Investigation of Tree Diameter and Volume Increments using Stochastic Differential Equations

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Abstract: - Diameter and stem volume current and mean annual increments for Scots pine are described using a stochastic differential equation methodology, where stochastic variability are defined by a one-dimensional standard Wiener process. The Gompertz stochastic homogeneous growth model is applied to analyze the trend of tree diameter. A generalized form of mean diameter is introduced which implicitly incorporates an age-dependent transition probability density function of tree diameter. The diameter analysis is based on measurements in Scots pine (Pinus sylvestris) stands at Lithuania. The results are implemented in the symbolic computational language MAPLE 11.

Key-Words: - Stochastic growth; Gompertz; Transition probability density function; Diameter; Volume; Current annual increment; Mean annual increment.

1 Introduction

Evolution of a forest ecosystem is a multiple process whose important stand level characteristic is biomass accumulation. Forest site productivity is the production that can be realized at a certain site with a specified management regime. Assessment of the production capacity is necessary for the scientifically grounded management. Forest productivity tends to decrease with increasing stand age. Diameter and volume increment models are highly useful for estimation of the biomass production. The rate of increments can be ascertained either by systematic measurements of standing trees or by a stem analysis of felled trees. Forest managers strive to know when the mean annual increment of a stand (MAI) reaches a maximum, and how MAI changes with age.

Basically, stand or tree growth models are formed by a mathematical equation which represents volume evolution over age or other predictor variables. Regression equations do not offer a great deal to commend themselves, and the differential equations should be considered since it is easy understand the increments of volume and diameter. Differential equations approach is based on assumptions which give an explanation of the mechanism of growth process. The forest growth is usually modelled with a logistic model [2]. The parameters of logistic diameter models are not directly measurable but they are estimated from the observed data set. Stand as a community of trees is the main component of the forest. Stand consists of trees with different diameters and heights. These differences depend on a lot of unsearchable genetic and environmental factors, therefore it leads to consideration that diameter of a tree is a random variable which depends on the age. Stochastic diameter growth models allow us to reduce the unexplained variability of a diameter and to implement the randomness phenomenon. Over the years an extensive amount of research has been devoted to the randomness of stand growth since the pioneer work of Suzuki [14] and the successive works of Tanaka [15], Rupšys [6], Rupšys and Petrauskas [9]-[13]. Diffusion models defined by means of stochastic differential equations are popular tools for modelling continuous time phenomena in many scientific disciplines such as economics [1], biology [7], [8], environment [4].

In this work we motivate the use of stochastic differential equations in prediction of the mean and current annual diameter and volume increments. The methodology is to consider a univariate distribution as arising from univariate diameter growth stochastic dynamical system. The system fluctuations, generally infiltrated from outside, are defined by a one-dimensional standard Wiener process.

2 Materials and methods
2.1 Growth data

The diameter analysis is based on measurements in pine (Pinus sylvestris) stands at Lithuania. The data were provided by the Lithuanian National Forest Inventory. We included full calliperings of permanent sample plots. Over 20 years period (1976-1996) in the even-aged uncut sample plots were re-measured at the most five times. The following variables were measured: age (t), number of trees per hectare, diameter at breast height (d), trees position (coordinates x, y), height (h). Approximately 20% of the sample trees were randomly selected for the height measurement. The measurements have been conducted in 30 occasions of permanent treatment plots and the initial planting densities are unknown. The age of stands ranges from 12 to 103 years. The diameter at breast height varies from 2.2 to 51.5 cm. Diameter was measured to the nearest 0.1 cm. For model estimation observations on 900 pines were used. The observed data sets of study plots are summarized in Table 1.

Table 1. Summary statistics of the data set

<table>
<thead>
<tr>
<th>Data</th>
<th>Number of trees</th>
<th>Min</th>
<th>Max</th>
<th>Mean</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>D(cm)</td>
<td>900</td>
<td>4.3</td>
<td>51.0</td>
<td>24.17</td>
<td>9.94</td>
</tr>
<tr>
<td>H(m)</td>
<td>900</td>
<td>3.2</td>
<td>33.5</td>
<td>19.98</td>
<td>6.22</td>
</tr>
<tr>
<td>A(yr)</td>
<td>12</td>
<td>103.0</td>
<td>53.74</td>
<td>19.60</td>
<td></td>
</tr>
</tbody>
</table>

2.2 Diameter, volume and annual increment models

Let study the dynamic behavior of tree diameter (diameter at breast height) and its relationship with diameter distribution law. For determination of diameter growth we suppose that dynamic of tree diameter is expressed in terms of the Gompertz shape stochastic differential equation with multiplicative noise. The Gompertz deterministic model is a classical continuous model useful in describing population dynamic. It was introduced by Benjamin Gompertz [3] to analyze population dynamic and to determinate life contingencies. We consider a univariate Gompertz model useful in describing population dynamic. It was introduced by Benjamin Gompertz [3] to analyze population dynamic and to determinate life contingencies. We consider a univariate Gompertz growth process facing stochastic fluctuations in the following Itô [5] stochastic differential equation

\[ dD(t) = \left[ \alpha D(t) - \beta D(t) \ln(D(t)) \right] dt + \sigma D(t) dW(t) \]  

where \( \alpha, \beta, \sigma > 0 \) are unknown real parameters to be estimated, \( D(t) \) is a breast height diameter (in the sequel – diameter) at the age \( t, d_0 \geq 0, \{ W(t); t \in [t_0, T] \} \) is a one-dimensional Wiener process and the differential \( dD(t) \) is to be understood in the sense of Itô [5]. In the sequel, the density \( p(d,t) \) of \( D(t) \) at \( t \) given \( D(t_0) = d_0 \) at \( t = t_0 \) is denominated as transition probability density function or conditional probability density function. Using Itô’s formula for the age dependent transformation \( X(t) = e^{\theta t} \ln(D(t)) \) we obtain the explicit solution of original stochastic differential equation (1) in the following form

\[ D(t) = \exp \left[ e^{\beta(t-t_0)} \ln(d_0) + \int_{t_0}^{t} \left( \alpha - \frac{\sigma^2}{2} \right) \frac{e^{-\beta(t-t_0)}}{2} dy \right] \times \exp \left[ e^{\beta(t-t_0)} dW(t) \right] \]  

which is a continuous homogeneous Markov (Gaussian) process with transition probability density function [4]

\[ p(d,t) = \frac{1}{\sqrt{2\pi \lambda^2(t)}} e^{-\frac{(d - \mu(t))^2}{2\lambda^2(t)}} \]  

where

\[ \mu(t) = e^{-\beta(t-t_0)} \ln d_0 + \frac{\alpha - \sigma^2}{2 \beta} \left( 1 - e^{-\beta(t-t_0)} \right) \]  

\[ \lambda^2(t) = \frac{\sigma^2}{2\beta} \left( 1 - e^{-2\beta(t-t_0)} \right) \]  

Therefore, the random variable \( D(t) / D(t_0) = d_0 \) has a one-dimensional lognormal distribution \( \Lambda(\mu(t), \lambda^2(t)) \).

Traditionally, the mean tree volume \( \bar{V} \) is estimated as an average of sample tree volumes

\[ \bar{V} = \frac{1}{n} \sum_{i=1}^{n} V(d_i, h_i) \] , where \( V(d,h) \) is an individual tree volume equation on diameter and height. Much greater accurateness is obtained by substituting (smoothing) a density function \( p(d) \) of tree diameter and integrating by all diameters \( d > 0 \). If the tree volume regression function \( V(d,h) \) and the density \( p(d) \) function additionally depend on age \( V(d,h,t) \), \( p(d,t) \), then the mean tree volume \( \bar{V} \) and standard deviation \( s_T \) can be rewritten as follows.
\[ V(t) = \int_{d>0} V(d, h(t), t) p(d, t) \cdot dd \quad (6) \]

\[ s_p = s_p(t) = \sqrt{\int_{d>0} (d - V(t))^2 \cdot p(d, t) \cdot dd} \quad (7) \]

The integral form (6) describes the mean tree volume as an explicit function of the age and can provide additional information about volume dynamic. The commonly used functional dependence for volume \( V(d, h(t), t) \) calculation takes the form of the power function \( V = \exp(\delta_0) d^\delta_1 h^\delta_2 \) and parameters \( \delta_0, \delta_1, \delta_2 \) to be estimated. The estimators and their standard deviations (in parenthesis) are: \( \hat{\delta}_0 = -9.5282 \) (0.0127), \( \hat{\delta}_1 = 1.9183 \) (0.0072), \( \hat{\delta}_2 = 0.8907 \) (0.0104), \( \hat{\delta}_3 = -0.0268 \) (0.0042) [11]. The height-diameter relationship takes the Chapman-Richards form \( h(d) = \beta_1(1 - \exp(-\beta_2 d))^{\beta_3} \), where \( \hat{\beta}_1 = 35.1740 \) (1.2386), \( \hat{\beta}_2 = 0.0030 \) (0.0039), \( \hat{\beta}_3 = 12.3991 \) (0.0518). Equations for predicting mean diameter \( \bar{d}(t) \) and standard deviation \( s_d(t) \) of diameter are expressed in the general forms

\[ \bar{d}(t) = \int_{d>0} d \cdot p(d, t) \cdot dd \quad (8) \]

\[ s_d(t) = \sqrt{\int_{d>0} (d - \bar{d}(t))^2 \cdot p(d, t) \cdot dd} \quad (9) \]

Using probabilistic mean diameter and volume growth models (Eq. 8, 6) we can define the current (the mean) annual diameter and volume increments \( z_d(t) \), \( z_v(t) \) of an average tree in the following form

\[ z_d(t) = \frac{d}{dt}(\bar{d}(t)) \quad (\bar{z}_d(t) = \frac{1}{t-t_0} \int_{t_0}^t z_d(s) \, ds) \quad (10) \]

\[ z_v(t) = \frac{d}{dt}(\bar{V}(t)) \quad (\bar{z}_v(t) = \frac{1}{t-t_0} \int_{t_0}^t z_v(s) \, ds) \quad (11) \]

### 3 Results and discussion

Using the observed data set summarized in Table 1 we calculated the parameter estimations of the stochastic diameter growth model defined by Equation (1). The estimate of parameters of the stochastic differential equation (1) is compound of the Least Squares Estimate (LSE) of the deterministic part (drift) and the MLE of the diffusion coefficient \( \sigma \). Hence, we estimate \( \sigma \) by keeping fixed the previously obtained drift parameter estimates \( \hat{\alpha}, \hat{\beta} \). The estimators (standard errors) are: \( \hat{\alpha} = 0.1120 \) (0.0047), \( \hat{\beta} = 0.0264 \) (0.0076), \( \hat{\sigma} = 0.15271 \) (0.0148). Figure 1 shows the transition probability density function of tree diameter defined by Equation (3). This density function indicates that the transition probability density function is steeper for the young stands and less steep for the mature stands.

![Fig. 1. Transition probability density function of tree diameter.](image)

Figure 2 illustrates the mean and standard deviation trajectories of the stochastic process \( D(t), t \in [t_0, T] \) of tree diameter. Both curves monotonically evolve to the steady state values. Figure 3 illustrates the mean and standard deviation trajectories of tree volume. Figures 2, 3, show that the standard deviations of diameter and volume increase steadily with the age.
Fig. 2. Plot of the mean and standard deviation dynamic of tree diameter with the parameterization data sets: mean (continuous curve), mean ± standard deviation (non-continuous curve).

Fig. 3. Plot of the mean and standard deviation dynamic of stem volume: mean (continuous curve), mean ± standard deviation (non-continuous curve).

Relationships between the current annual diameter (Eq. 10) and volume (Eq. 11) increments against the age of a tree are illustrated in Figures 4, 5. As we see in Figures 4, 5 the age and the mean diameter of a tree exert a strong influence on current and mean annual diameter and volume increments. The effect of the age on current annual diameter and volume increments becomes negligible above 150 yr age. From Figure 4a we see that the culmination of mean annual increment of diameter is reached even later than that of current annual increment. The peak in current and mean annual diameter increments occurred at 43 and 79 years of age, respectively (Figure 4a) or at 16.9 and 31.3 cm of mean diameter, respectively (Figure 4b). The current annual diameter increment becomes equal to mean annual diameter increment at 62 years of age (at 25.1 cm of mean diameter).

Figures 5a and 5b illustrate the current and mean annual volume increments against the age and the mean diameter. From Figures 5a and 5b we see that the culmination of volume increment is reached even later than that of diameter increment. The peak in current and mean annual diameter increments occurred at 43 and 79 years of age (at 16.9 and 31.3 cm of mean diameter), respectively (Figures 4a and 4b), and current and mean annual volume increments peaked at 73 and 115 years of age (at 29.3 and 39.8 cm of mean diameter), respectively (Figures 5a and 5b). The current annual volume increment becomes equal to mean annual volume increment at 115 years of age (at 39.8 cm of mean diameter). If an objective of forest management is to maximize the produced stem volume, the trees should be retained until they attain their maximum mean annual volume increment at the age 115 years (Figure 5b) or at the mean diameter 39.8 cm (Figure 5b).
Fig. 4. Relationship between the current and mean annual diameter increments of an average tree: current annual diameter increment (black, solid line), mean annual diameter increment (blue, dash line), annual diameter increments against the age (a), annual diameter increments against the mean diameter (b).

Fig. 5. Relationship between the current and mean annual volume increments of an average tree: current annual volume increment (black, solid line), mean annual volume increment (blue, dash line), annual volume increments against the age (a), annual volume increments against the mean diameter (b).

4 CONCLUSIONS FUTURE WORK
Given the importance of stochastic analysis in modern forestry, we consider the case where the governing tree diameter dynamic is defined by an elementary stochastic differential equation. A theoretical prerequisite of our presented approach was the stochastic Gompertz diameter growth law driven by one-dimensional standard Wiener process. The results obtained here have shown that it is possible to relate nonlinear stochastic diameter growth model and volume increment model.

The approach used in this study is purely empirical, and the parameters in the transition probability density function may therefore be valid only for the tested conditions.

This study indicates the usefulness of an approach that combines the Gompertz curve, a stochastic differential equation, and probabilistic characteristics of the diameter growth. Thus, the proposed method could be continued in terms of properly modifying the drift and diffusion functions of the stochastic diameter growth process and choosing predictor variables. The accuracy of the age-height-dependent diameter distribution (Eq. 5) depends on the amount of information available from the stand. Our methodology extends some way to inclusion of the basal-area or/and density of a stand as an exogenous factor or as an independent variable.
References:


