# Autocorrelation function for a noisy fractional oscillator

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*Abstract:* The temporal behavior of the autocorrelation function for the output signal of a fractional oscillator with fluctuating eigenfrequency subjected to a periodic force is considered. The influence of a fluctuating environment is modeled by a multiplicative white noise and by an additive noise with a zero mean. The viscoelastic type friction kernel with memory is assumed as a power-law function of time. On the basis of exact formulas it is demonstrated that the temporal behavior of the normalized autocorrelation function is qualitatively different for the cases of internal and external additive noise. This provides a simple criterion enabling one to analyse the output signal for alternative information about the physical nature of the additive noise.

*Key–Words:* Fractional oscillator, autocorrelation function, internal noise, external noise, multiplicative noise, viscoelastic type friction

## 1 Introduction

Noise-induced phenomena in nonequilibrium systems present a fascinating subject of investigation since environmental randomness may cause unexpected behaviors of the system [1, 2]. Diffusion is one of the fundamental mechanisms for non-equilibrium transport phenomena in physical systems. Normal diffusion is characterized by a mean-square displacement that is asymptotically linear in time and is well described in the theory of Brownian motion as a Gaussian process that is local in both space and time. However, the Brownian motion theory cannot account for anomalous diffusion processes, in which the mean-square displacement is not proportional to time. Examples of such systems are supercooled liquids, glasses, colloidal suspensions, dense polymer solutions [3, 4], viscoelastic media [5], and amorphous semiconductors [6]. Even anomalous diffusive dynamics of atoms in biological macromolecules and intrinsic conformotional dynamics of proteins can be subdiffusive [7,8]. There are several approaches to describe anomalous diffusion processes, where the dynamical origin of the phenomenon is considered as a nonlocality, either in space or time [9]. One of the objects of special attention in this context is the noise-driven fractional oscillator. The dynamical equation for such an oscillator is obtained by replacing the usual friction term in the dynamical equation for a harmonic oscillator by a generalized friction term with a power-law-type memory [7, 9–12].

Although the behavior of the fractional oscillator with an additive noise has been investigated in some detail [10, 12], it seems that analysis of the potential consequences of interplay between eigenfrequency fluctuations and memory effects is rather missing in literature. This is quite surprising in view of the fact that the importance of multiplicative fluctuations and viscoelasticity for biological systems, e.g. living cells, has been well recognized [13, 14]. Thus motivated, the authors of [11, 15] have recently considered a fractional oscillator with fluctuating eigenfrequency subjected to an external periodic force and an additive noise. These models demonstrate that an interplay of noises and memory can generate a variety of cooperative effects, such as memory-enhanced energetic stability, stochastic resonance versus noise parameters, as well as friction-induced resonance. However, in these works the dependence of the autocorrelation function of the oscillator displacement on the lag-time has not been investigated.

As from an experimental point of view, information about the dynamics of the observed subdiffusive system can be extracted from the normalized autocorrelation function and from the mean-square displacement [7, 8, 12], we investigate the behavior of the autocorrelation function of the output signal of the fractional oscillator with multiplicative white noise subjected to an external periodic force and an additive driving noise (i.e. a model similar to the one presented in [15]). The main purpose of this paper is to demonstrate, partially on the basis of the exact expressions of output characteristics found in [15], that the dependence of the normalized autocorrelation function on the lag-time depends crucially on the physical nature of the additive driving noise; i.e. the results are qualitatively different for internal and external noises. Moreover, we aim at providing a simple criterion that permits to distinguish between these two types of output processes and also to verify the physical nature of the additive noise.

## 2 Model

As a model for an oscillatory system with memory, strongly coupled with a noisy environment, we consider a fractional oscillator with a fluctuating eigenfrequency [15]

$$\ddot{X} + \frac{\gamma}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{X}(t')\,dt'}{(t-t')^{\alpha}} + [\omega^2 + Z(t)]X = \xi(t) + A_0 \sin \Omega t, \quad (1)$$

where $\dot{X} \equiv dX/dt$, $X(t)$ is the oscillator displacement, $\gamma$ is a friction constant , $\Gamma(y)$ is the Gamma function, $A_0$ and $\Omega$ are the amplitude and the frequency of the harmonic driving force, respectively, and the parameter $\alpha$, $0 < \alpha < 1$ denotes the memory exponent. Fluctuations of the eigenfrequency $\omega$ are expressed as Gaussian white noise $Z(t)$ with a zero mean and a delta-correlated correlation function:

$$\langle Z(t) \rangle = 0, \; \left\langle Z(t)Z(t') \right\rangle = 2D\delta(t-t'), \quad (2)$$

where $D$ is the noise intensity. The zero-centered random force $\xi(t)$ with a stationary correlation function

$$\left\langle \xi(t)\xi(t') \right\rangle = \frac{k_B T \gamma}{\Gamma(1-\alpha)|t-t'|^{\alpha}} + 2D_1\delta(t-t'), \quad \langle \xi(t) \rangle = 0 \quad (3)$$

is assumed as statistically independent from the noise $Z(t)$. If $D_1 = 0$, the driving noise $\xi(t)$ can be regarded as an internal noise, in which case its stationary correlation function satisfies Kubo's second fluctuation-dissipation theorem, where $T$ is the absolute temperature of the heat bath, and $k_B$ is the Boltzmann constant. In the case of $T = 0$ the driving white noise $\xi(t)$ with an intensity $D_1$ and the dissipation have different origins and $\xi(t)$ will be referred to as "external noise".

From Eq. (2) it follows that fluctuations of the frequency $\omega$ do not affect the first moment $\langle X(t) \rangle$ of the oscillator displacement, and $\langle X(t) \rangle$ remains equal to the noise-free solution. By means of the Laplace transformation to Eq. (1) one can easily obtain that in the long-time limit, $t \to \infty$, the memory about the initial conditions will vanish and the first moment $\langle X(t) \rangle$ is given by

$$\langle X(t) \rangle_{as} \equiv \langle X(t) \rangle_{|t\to\infty} = A \sin(\Omega t + \varphi). \quad (4)$$

For the amplitude $A$ of the output signal we obtain

$$\frac{A^2}{A_0^2} = \left\{ \left[ \omega^2 - \Omega^2 + \gamma \Omega^{\alpha} \cos\left(\frac{\alpha\pi}{2}\right) \right]^2 + \gamma^2 \Omega^{2\alpha} \sin^2\left(\frac{\alpha\pi}{2}\right) \right\}^{-1}. \quad (5)$$

Here we note that the formulas (4) and (5) for a deterministic fractional oscillator have been previously represented in [10]. From an experimental point of view an important output characteristic is the time-homogeneous part of the variance of the oscillator displacement $X$ expressed as

$$\sigma^2 := \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \left[ \left\langle X^2(t) \right\rangle_{as} - \langle X(t) \rangle_{as}^2 \right] dt, \quad (6)$$

with $\mathcal{T} = 2\pi/\Omega$. Alternative information about the experimentally observed stochastic behavior of the output signal can be extracted from the one-time normalized autocorrelation function [8, 12], defined as

$$K(\tau) = \frac{1}{\mathcal{T}\sigma^2} \int_0^{\mathcal{T}} \left[ \langle X(t+\tau)X(t) \rangle_{as} - \langle X(t+\tau) \rangle_{as} \langle X(t) \rangle_{as} \right] dt. \quad (7)$$

Using the Laplace transformation technique and the results of Ref. [15], one gets

$$K(\tau) = 2D_{cr}\Psi(\tau)\left(1 - \frac{k_B T}{\omega^2\sigma^2}\right) + \frac{k_B T}{\omega^2\sigma^2} F(\tau), \quad (8)$$

where

$$D_{cr} = \frac{1}{2\Psi(0)}. \quad (9)$$

The functions $\Psi(\tau)$ and $F(\tau)$ are given by

$$\Psi(\tau) = \frac{e^{-\beta\tau}}{u^2+v^2} \left\{ \frac{1}{\beta}\cos(\omega^*\tau) - \frac{1}{\beta^2+\omega^{*2}} \times [\beta\cos(\omega^*\tau + 2\theta) - \omega^*\sin(\omega^*\tau + 2\theta)] \right\} + \frac{\gamma\sin(\alpha\pi)}{\pi} \int_0^{\infty} \frac{dr\, r^{\alpha}}{B(r)} \left\{ \frac{e^{-r\tau}}{r^2 + \gamma r^{\alpha} + \omega^2} + \frac{2e^{-\beta\tau}}{\sqrt{u^2+v^2}\left[(r+\beta)^2 + \omega^{*2}\right]} \times [\omega^*\cos(\omega^*\tau + \theta) + (r+\beta)\sin(\omega^*\tau + \theta)] \right\}, \quad (10)$$

$$F(\tau) = \frac{2\omega^2 e^{-\beta\tau}}{\sqrt{u^2+v^2}\left(\beta^2+\omega^{*2}\right)} \times \left[\omega^* \cos(\omega^*\tau+\theta) + \beta \sin(\omega^*\tau+\theta)\right] + \frac{\omega^2\gamma}{\pi}\sin(\alpha\pi)\int_0^\infty \frac{dr\, e^{-r\tau}}{r^{1-\alpha}B(r)}, \tag{11}$$

where $s_{1,2} = -\beta \pm i\omega^*$, $(\beta > 0, \omega^* > 0)$, are the pair of conjugate complex zeros of the equation

$$G(s) \equiv s^2 + \gamma s^\alpha + \omega^2 = 0; \tag{12}$$

here, $G(s)$ is defined by the principal branch of $s^\alpha$. The quantities $u$, $v$, $\theta$, and $B(r)$ are determined by

$$u = -2\beta + \frac{\alpha\gamma\cos\left\{(1-\alpha)[\arctan(-\omega^*/\beta)+\pi]\right\}}{\left(\beta^2+\omega^{*2}\right)^{(1-\alpha)/2}},$$

$$v = 2\omega^* - \frac{\alpha\gamma\sin\left\{(1-\alpha)[\arctan(-\omega^*/\beta)+\pi]\right\}}{\left(\beta^2+\omega^{*2}\right)^{(1-\alpha)/2}},$$

$$\theta = \arctan\left(\frac{u}{v}\right),$$

$$B(r) = \left[r^2 + \gamma r^\alpha \cos(\pi\alpha) + \omega^2\right]^2 + \gamma^2 r^{2\alpha}\sin^2(\pi\alpha). \tag{13}$$

It should be emphasized that the functions $\Psi(\tau)$ and $F(\tau)$ are independent of the driving force parameters $A_0$ and $\Omega$ as well as of the noise intensities $D$, $D_1$, and $T$.

For the time-homogeneous part of the variance $\sigma^2$ we have

$$\sigma^2 = \frac{1}{D_{cr} - D}\left[\frac{DA^2}{2} + \frac{D_{cr}k_BT}{\omega^2} + D_1\right]. \tag{14}$$

From Eq. (14) we can see that the stationary regime is possible only if $D < D_{cr}$. As the intensity of the multiplicative noise $D$ tends to the critical value $D_{cr}$ the variance $\sigma^2$ increases to infinity. This is an indication that for $D > D_{cr}$ energetic instability appears, manifested in an unlimited increase of second-order moments of the output of the oscillator with time, while the mean value of the oscillator displacement remains finite [16, 17].

## 3 Results

In Fig. 1 we depict, on two panels, the behavior of the critical noise intensity $D_{cr}$ and the variance $\sigma^2$ by variations of the memory exponent $\alpha$. Panel (a) shows a typical resonance-like behavior of $D_{cr}(\alpha)$. As a rule, the maximal value of $D_{cr}/\gamma$ increases as the value of the friction coefficient $\gamma$ increases, while the positions of the maxima are monotonically shifted to a lower $\alpha$ as $\gamma$ rises. In the case considered in panel (b) the intensity of the multiplicative noise is in the interval $\omega^2\gamma < D < D_{cr\ \max}$, where $D_{cr\ \max}$ is the maximal value of $D_{cr}(\alpha)$ by variations of $\alpha$. In this case the variance $\sigma^2$ decreases rapidly from infinity at $\alpha_1$, $D_{cr}(\alpha_1) = D$, to a minimum and next increases to infinity at $\alpha_2$, $D_{cr}(\alpha_2) = D$. Thus the fractional oscillator is energetically stable only in the interval $\alpha_1 < \alpha < \alpha_2$.

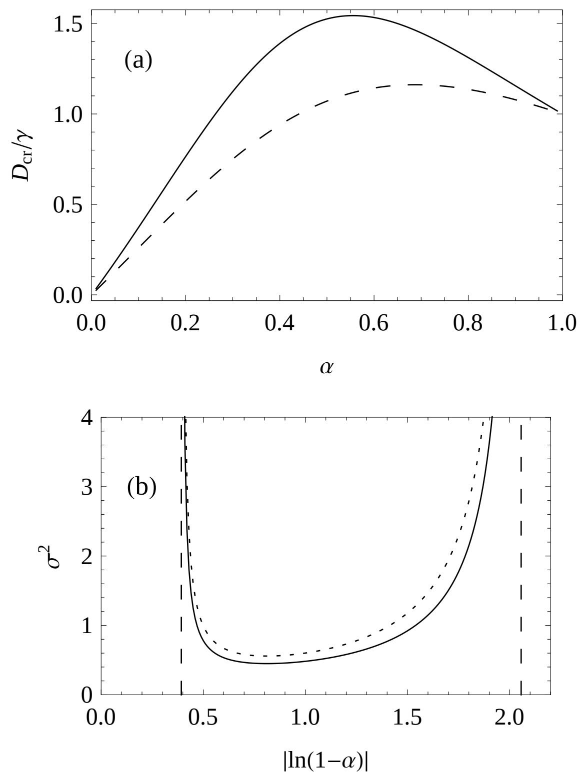


Fig. 1. Critical noise intensity $D_{cr}$ and variance $\sigma^2$ as functions of the memory exponent $\alpha$, obtained from Eqs. (5), (9), (10), and (14) for $A_0 = \omega = 1$. Panel (a): solid line, $\gamma = 4$; dashed line $\gamma = 1.62$. Panel (b): $\gamma = 4$, $D = 4.8$, $D_1 = 0$, and $k_BT/\omega^2 = 0.1$. Solid line, $\Omega = 10$; dotted line, $\Omega = 1$. The dashed lines depict the positions of the critical memory exponents $\alpha_1 \approx 0.325$ and $\alpha_2 \approx 0.872$ between which the oscillator is energetically stable.

From Fig. 1(b) one can see that the values of the variance $\sigma^2$ depend on the frequency $\Omega$ of the harmonic driving force. As in the case without multiplicative noise such a dependence is absent (cf. also Eq. (14)), this phenomenon offers a simple experimental possibility to verify the existence of a multiplicative noise in oscillatory systems described by model (1).

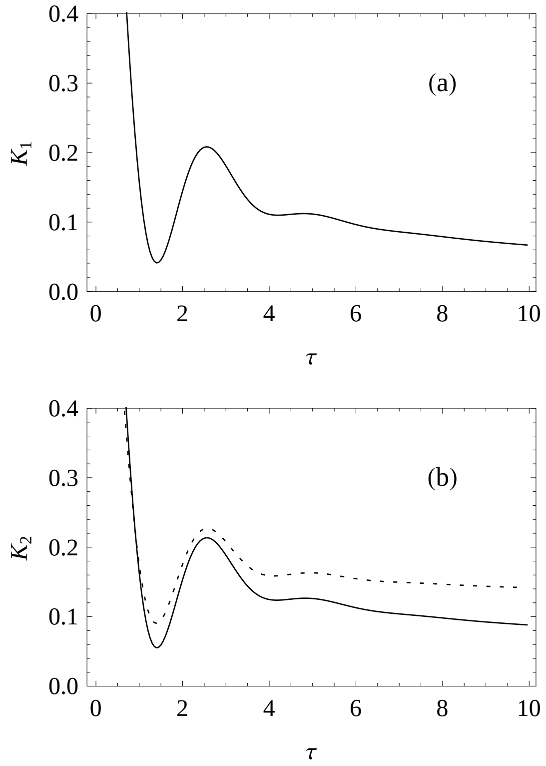


Fig. 2. Normalized autocorrelation function $K(\tau)$ vs. time lag $\tau$ computed from Eq. (8) for $\omega = 1$, $\gamma = 4$, and $\alpha = 0.45$. Panel (a): an external noise (i.e., $T = 0$). Panel (b): internal noise (i.e., $D_1 = 0$). The system parameter values: $A_0 = 1$, $D = 4.8$, and $k_B T/\omega^2 = 0.01$. Solid line, $\Omega = 1$; dotted line, $\Omega = 10$.

Now we will consider the behavior of the normalized autocorrelation function $K(\tau)$ (see Eqs. (7) and (8)). In contrast to the results for variance $\sigma^2$, here the role of the additive driving noise $\xi(t)$ is crucial. If the driving noise is external (i.e. $T = 0$), the typical form of the graph $K(\tau)$ is represented in Fig. 2(a). Note that the exact solution exhibits exponentially damped oscillations around a curve which for large $\tau$ decays totally monotonically like a power-law. Consequently, the normalized autocorrelation function has only a finite number of zeros and decays, in the end, as $\tau^{-(1+\alpha)}$ to the $\tau$-axis. Note that in this case the function $K(\tau)$ is independent of the driving force parameters $A_0$ and $\Omega$.

In the case of an internal noise $\xi(t)$ (i.e., $D_1 = 0$) the picture of the dependence of $K(\tau)$ on $\tau$ is different (see Fig. 2(b)). First, the autocorrelation function $K(\tau)$ relaxes asymptotically like $\tau^{-\alpha}$. This is in sharp contrast with the result for the external noise that exhibits a much faster decay. Second, the most important difference is the dependence of $K(\tau)$ on the amplitude $A$ of the output signal (cf. Eqs. (8), (14), and (5)). Thus, in the case of internal noise the exact form of $K(\tau)$ is sensitive to the values of the frequency $\Omega$ of the external harmonic driving force.

In conclusion, on the basis of the autocorrelation function of the fractional oscillator with fluctuating frequency subjected to an external periodic force and an additive noise we have found convenient criteria that enable us to distinguish the presence of multiplicative and internal noises in such systems. The advantage of these criteria is that the control parameter is the frequency (or the amplitude) of the external periodic force, which can be easily varied in possible experiments as well as potential technological applications, e.g., in electric oscillator devices with circuit elements of a fractional type (i.e. tree or chain fractances) [18].

We believe that the results discussed here may be also useful in modeling oscillations of population sizes in some ecosystems, as the model considered includes memory effects that are essential in natural predator–prey communities [19].

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*References:*

[1] P. Reimann, Brownian Motors: Noisy Transport Far from Equilibrium, *Physics Reports*, 2002, Vol. 361, No. 2–4, pp. 57 – 265.

[2] R. Mankin, E. Soika, A. Sauga and A. Ainsaar, Thermally Enhanced Stability in Fluctuating Bistable Potentials, *Phys. Rev. E*, 2008, Vol. 77, No. 5, pp. 051113 (1)–(9).

[3] W. Götze and L. Sjögren, Relaxation Processes in Supercooled Liquids, *Reports on Progress in Physics*, 1992, Vol. 55, No. 3, pp. 241–376.

[4] T. Carlsson, L. Sjögren, E. Mamontov and K. Psiuk-Maksymowicz, Irreducible Memory Function and Slow Dynamics in Disordered Systems, *Phys. Rev. E*, 2007, Vol. 75, No. 3, pp. 031109 (1)–(8).

[5] I. Golding and E. C. Cox, Physical Nature of Bacterial Cytoplasm, *Phys. Rev. Lett.*, 2006, Vol. 96, No. 9, pp. 098102 (1)–(4).

[6] I. Goychuk, Anomalous Relaxation and Dielectric Response, *Phys. Rev. E*, 2007, Vol. 76, No. 4, pp. 040102 (R) (1)–(4).

[7] S. C. Kou and X. S. Xie, Generalized Langevin Equation with Fractional Gaussian Noise: Subdiffusion within a Single Protein Molecule, *Phys.*

*Rev. Lett.*, 2004, Vol. 93, No. 18, pp. 180603 (1)–(4).

[8] W. Min, G. Luo, B. J. Cherayil, S. C. Kou and X. S. Xie, Observation of a Power-Law Memory Kernel for Fluctuations within a Single Protein Molecule, *Phys. Rev. Lett.*, 2005, Vol. 94, No. 19, pp. 198302 (1)–(4).

[9] E. Lutz, Fractional Langevin Equation, *Phys. Rev. E*, 2001, Vol. 64, No. 5, pp. 051106 (1)–(4).

[10] S. Burov and E. Barkai, Fractional Langevin Equation: Overdamped, Underdamped, and Critical Behaviors, *Phys. Rev. E*, 2008, Vol. 78, No. 3, pp. 031112 (1)–(18).

[11] E. Soika, R. Mankin and A. Ainsaar, Resonant Behavior of a Fractional Oscillator with Fluctuating Frequency, *Phys. Rev. E*, 2010, Vol. 81, No. 1, pp. 011141 (1)–(11) .

[12] M. A. Despósito and A. D. Viñales, Subdiffusive Behavior in a Trapping Potential: Mean Square Displacement and Velocity Autocorrelation Function, *Phys. Rev. E*, 2009, Vol. 80, No. 2, pp. 021111 (1)–(7).

[13] R. Astumian and M. Bier, Mechanochemical Coupling of the Motion of Molecular Motors to ATP Hydrolysis, *Biophys. J.*, 1996, Vol. 70, No. 2, pp. 637–653.

[14] I. M. Tolić-Nørrelykke, E.-L. Munteanu, G. Thon, L. Oddershede and K. Berg-Sørensen, Anomalous Diffusion in Living Yeast Cells, *Phys. Rev. Lett.*, 2004, Vol. 93, No. 7, pp. 078102 (1)–(4).

[15] R. Mankin and A. Rekker, Memory-enhanced Energetic Stability for a Fractional Oscillator with Fluctuating Frequency, *Phys. Rev. E*, 2010, Vol. 81, No. 4, pp. 041122 (1)–(10).

[16] K. Laas, R. Mankin and A. Rekker, Constructive Influence of Noise Flatness and Friction on the Resonant Behavior of a Harmonic Oscillator with Fluctuating Frequency, *Phys. Rev. E*, 2009, Vol. 79, No. 5, pp. 051128 (1)–(7).

[17] R. Mankin, K. Laas, T. Laas and E. Reiter, Stochastic Multiresonance and Correlation-time-controlled Stability for a Harmonic Oscillator with Fluctuating Frequency, *Phys. Rev. E*, 2008, Vol. 78, No. 3, pp. 031120 (1)–(11).

[18] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[19] R. Mankin, T. Laas, E. Soika and A. Ainsaar, Noise-Controlled Slow–Fast Oscillations in Predator–Prey Models with the Beddington Functional Response, *Eur. Phys. J. B*, 2007, Vol. 59, No. 2, pp. 259–269.