Euclidian Edge Length and Topology of Networks Generated by Random Transports on One-Dimensional Lattice

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Abstract: In geographical graphs, restriction on the Euclidian length of edges between nodes far from each other plays an important role in determining their structures. We propose a stochastic network model developed in one-dimensional lattice, in which distribution of Euclidian length of edges is determined by the prosperity of random transports represented by random walkers and ageing of edges. We showed by numerical calculation that the increase rate of typical length of edges changes from a rate proportional to $\hat{B}/\hat{D}^{8/3}$ to the normal diffusion rate $\hat{D}$ as time $\hat{t}$ passes. As a result, the evolved network is able to spread in the one-dimensional lattice in a range determined by the lifetime of vertices with large degree. The edge length distribution is supported by the smallness of the mean shortest path length and largeness of the local clustering of vertices regardless of the value of vertex degree.

Key–Words: Network modelling, Euclidian length of edges, Random walk, Clustering coefficient

1 Introduction

Diverse systems in the real world, such as the Internet, social interacting species, technological systems, and biological systems, can be modeled by networks where individuals constituting the system are regarded as vertices and interactions between them as edges. For the last ten years, common network structures in different systems, for example power-law distribution of vertex degree (the scale-free property) and the small-world phenomenon, have been discovered through stimulations in investigations on structure and functions of networks. Studies on networks have elucidated that network structures play an important role in their functions such as spreading processes, synchronization, tolerance to errors and attacks [1, 2].

Network modeling is an effective tool for elucidating the principle for explaining such typical properties of various networks. For example, the scale-free property can be described by a simple rule, that is, vertices of large degree tend to gain new links more than those of small degree [3]. Although random networks intrinsically have small mean shortest path lengths with respect to network size, adding small fractions of random links can reduce the mean shortest path length even for geographical networks [4]. In real geographical networks such as airport networks and highway networks [5, 6], Euclidian edge length distribution is an important characteristic because it reflects spatial constraints peculiar to the system on the network topology. As for rewiring between vertices in a lattice, an interesting problem, “what distribution of length of edges added to a lattice can lead to small-world property in geographical networks?” was studied [7]. However, restriction on distribution of length is introduced in the first place in most cases. Emergence process of distribution Euclidian length of edges has not been understood so far. In this paper, we investigate the time evolution of edge length distribution in a network model where random walkers’ movements on the network stimulate short-cuts between vertices. Although the presented model is considered as artificial one, the model presents an example of determining length distribution using a dynamical process in a network system.

This paper is organized as follows. In the next section, after explaining the model, fundamental properties of the graph, time evolution of graph and degree distribution, already examined in previous works [8], are reviewed. The main results newly reported in the paper are presented in Section 3 and Section 4. In Section 3, dependence of Euclidian distribution of edges on time and $\rho_1$, the control parameter for lifetime of edges, is calculated. In Section 4, numerical results for clustering coefficient and harmonic mean of shortest path length between vertices are presented. In Section 5, results are summarized and discussed.
2 Fundamental Properties of Graph

The rules that generate the graph investigated in this paper are as follows [8]. Initially, a random walker starts from one vertex in a one-dimensional lattice. The walker can move randomly from a vertex where the walker currently stays to one of adjacent vertices per one time step. The process that regulates the adding and removing of edges is divided into the following three parts.

- Creation of edges. At each time step, a short-cut is created between the vertex where the walker currently stays and the vertex where the walker stayed two times steps before as long as an edge does not already exist between them. The random walker can pass not only through edges in the original lattice but newly created edges. In the following text, we call these vertices that have gained added edges by the passing of the walker as "vertices with created edges".

- Strengthen of edges. Each edge is associated with integer ‘strength’. Strength 1 is given to newly added edges initially. At each time step, the strength of edge that the walker has passed or that connects the vertex where the walker currently stays and the vertex where the walker stayed two times steps before is increased by 1 if such an edge already exists.

- Aging of edges. At each time step, each strength of the edge is decreased by 1 with probability \( p_{A} \). Edges that attain strength 0 are removed except edges constituting the one-dimensional lattice that has existed from the initial time. These exceptional edges are assumed to maintain their strength with minimum value 1. This assumption guarantees permanent maintenance of one-dimensional structure in resulting networks.

In this model, movement of random walker is always restricted by the last movement of walker because of short-cuts created by the walker’s passage. This restriction strongly affects the time evolution of networks especially for one-dimensional cases. Consequently, the subgraph consisting of all the vertices with created edges evolves through distinct three stages.

The first stage is characterized by the spread of nearly complete subgraph in the way that \( N = (6t)^{1/3} \), where \( N \) is the number of vertices with created edges and \( t \) is an integer denoting time steps (Figure 1 (a)). The next stage of network evolution begins after the condition \( N = 2\sqrt{1/p_{A}} \) is reached. This stage is characterized by the collapse of the nearly complete subgraph created in the first stage owing to the impossibility of keeping complete subgraph in the condition \( N > 2\sqrt{1/p_{A}} \). This is because the decrease rate of the sum of edge strength \( p_{A} N (N−1)/2 \) is larger than the increase rate of the edge strength per time 2 in this condition. The balance of decrease and increase of strength on edges leads to a stagnation in the increase of the number of created edges. Consequently, the number of created edges fluctuates around an equilibrium value while the number of vertices with created edges continues to increase little by little.

In the third stage of the network evolution, the number of vertices with created edges also reaches an equilibrium value (Figure 1 (b)). The equilibrium state of number of created edges and number of vertices with created edges implies a stationary vertex degree distribution with constant mean vertex degree. Figure 1(c) shows one example of such a degree distribution, in which the power-law behavior can be only seen in a small region of vertex degree.

![Figure 1](image_url)

Figure 1: (a) Complete subgraph created by the wandering of random walkers after 3,000 time steps (in the first stage of network evolution) when \( p_{A} = 0.004 \). The edges aligned in the circle denote edges that constitute the original one-dimensional lattice, and other edges are edges created by a walker. (b) Wide spread subgraph at \( t = 800,000 \) (in the third stage of network evolution) when \( p_{A} = 0.004 \). (c) Numerically obtained degree distribution of the subgraph (Number of vertices in the subgraph of degree \( k \)) at \( t = 9,500,000 − 10,000,000 \) when \( p_{A} = 0.001 \).
ever, the Euclidian length between two vertices is defined by the distance measured in the original one-dimensional lattice in which all vertices are located with constant interval, 1, on a straight line. The shape of the circle indicated in Figure 1, in which vertices are located, is employed only for visualization.

Figure 2: (a) One example of time series obtained for mean edge length (dotted line) and maximum length (solid line) when \( p_d = 0.04 \). (b) Edge length distributions at \( t = 3,000,000 \) (in the third stage of network evolution) obtained by 50 times repeated calculations when \( p_d = 0.04 \).

3 Distribution of Euclidian length of edges

In the first stage of the network evolution, an analytical form of Euclidian edge length distribution \( P(l) \) (ratio of edges of Euclidian length \( l \)) can be easily obtained as, \( P(l) \sim 2(N - l) - 1 \) for \( 1 \leq l \leq N - 1 \) and \( P(l) = 0 \) for \( N \leq l \) due to the simple structure of the nearly complete subgraph. Therefore, the maximum length of edges is subject to time dependence proportional to \( N \sim t^{1/3} \). Although the distribution begins to change to broad-type after entering the second stage of the network evolution, our interest is on the final form of distribution in the third stage.

Figure 2(a) illustrates one example of time series of mean and maximum edge length. The shape of the distribution changes with time irregularly corresponding to irregular changes in the maximum length with time. Although the exact form of distribution is yet to be identified only by numerical calculation, repeated calculations reveal edge length distribution which can be approximated by exponential decay (Figure 2(b)). As \( p_d \) tends to small, a detailed picture of the changes in the maximum length begins to appear as shown in Figure 3(a). The figure shows sudden decrease in the maximum length of edges after slow increase and repetition of such a behavior. This sudden decrease means that the longest edge which is incident to the vertex far from other vertices with created edges had been removed. This result implies that edges with a length longer than a certain length can exist only in an unstable state, although the distribution obtained by repeated calculation shows finite probability of the existence of such long edges (Figure 3(b)).

This behavior of the maximum length of edges implies that the typical length of edges is determined by a balance between the life time of a vertex with the longest edges and the expansion rate of length of edges incident to the vertex. The expansion rate of length of edges incident to a vertex far from other vertices with created edges can be estimated by the calculation of temporal rate of increase in mean length of edges when \( p_d \) is so small that the sudden decrease of the maximum length can be hardly observed. This result is indicated in Figure 4(a), which shows that the mean edge length and the maximum edge are ap-
rapidly with respect to decrease in life time of vertices with large degree increases, which shows that the numerically calculated approximately proportional to that of vertices with long edges.

Figure 4: (a) Log-log plot of time dependence of mean edge length and maximum edge length when \( p_{\mu} = 0.001 \). Slopes are estimated approximately as 1/2. (b) Characteristic length defined by reciprocal number of slopes in semi-log plots of edge length distribution (in the third stage of network evolution) versus 1/\( p_{\mu} \).

In other words, the vertex with the longest edges behaves like a normal diffusion. This is not a trivial result, because this is not the diffusion of random walker, but of vertices with created edges.

However, the exponential decay in the edge length distribution indicated in Figure 2(b) and 3(b) implies that the diffusion does not lead to infinite length of edges. Figure 4(b) shows that the typical length of edges for different \( p_{\mu} \), which is estimated from distribution obtained numerically, is proportional to 1/\( p_{\mu} \). Therefore, if we denote the typical life time of vertices with long edges as \( \tau \), \( \tau^{1/2} \) should be proportional to 1/\( p_{\mu} \). As a result,

\[
\tau = 1/p_{\mu}^2. \tag{1}
\]

This result is consistent with a result reported in ref. [8], which shows that the numerically calculated life time of vertices with large degree increases rapidly with respect to decrease in \( p_{\mu} \), since the life time of vertices with long edges must be associated with that of vertices with large degree.

4 Other properties

In this section, two properties, “efficiency of graph” \( EG \) and “mean local clustering taken over vertices of degree \( k \)” \( C(k) \), which support the connected structure with wide spread range in one-dimensional lattice are calculated. The efficiency of graph \( EG \) is defined as [9],

\[
EG = \frac{1}{N(N-1)} \sum_{i \neq j} \frac{1}{d_{ij}}, \tag{2}
\]

where \( d_{ij} \) denotes the shortest path length between vertex \( i \) and \( j \) with created edges (Note that \( d_{ij} \) is quite different from the Euclidian length.). The reciprocal of \( EG \) is the harmonic mean of \( d_{ij} \), which indicates traffic capacity of the subgraph with avoiding infinite value of \( d_{ij} \). Local clustering coefficient \( c_i \) of a vertex \( i \) of degree \( k_i \) is defined as,

\[
c_i = \frac{2e_i}{k_i(k_i - 1)}. \tag{3}
\]

where \( e_i \) denotes the number of edges that directly connect two adjacent vertices of vertex \( i \). \( C(k) \) is defined as an average of \( c_i \) taking over all the vertices of a given degree \( k \). \( C(k) \) can be related to the frequency of the walker’s passing near a vertex of degree \( k \) by the consideration of the balance between the increase \( \Delta S_i \) and decrease in the sum of strength over edges that directly connect two adjacent vertices of vertex \( i \),

\[
\Delta S_i = \frac{k_i(k_i - 1)C(k_i)p_{\mu}}{2}. \tag{4}
\]

Of course, the identity (4) does not always hold for all vertices \( i \). The identity is only valid for the cases where changing rate of the right hand side is far smaller than the time scale 1/\( p_{\mu} \). The increase rate \( \Delta S_i \) is approximately proportional to the frequency of the walker’s passing near a vertex of degree \( k \) because the walker’s passing near a vertex gives strength to edges that directly connect two adjacent vertices of vertex \( i \) (See Figure 5).

Figure 6(a) illustrates one example of time dependence of 1/\( EG \). Compared to Figure 3, it can be seen that 1/\( EG \) can keep a stationery value regardless of the width of spread of the subgraph on the one-dimensional lattice. The smallness of the mean shortest path length enables the walker to visit every vertex in the subgraph, even when the subgraph is spread over a wide range in the one-dimensional lattice. In the region in the lattice where the subgraph is spread, there are however many vertices with no created edges, which are not considered in the calculation of (3).

Figure 6(b) shows the dependency of \( C(k) \) on the value of vertex degree \( k \). Roughly speaking, the figure shows that \( C(k) \) can take large values regardless
Figure 5: Approaching ways adding strength to edges connecting two adjacent vertices of vertex $i$. (a) Walker’s passing through an edge that connects two adjacent vertices, $a$ and $b$, of vertex $i$. This process strengthens the edge between $a$ and $b$ by 1. (b) Walker’s passing vertex $i$. This process strengthens not only edges where the walker passes but edges between $a$ and $b$. (c) Walker’s visit to $i$ via two vertices, $a$ and $b$. This process not only creates an edge between $a$ and $i$ but strengthens the edge between $a$ and $b$. Walker’s departure also has similar effects on the strength of edges.

Figure 6: (a) Time dependence of $1/EG$ when $\rho_1 = 0.004$. (b) Dependence of $C(k)$ on value of vertex degree $k$ when $\rho_1 = 0.004$.

of the value of vertex degree $k$. Especially for large $k$, the value of $k$ determines the value of $C(k)$ deterministicly. Therefore, the identity (4) must be valid for vertices with large degrees. From the constancy of $C(k)$ with respect to $k$ and (4), the frequency of the walker’s passage near vertex $i, \Delta S_i$, depends on $k$ in the form $k^2$. Such an ability of vertices with large degrees to gather a walker is thought to be needed for the long life time of vertices with large vertex degrees.

5 Concluding Remarks

We have investigated the distribution of Euclidian edge length in networks generated by short-cuts created after tracing a random walk on a one-dimensional lattice. The distribution changes with time from a triangle corresponding to the created nearly complete subgraph to a broad-type distribution with typical edge length $1/\rho_1$. The distribution can be understood by the balance between the increase rate of the maximum length of edges and life time of vertices with the longest edges. Numerical calculation indicates that the increase rate of the maximum length of edges is subject to the rate of normal diffusion in the one-dimensional lattice, which corresponds to the diffusion of vertices with the longest edges. Note that such a diffusion of vertices with large vertex degree will never be realized without the frequent visiting of the walker to every vertex, because edges incident to a vertex will only be removed without the frequent vis-
iting of the walker to the vertex. Numerical calculation shows a consistency between the typical length of edges and dependence of the life time $\tau$ of vertices with large degree on $p_H$ in the form $\tau \sim 1/p_H^2$.

The wide spread structure in the one-dimensional lattice and long life time of vertices of large degree are expected to be supported by a large traffic capacity of the subgraph. The large traffic capacity of the subgraph enables the walker to visit every vertex in the subgraph regardless of the geographical location of vertices in the one-dimensional lattice. Large $C(k)$ regardless of $k$ is an indicator of such a large traffic capacity of the subgraph. As a result, the network can possess not only the broad-type degree distribution but the small-world property. Note that, however, this structure provides frequent visits of the walker only to vertices with created edges. In the region where the subgraph is spread, there are many vertices with no created edges in the lattice.

It should be noted that one-dimensional lattices are a very special case. According to previous works [10], similar models on two-dimensional and three-dimensional lattice yield different behaviors on the traffic capacity of the subgraph even when $p_H = 0$. Therefore, for these cases, the traffic capacity of the subgraph and life time of vertices with large degree are different from that for one-dimensional cases. Consequently, it should be difficult to keep long edges for a long time in two- and three-dimensional cases, because vertices far from other vertices will lose their edges rapidly.

It should also be noted that the random transports modeled by a random walker is extreme in that vertices are only waiting for visits by the transports. It is obvious that transports (or transmission of information) in networks can be modeled in various ways. The development of stochastic models of networks considering such transports should thus be an interesting subject to study.

References:


