Some properties of Nonlinear Continuous Time Generalized Predictive Control

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Abstract: - Some properties of NCGPC are discussed (Nonlinear Continuous Time Generalized Predictive Control). The demonstration that, the selection of particular design parameters, such as control order and predictor order leads to well known feedback linearization. The response of closed loop is influenced by the prediction horizon and the reference model. Simulations are presented in order to demonstrate the features of NCGPC.

Key-Words: - Nonlinear control, Predictive Control, Feedback linearization, Relative Degree, Continuous system.

1 Introduction

Predictive control [3-14] and feedback linearization [1] and [2] constitute two of the most important research lines in nonlinear control. Geometric linearization theory has permitted the applications of linear control algorithms to nonlinear systems. It is done by using feedback linearization techniques [1], through differential geometry approach to transform a nonlinear input-output system into a linear system; then, a linear controller can be applied to the linearized model. One of the most important questions in NMPC (Nonlinear Model Predictive Control) is if a finite horizon NMPC strategy does guarantee stability of the closed-loop or not. Another problem is the algorithms demand tremendous computational power for solving a nonlinear dynamic optimization.

The NCGPC, [10, 11] is an alternative nonlinear predictive controller; this controller was developed in a different way than conventional nonlinear predictive controllers. The NCGPC [10, 11] is based in the prediction of the system output and due to the fact that it was not derived with the objective of canceling nonlinearities, as feedback linearization techniques do, the NCGPC has three advantages: First, it can constrain the predicted control through $N_a$ additionally the response becomes slow and the control is not very active-, and second, when $N_a < N_f - r$, there is not zero dynamics cancellation and then the internal stability is preserved. Also, the NCGPC [12] provides a nice way of handling systems with unstable zero dynamics. And the last advantage is the control weight $\lambda$, it plays a very important role in the cost function. In [14], the non-regular nonlinear system is treated by using the last two advantages of the NCGPC. Another of the main advantages of NCGPC control schemes is that, when $N_a = N_f - r$ they do not require on-line optimization and asymptotic tracking of the smooth reference signal is guaranteed. Additionally, it is possible to demonstrate that the control law is another feedback linearization, thus closed-loop stability is ensured.

2 Review of NCGPC

This paper considers the nonlinear SISO systems with all system states assumed to be accessible, affine in the input with the following state-space representation:

$$\begin{align*}
  x(t) &= f(x) + g(x)u \\
  y(t) &= h(x)
\end{align*}$$

where $f$, $g$ and $h$ are differentiable $N_f$ times with respect to each argument. $x \in \mathbb{R}^n$ is the vector of the state variables, $u \in \mathbb{R}$ is the manipulated input and $y \in \mathbb{R}$ is the output to be controlled. It has a well-defined relative degree and its zero dynamics are stable.

2.1 Development of the NCGPC

The development of the NCGPC [10, 11] was carried out following the receding horizon strategy of its
linear counterpart [13]. The output prediction, reference trajectory and cost function are given in this section.

### 2.1.1 Prediction of the output
The output prediction is approximated for a Maclaurin series expansion of the system output as follows.

\[
y'(t, T) = y(t) + y(t)T + \frac{y^{(2)}(t)}{2!}T^2 + \cdots + \frac{y^{(N_s)}(t)}{N_s!}T^{N_s},
\]

or

\[
y'(t, T) = T_{N_s} Y_{N_s},
\]

where

\[
Y_{N_s} = \begin{bmatrix} y & y^{(2)} & \cdots & y^{(N_s)} \end{bmatrix}^T
\]

and

\[
T_{N_s} = \begin{bmatrix} 1 & T & \cdots & \frac{T^{N_s}}{N_s!} \end{bmatrix}
\]

The predictor order \(N_s\) is the number of the times that the output has to be predicted.

### 2.1.2 Prediction of the reference trajectory
In order to drive the predicted output along a desired smooth path (reference trajectory) to a set point. Two different reference trajectories are chosen in order to demonstrate the properties of NCGPC. The first reference trajectory is the output of the following reference model [13]

\[
W_r(t, s) = \frac{R_r(s)}{R_r(s)} \frac{w(t) - y(t)}{s}
\]

Considering the following approximation

\[
\frac{R_r(s)}{R_r(s)} = \sum_{i=0}^{N_r} r_i s^{-i}
\]

The reference trajectory is given by

\[
w_r'(t, T) = T_{N_r} w_r + y(t)
\]

where

\[
w_r = [r_0, r_1, \cdots, r_{N_r}]^T (w - y(t))
\]

and \(T_{N_r}\) is given by (5).

### 2.1.4 Cost function minimization
The function is not defined with respect current time, but respect a moving frame, which origin is in time \(t\) and \(T\) is the future variable. NCGPC calculates the future controls from a predicted output over a time frame. The first element \(u(t)\) of the predicted controls is then applied to the system and the same procedure is repeated at the next time instant. Thus predicted output depend on the input \(u(t)\) and its derivatives, and the future controls being function of \(u(t)\) and its \(N_r\)-derivatives. The cost function is:

\[
J(u_{N_r}) = \int_{\tilde{t}}^{\tilde{T}} \left[ y'(t, T) - w_r(T, t) \right]^2 dT
\]

With the substitution of equations (3) and (7) the cost function becomes

\[
J(u_{N_r}) = \int_{\tilde{t}}^{\tilde{T}} \left[ T_{N_r}O + T_{N_r}H_{u_{N_r}} - T_{N_r}w_r \right]^2 dT
\]

and the minimization results in
\[ u_{N_y} = K(w_r - O) \]  (15)

where
\[ T_s = \frac{\partial}{\partial t}T_{N_y}dT \]  (16)

The \( i \)th element of \( T_s \) is:
\[ T_i = \sum_{j=0}^{(i-1)!} T_s \left( i-j-1 \right)! T_s \]  (17)

As explained above, just the first element of \( u_{N_y} \) is applied. The control law is given by
\[ u(t) = k[w_r - O] \]  (18)

3 Geometric Interpretation

The input-output feedback linearization \[1\] and \[2\] is shown to be equivalent to NCGPC, if the following assumptions are satisfied:
- The system is given by equation (1).
- It has stable zero dynamics
- \( N_y = N_r \cdot r \)
- The trajectory reference is given by equation (12).
- The system states must be measurable.

The control law given by equation (18) is analyzed; the matrix \( H \) equation (8) is decomposed as
\[ H = \begin{bmatrix} H_1 & 0 \\ H_2 & 0 \end{bmatrix} \]  (19)

where \( H_1 \) is a zero matrix with dimension \( r \times (N_y - r + 1) \), and \( H_2 \) is a lower triangular matrix with dimension \( (N_y - r + 1) \times (N_y - r + 1) \) given by
\[ H_2 = \begin{bmatrix}
L_y L_y^{-1} h(x) & 0 & \ldots & 0 \\
J_1(x) & L_y L_y^{-1} h(x) & \ldots & 0 \\
J_2(x) & 1 & \ldots & \ldots \\
\vdots & \vdots & \ddots & \ddots \\
J_{r-1,y}^{N_y-r}(x) & 0 & \ldots & L_y L_y^{-1} h(x)
\end{bmatrix} \]  (20)

The matrix \( T_s \) equation (5) is decomposed as
\[ T_s = \begin{bmatrix}
T_{si1} & T_{si2} \\
T_{si2} & T_{si1}
\end{bmatrix} \]  (21)

where
\[ T_{si1} = r \times r \]
\[ T_{si2} = r \times (N_y - r + 1) \]
\[ T_{si3} = (N_y - r + 1) \times r \]
\[ T_{si4} = (N_y - r + 1) \times (N_y - r + 1) \]

The equation (22) can be written as
\[ K = H^{-1}_2 \sum_{j=2}^{N_y} T_{j21} I \]  (22)

\( I \) is the unitary matrix with dimension \( (N_y - r + 1) \times (N_y - r + 1) \). The first row of the inverse of \( H_2 \) is given by
\[ h_2^{-1} = \begin{bmatrix} I / L_y L_y^{-1} h(x) & 0 & \ldots & 0 \end{bmatrix} \]  (23)

Then, the first row of \( K \), which will be called \( k \)
\[ k = \frac{1}{L_y L_y^{-1} h(x)} [t_1, t_2, \ldots, t_r, 0, 0] \]  (24)

where \( t_1, t_2, t_3, \ldots, t_r \) are elements of the first row of \( T_{j22} \). The control law is given by
\[ u(t) = \frac{(w - y(t)) - \sum_{i=1}^{r+1} \beta_i L_y h(x)}{\beta_i L_y L_y^{-1} h(x)} \]  (25)

where
\[ \beta_i = 1/(t_0, t_0 + t_r, t_0 + t_r + \ldots, t_0 + t_{r-1}) \]
\[ \beta_i = t_i/(t_0, t_0 + t_r, t_0 + t_r + \ldots, t_0 + t_{r-1}) \]

We can notice that, incredible as it may seem, large \( N_y \) does not require a bigger computational effort, because as we can see from equation (30), the control depends just on the \( r \)-first derivatives, thus the rest of the derivatives only have influence in obtaining the parameters of \( t_r \) which just depends on \( T \). Moreover, \( N_y \) can be chosen as the smallest predictor order, which is such that the predicted output depends on \( u(t) \). The relative degree \( r \) of the system is exactly equal to the number of times the output has to be differentiated in order for the input to explicitly appear. Because of this, the relative degree \( r \) will be the smallest predictor order \( N_y \). We can conclude if \( N_y = N_r \cdot r \), the control law is independent of the last \( N_r \cdot r \) derivatives. Then it is possible to calculate the parameters \( \beta_i \) considering the largest \( N_y \) without the use of the remaining derivatives. We will consider this case, in all the process, except in the non minimum phase systems.

Substituting equation (30) into the rth derivative given by equation (6) leads to:
\[ y'(t) = \frac{1}{\beta_i} (w - y) - \sum_{i=1}^{r+1} \frac{\beta_i}{\beta_j} y' \]  (28)

Rearranging and taking Laplace transforms, the resulting closed-loop transfer function is given by:
\[ Y(s) = G(s)W(s) \]
\[ G(s) = \frac{1}{\beta_i s' + \beta_{r+1} s'^{r+1} + \ldots + \beta_1 s + 1} \]  (29)

Note that, by using the Routh-criterion, we can show that the systems are stable only for systems with \( r \leq 4 \).

4 Simulations

In order to show the effectiveness of proposed controller and to study the effects of the NCGPC
design parameters \((N_y, N_u, T_1, T_2, R_f/R_d)\) simulation will be presented. The example used in the simulation is as follows:
\[
\begin{align*}
  x_1 &= -x_1 - x_2 \\
  x_2 &= \exp(-x_1) - 1 - u \\
  y &= x_1
\end{align*}
\]  
\[(30)\]

\[4.1 \text{ The predictor order } N_y\]

As we explained before, the future output is approximated by an \(N^y\) order truncated Maclaurin series. It is clear that \(N_y\) needs to be chosen such that a good approximation can be obtained over the range in which \(T\) varies. We will choose \(N_y\) so that a good approximation of the open-loop system step response over the range \(0<T<T_2\) is obtained. The output derivatives of system the output equation \((30)\) are given by
\[
\begin{align*}
  \dot{y} &= -x_1 - x_2 \\
  \ddot{y} &= x_1 + x_2 - \exp(-x_1) + 1 + u \\
  \dot{y} &= -x_1 - x_2 + (\exp(-x_1) - 1)(\exp(-x_1) + 1) - (\exp(-x_1) + 1)\dot{u} + \dot{u}
\end{align*}
\]

The actual and approximate step responses of this system for various \(N_y\) over the range \(0<T<2\) is shown in Figure (1). We can see from the figure, for a larger \(T_2\) a larger \(N_y\) is needed.

In figure (2) it is shown the outputs systems when \(N_y\) is varied and control law given by equation (26). In this simulation design parameters are chosen as: setpoint equal 1, \(r=2\), \(N_y=3\), \(N_y=N_r-r=1\), \(N_u=2\), \(N_u=N_r-r=0\) \(R_f/R_d=1/(s+1)\), \(T_1=0\) and \(T_2=1\). As we conclude before if \(N_y=N_r-r\), the control law is independent of the last \(N_r-r\) output derivatives. Then it is possible to alculate the parameters \(\beta_i\) considering the largest \(N_r\), without the use of the remaining derivatives. It is possible to see, when \(N_r=3\), the response has better behaviour that if \(N_r=2\).

\[4.2 \text{ The control order } N_u\]

The control order has the function to constrain the predicted input. When the \(N_u=0\) the predicted control input is constrained to be constant in the future, i.e. derivatives of \(u(t)\) are taken zero, when \(N_u=1\) the predicted input will be a ramp. Small value of \(N_u\) gives less active control \(u(t)\) and slow response, as it is possible to see from figures (3) and (4).

\[4.3 \text{ The maximum prediction horizon } T_2\]

In this simulation design parameters are chosen as: setpoint equal 1, \(r=2\), \(N_y=3\), \(N_u=N_r-r=1\), \(R_f/R_d=1/(s+1)\) and \(T_1=0\). Variations of \(T_2\) are chosen to vary from 0.3, 0.5, 1, 2. From figures (5) and (6), we can deduce that small value \(T_2\) fastest responses are obtained.
and predictor order leads to well known feedback linearization. The response of closed loop is influenced by the prediction horizon and the reference model. Simulations are presented in order to demonstrate the features of NCGPC.

Another of the main advantages of NCGPC control schemes is that, when $N_u = N_y - r$ they do not require on-line optimization and asymptotic tracking of the smooth reference signal is guaranteed. We can show, that incredible as it may seem, large $N_u$ does not require a bigger computational effort, because the control depends just on the $r$-first derivatives, thus the rest of the derivatives only have influence in obtaining the parameters of $t_i$, which just depends on $T$. Then it is possible to calculate the parameters $\beta_i$ considering the largest $N_u$, without the use of the remaining derivatives. Additionally, it is shown that the control law is another feedback linearization, thus closed-loop stability is ensured.

A closed-loop transfer function was found, it is possible to infer that, by using the Routh-Horwitz criterion, and systems are stable only for systems with $r \leq 4$.

References:


