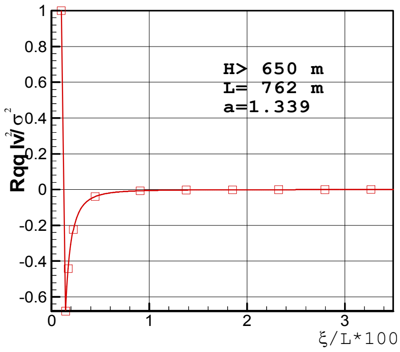

Fig. 7 Correlation function for rotation velocities

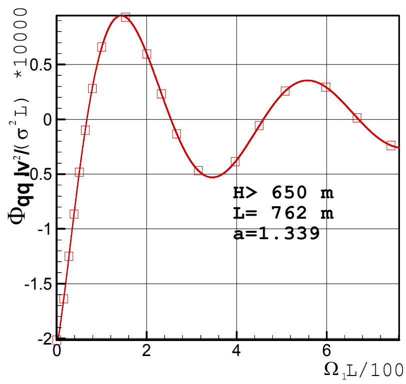

Fig. 8 Spectral function for the rotation velocity

Thus, for the longitudinal component we obtained:

$$R_{uu}(\tau)=<u_g(t)u_g(t+\tau)>=<u_0u'_0>=R_{11}(V\tau,0,0)=\sigma^2 f(\xi_1) \quad (24)$$

For the lateral component we obtained:

$$R_{vv}(\tau)=<v_g(t)v_g(t+\tau)>=<v_0v'_0>=R_{22}(V\tau,0,0)=\sigma^2 g(\xi_1) \quad (25)$$

$$R_{ww}(\tau)=<w_g(t)w_g(t+\tau)>=<w_0w'_0>=R_{33}(V\tau,0,0)=\sigma^2 g(\xi_1) \quad (26)$$

Relations (24)…(26) show that for linear model we found the same characteristic *f* and *g* from Figure 3 and the spectral functions shown in Figure 4. This means that the translation velocities for the liner model are the same with those from the uniform model and for these we can use theoretical relation (14) or the experimental relations (16) and (17).

Relations (23) for the angular velocities may be used only in the case of the theoretical model defined by the characteristic functions (14). In the case of axis-symmetric rockets, the only supplementary term specific to relations (23) is produced by the angular pitch/yaw velocities. If the aerodynamic length is comparable with respect to the characteristic length, the influence of this term is small and can be neglected. In the case of short rockets, like fire-extinguishing rocket, the term is important as we will see from the numerical comparison.

5 Simulation of Atmospheric Turbulence

The simulation of an atmospheric turbulence is based on the method of sinusoidal sum, using the spectral function modulus of the turbulence velocity to obtain the oscillation amplitude specific to a frequency bandwidth. So, for the spectral functions values presented in Figure 4, we determine by integration the value of amplitude corresponding to a frequency bandwidth, which can be approximated with the frequency corresponding to the centre of the integrated domain. To complete the model, to each frequency we associated an initial arbitrary phase.

Using the notation S_ϕ for the portion of the area obtained by integrating the modulus of a spectral function from Figure 4 and considering that the function has been represented only for the positive values of the variable Ω_1, the amplitude has the value:

$$A=\sigma\sqrt{2S_\phi} \quad (27),$$

Two of the translation velocities generated by turbulence in a point ξ are given by:

$$u_g(\xi)=A_1\sum_i \sin(\Omega_1^i\xi+\phi_1^i)\,;$$

$$w_g(\xi)=A_2\sum_i \sin(\Omega_2^i\xi+\phi_2^i)\,, \quad (28)$$

where the index "1" was given to amplitude, pulsation and initial phase obtained from the modulus of the longitudinal spectral function, and the index "2" – given to the elements obtained from the modulus of the lateral spectral function. The third component of the velocity is the same as the second, excepting the initial phase:

$$v_g(\xi)=A_2\sum_i \sin(\Omega_2^i\xi+\phi_3^i) \quad (29)$$

Making the assumption that the rocket flies on a linear trajectory with a constant velocity *V*:

$$\xi = V\tau , \quad (30)$$

Consequently, the velocities originated by turbulence may be expressed as functions of time (Figure 9)

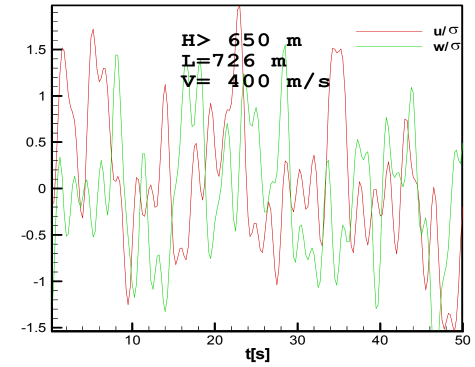

Fig. 9 The axial u_g and normal w_g velocities produced by turbulence

The modulus of the spectral functions presented in Figure 8 was used to find the angular velocities. By applying similar relations to (27) and (28) we obtained a form of the angular velocities originated by turbulence with respect to time. Because the spectral functions are equal (Fig.8), the yaw and pitch velocities will be approximately equal $r_g \approx q_g$, differing only by initial phases. In this case, in Figure 10, we presented only the pitch velocity function q_g.

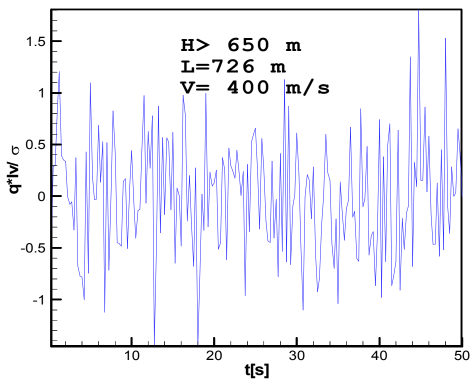

Fig. 10 The pitch velocity produced by turbulence

6 Fire-extinguishing Rocket Trajectory

Starting from papers [3] and [4], some details regarding the procedure of building the simulation model for the fire-extinguishing rocket are presented in this section.

For the turbulent atmosphere we used in order to make a comparison, the uniform and the linear turbulence model. The initial values of the sinusoidal sums phases were obtained by generating random numbers, and the amplitude and frequency were determined starting from the spectral density functions using the procedure described in the previous section. Because of the sequential initial random input of the phases, a fascicle of trajectories was obtained. The results are presented in the next tables.

In order to analyse trajectory parameters, we define four check points: 1-Start point, 2-End of rocket motor burn, 3-Trajectory apex, 4-Impact point

Table 1. Nominal trajectory parameter, $\sigma = 0$

Stage of flight	1	2	3	4
T [s]	0	9.5	26.1	58.
V [m/s]	40	625.4	209	206
γ [deg]	45	29.2	0.20	-68.1
X [m]	0	2654	7882	12210
Y [m]	1	1788	3677	0

Table 2. Average and standard deviation trajectory parameters, low turbulence $\sigma = 1.53$

Stage of flight	1	Uniform model			Linear model		
		2	3	4	2	3	4
T [s]	0	9.5	26.9	57.9	9.5	26.6	58.06
	0	0.0	0.1	0.59	0.0	0.1	0.97
V [m/s]	40	628.6	201.8	206.0	625.6	201.2	200.3
	0	0.3	0.9	0.73	0.3	1.9	1.6
γ [deg]	45	28.4	0.28	-68.03	29.5	-1.	-68.7
	0	0.24	0.19	0.47	0.54	0.39	1.24
X [m]	0	2672.2	7907.9	12207.7	2644.3	7846.0	12018.0
	0	7.9	2.5	17.4	14.6	19.5	64.7
Y [m]	1	1755.5	3980.1	0	1801.0	3862.1	0
	0	11.17	30.8	0	17.7	41.2	0

In table 2 for each parameter are two rows: first for average and the second for standard deviation values. For a better analyse the main results are presented in a comparative form.

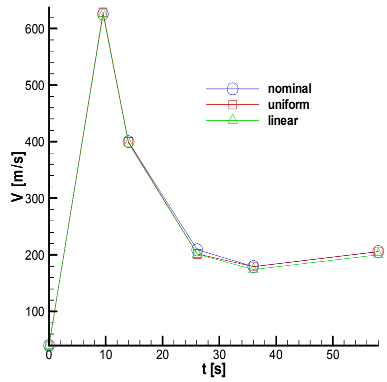

Fig. 11 Velocity diagram average values, low turbulence

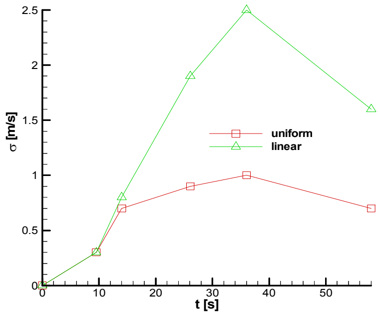

Fig. 12 Velocity diagram standard deviation values, low turbulence

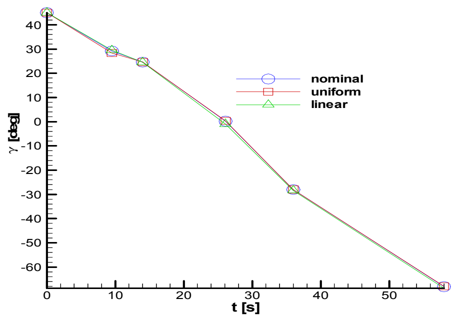

Fig. 13 Climb angle diagram, average values, low turbulence

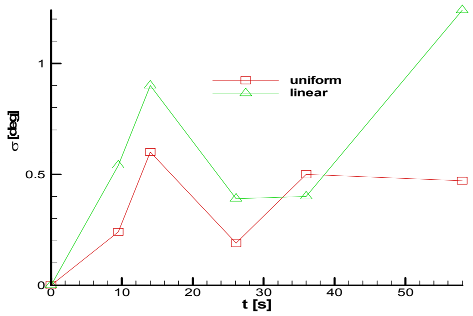

Fig. 14 Climb angle diagram, standard deviation values, low turbulence

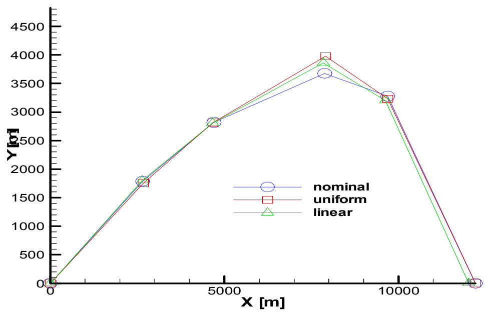

Fig. 15 Trajectory diagram, average values, low turbulence

The average parameters from nominal trajectory, uniform model and linear model are relatively closed. Different from these, the parameters dispersion obtained with uniform model and linear model are different, therefore the use of the linear model are rightful.

7 Conclusions

Two turbulence models were presented in the paper. In the uniform turbulence model, only the translation velocities of the mass centre are considered. In the linear turbulence model, besides the linear velocities, the angular velocities of pitch and yaw with respect to mass centre are added.

In the case of axis-symmetric rockets, the term depending on angular velocities is the only which couples the equations. A further simplification could be made in the case of short rockets to which the aerodynamic length is much smaller than the characteristic length. In this situation, the coupling term due to rotations is small and may be neglected, so that the uniform turbulence model is an acceptable approximation.

Considering the linear turbulence model developed in our study, we have analysed in a simulation example the flight dynamics of a rocket, in which the influence of turbulence was evaluated in four characteristic stages of the flight.

The work novelty consists in technique used in modelling the turbulent atmosphere influence on unguided rocket flight:

- two point model used for the linear turbulence
- the technique used to obtain the turbulence velocity
- the comparison between linear and uniform turbulence model

References:

[1] Etkin, Bernard, *Dynamics of Atmospheric Flight*, John Wiley & Sons, Inc., New York, 1972.

[2] Etkin, Bernard, Lloyd Duff Reid, *Dynamics of Flight*: *Stability and Control* -3rd ed, John Wiley & Sons, Inc., New York, 1995.

[3] Chelaru T.V., Barbu C. *Mathematical model and technical solutions for multi-stage sounding rockets, using the rotation angles,* Proceedings of the 3rd International Conference on Applied Mathematics, Simulation, Modeling (ASM'09), pp.94-99, ISBN: 978-960-474-147-2, NAUN, Vouliagmeni, Athena, Greece, December 29-31, 2009.

[4] Chelaru T.V.**,** Barbu C., Chelaru A., *Mathematical model and technical solution for small multistage suborbital launchers,* Proceedings of the 4th International Conference on Recent Advances in Space Technologies-RAST 2009, pp. 186-191, ISBN: 978-1-4244-3626-2, Istanbul, Turkey, 11-13 Jun. 2009.