General q-exponential Model for Tree Height, Volume and Stem Profile

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Abstract –Two new q-exponential height and q-exponential segmented taper models were developed using q-exponential function. Four height models, six stem volume models, and three taper models were compared to the observed values of height, stem volume and diameter outside bark. The performance statistics of the height, volume and taper equations included four fit statistics: mean bias, mean absolute bias, mean percentage of absolute bias and a coefficient of determination. Data used in this study came from stem analysis on 1925 pine trees. Results show that the q-exponential models were superior to the other used models in predicting height, volume and diameter outside bark. The results are implemented in the symbolic computational language MAPLE.

Key-Words: - Stem taper model, q-exponential function, Height, Volume, Diameter.

1 Introduction

Forest productivity is an intricate biological concept that cannot be described directly with mathematical expressions. Complex inventory of forests requires accurate estimates of the diameter, length and volume of each tree. For some decades back, modelling of individual tree height, taper and volume as a function of individual tree attributes has received much attention in forest science. Traditionally, the relationship between height, taper and volume and predictor variables has been modelled based on simple nonlinear regression. As was pointed out by Vanclay [16], the allometric models assist forest researchers and managers in many ways. Using a growth model, we can explore management alternatives and get well-founded insights into stand dynamics. Height-diameter relationships are used to estimate the heights of trees measured only for diameter at breast height. In most sample plot systems diameter at breast height is conventionally measured for all trees sampled, but height is measured for only a sub-sample. The height-diameter regression model affects the estimate of individual tree volume. To improve inventory efficiency, the merchantable tree height and both total and merchantable tree stem volume can be estimated from tree taper models [6]. Therefore, taper functions become the primary tool for estimating the volume at any part of the trunk, by means of the mathematical integration of the section area along the tree taper function.

Numerous height-diameter equations have been developed using only diameter at breast height as the predictor variable for estimating total height [1], [2], [7], [14]. Most volume equations have been developed using diameter at breast height and total height as the predictor variables [3], [13], [17, [18].

Taper functions have been used to precisely describe tree form and to estimate stem volume at any part of the trunk. In last decades, various types of taper models have been proposed including segmented taper models [6], variable-exponent taper models [4] and other forms [5].

Recently, the generalizations of the exponential and logarithmic functions have attracted the attention of many researches [15]. In this paper by using the q-exponential function new tree height, volume and stem profile equations are proposed.

The main aim of our contribution is to expand height and taper models by the q-exponential function and compare their performance in predicting height, diameter outside bark at any given height and volume of a tree.

2 Material and methods

2.1 Data

Summary statistics for diameter at breast height (D) and total height (H) of all trees used for fitting and comparing height, volume, taper equations are presented in Table 1.

Table 1. Summary statistics of the data set

Data	Number of trees	Min	Max	Mean	St. Dev.
D (cm)	1925	3.8	62.8	26.5	10.06
H (m)	1925	3.8	35.2	21.7	5.13

Using observed diameters for each tree section between two adjacent diameter measurements was calculated its volume. Volume of the top section is assumed to be as conic in shape and volumes of all other sections is assumed to be as a truncated cone in shape. The Smalian's formula for *i-th* tree is defined as:

$$V_i = \frac{\pi}{40000}\left(\sum_{k=1}^{n_i-2}\frac{\left(d_{ik}^2 + d_{ik+1}^2\right)\cdot L_{ik}}{2} + \frac{d_{in_i-1}^2 \cdot L_{in_i-1}}{3}\right)$$

where d_{ik}^2 is the diameter (cm) for section k of tree i, L_{ik} is the section length (m). Unfortunately, the rate of a tree taper from the base to the tip is not uniform throughout all parts of the stem. The greater the difference between the two adjacent diameter measurements, the less reliable will be the volume obtained using Smalian's formula. The volume of tree stem sections can also be derived using truncated cone formula

$$V_i = \frac{\pi}{40000}\left(\sum_{k=1}^{n_i-2}\frac{\left(d_{ik}^2 + d_{ik+1}^2 + d_{ik}\cdot d_{ik+1}\right)\cdot L_{ik}}{3} + \frac{d_{in_i-1}^2 \cdot L_{in_i-1}}{3}\right) \qquad (1)$$

In this paper the volume defined by Eq. (1) was considered as the observed volume.

2.2 q-exponential function

The one-parameter generalization of exponential function obtained from non-extensive statistical physics is suitable to describe sigmoidal growth phenomena. The q-exponential function appears in the solutions of differential equations. A general growth process defined by differential equation takes the following form [15]:

$$\frac{dy}{dx} = \alpha y - \beta y^q \quad y(0) = \delta$$

which solution is defined by q-exponential function as:

$$y = \left[\delta - \frac{\beta}{\alpha}\left(1 - \exp\left(\left(1-q\right)\alpha x\right)\right)\right]_+^{\frac{1}{1-q}}, \left[a\right]_+ = \begin{cases} a, & if\ a \ge 0, \\ 0, & if\ a < 0 \end{cases}$$

2.3 Height models

In this paper, four alternative models (described below) were used for the height–diameter relationship, namely the three parameter Hill model (Eq. (2)), the four parameter Morgan–Mercer–Flodin model (Eq. (3)), the three parameter Chapman-Richards model (Eq. (4)), and the four parameter q-exponential model (Eq. (5)). These models were chosen because it has been found to have better overall performance than many other height-diameter models [8]. Height-diameter models:

$$H = \frac{\beta_1 D^{\beta_2}}{\beta_3 + D^{\beta_2}} \qquad (2)$$

$$H = \frac{\beta_1 + \beta_3 D^{\beta_4}}{\beta_2 + D^{\beta_4}} \qquad (3)$$

$$H = \beta_1\left(1 - \exp\left(-\beta_2 D\right)\right)^{\beta_3} \qquad (4)$$

$$H = \left[\beta_1 - \frac{\beta_2}{\beta_3}\left(1 - \exp\left(\left(1-\beta_4\right)\beta_3 D\right)\right)\right]_+^{\frac{1}{1-\beta_4}} \qquad (5)$$

2.4 Volume models

The wealth of allometric equations that relate stem volume to diameter at breast height and to tree height has been summarised for European tree species [18]. Three alternative models (described below) were used for the stem volume modelling, namely the three parameter Schumacher model (Eq. (6)), the two parameter Honer model (Eq. (7)), and the four parameter Vuokila model (Eq. (8)).

$$V = \beta_1 D^{\beta_2} H^{\beta_3} \qquad (6)$$

$$V = \frac{D^2}{\beta_1 + \beta_2/H} \qquad (7)$$

$$V = \beta_1 DH + \beta_2 D^2 H + \beta_3 DH^2 + \beta_4 D^3 \qquad (8)$$

2.5 Taper models

The most commonly used models were developed by [6] and [4]. In this study, two known taper models and one new developed model were utilized for evaluation.

Max and Burkhart's [6] segmented polynomial model was written as:

$$\frac{d^2}{D^2} = \beta_1(z-1) + \beta_2(z^2-1) + \beta_3(\alpha_1 - z)^2 I_1(\alpha_1 - z) + \beta_4(\alpha_2 - z)^2 I_2(\alpha_2 - z) \quad (9)$$

where d is the diameter outside bark at any given height h, D is the diameter at breast height outside bark, H is the total tree height from ground to tip, α_1 and α_2 are the segmented join points of tree segments, $\beta_1 - \beta_4$ are the parameters to be estimated from data,

$$z = \frac{h}{H}$$

$$I_i(\alpha_i - z) = \begin{cases} 1 & \text{if } \alpha_i - z \ge 0, \\ 0 & \text{otherwise} \end{cases}, \quad i = 1,2$$

A few different formulations of taper models were proposed by Kozak [4]. In this study, one variable-exponent single continuous function taper model was utilized in the following form [4]:

$$d = \beta_1 D^{\beta_2} X^{\beta_3 + \beta_4 \exp\left(-D/H\right) + \beta_5 D^X + \beta_6 X^{D/H}} \quad (10)$$

where

$$X = \frac{1 - \left(h/H\right)^{1/4}}{1 - (0.01)^{1/4}}$$

$\beta_1 - \beta_6$ are the parameters to be estimated from data.

Using q-exponential function and parabola a segmented taper model was defined in the following form:

$$d = \beta_1 D^{\beta_2} \begin{cases} \beta_3(z-1) + \beta_4(z^2-1), & \text{if } z \ge \alpha_1 \\ \left[\beta_5 - \dfrac{\beta_6}{\beta_7}\left(1 - \exp\left((1-\beta_8)\beta_7 z\right)\right)\right]_+^{\frac{1}{1-\beta_8}} \end{cases} \quad (11)$$

where $\beta_1 - \beta_8$ are the parameters to be estimated from data.

The main advantage in using of taper curves is that, if tree profile can be accurately described, the volume for any merchantability segment can be computed by integrating. The taper equation of stem diameter, $d = d(D,H,h)$, allows us to revise prediction of total stem volume, $V(D,H)$, in the following form:

$$V(D,H) = \frac{\pi}{40000}\int_0^H \left(d(D,H,h)\right)^2 \cdot dh \quad (12)$$

2.6 Comparison of model performance

All models were compared to the observed values of diameter outside bark and stem volume. Numerical and graphical analyses of the residuals were used as criteria for comparing of models. The performance statistics of the height, volume and taper equations included four statistical indices: mean bias (B), mean absolute bias (MAB), mean percentage of absolute bias (MPB), and a coefficient of determination (R^2):

$$B = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \hat{y}_i\right), \quad MAB = \frac{1}{n}\sum_{i=1}^{n}\left|y_i - \hat{y}_i\right|$$

$$MPB = \frac{\sum_{i=1}^{n}\left|y_i - \hat{y}_i\right|}{\sum_{i=1}^{n} y_i} * 100, \quad R^2 = 1 - \frac{\sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2}{\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2}$$

where $n = \sum_{i=1}^{m} n_i$ is the total number of observations used to fit the model, m is the number of trees, y_i, $\hat{y}_i$, $\bar{y}$ are the measured, estimated and average values of the dependent variable (height, volume, diameter outside bark).

3 Results and discussion

The coefficients of all used models for data set summarized in Table 1 are presented in Table 2. As we can see all parameters are highly significant ($\alpha = 0.05$) with exception Honer's [3] volume model (7) (the parameter β_2), Kozak's [4] model (3) (the parameter β_5).

As seen in Table 2, all used height models account for at least 75% of the variation in height, all used volume models account for at least 98% of the variation in volume, and all used taper models account for at least 98% of the variation in diameter outside bark. Fit characteristics (B, MAB, MPB, R^2) of the q-exponential height model were better than other used models. In terms of all used fit statistics, q-exponential segmented taper model (11) appears to be superior for prediction of diameters outside bark and stem volume.

Figure 1 shows the residuals plotted against prediction of height, volume and diameter at any given height. Graphical diagnostics of residuals for the height, volume and diameter predictions showed that the residuals of the q-exponential models had more homogeneous variance than the other taper models. These results are very similar to the evaluation obtained using fit statistics. However, for the predictions of stem volume the residuals increased steadily with stem volume. The trend becomes less obvious when the predicted volume is less than 1.5 m^3. To further improve the predictions of the volume, the first step could be detection and modelling a variance function and addition new predictors such as stand density, ratio of crown length.

Taper profiles for three randomly selected pine trees with diameters outside bark of 16.2 cm, 27.5 cm, 35.0 cm and total tree heights of 19.5 m, 24.0 m, 27.2 m, respectively, using three different taper equations (Eqs. (9), (10), (11)) are plotted in Figure 2. It is clear that all tree profiles followed the stem data very closely. Graphical examination leads to the conclusion that q-exponential taper model (11) describes stem profile quite well and is comparatively superior than commonly used taper models. More appropriate methods for the incorporation of structured variance in prediction exist and need to be explored by the means of stochastic differential equations [8]-[12].

Table 2. Estimated parameters (standard errors in parentheses) of all used models

Eq.	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	B (m)	MAB (m)	MPB (%)	R^2
						Height models						
(2)	46.3601	0.9954	28.2866	-	-	-	-	-				
	(3.2532)	(0.0596)	(1.9280)						-0.003	2.086	10.10	0.750
(3)	75.1482	-1.0000	-75.1482	$-.1*10^{-6}$	-	-	-	-				
	(21.398)	(0.0033)	(21.400)	(0.0003)					0.005	2.079	10.06	0.751
(4)	33.6798	0.0341	0.9133	-	-	-	-	-				
	(1.2386)	(0.0039)	(0.0518)						-0.005	2.091	10.12	0.749
(5)	-217.679	21.5405	-0.0034	-1.2036	-	-	-	-	0.005	2.073	10.04	0.752
	(113.74)	(9.1341)	(0.0013)	(0.1858)								
						Volume models						
(6)	-9.5898	1.7784	1.0384	-	-	-	-	-				
	$(.3*10^{-6})$	$(.4*10^{-6})$	$(.7*10^{-6})$						-0.003	0.042	6.77	0.985
(7)	271.146	21672.5	-	-	-	-	-	-				
	(204.33)	(5544.4)							0.008	0.044	7.25	0.982
(8),	0.0002	$.6*10^{-4}$	$-.2*10^{-2}$	$-.1*10^{-4}$	-	-	-	-				
(12)	$(.2*10^{-4})$	$(.5*10^{-5})$	$(.1*10^{-5})$	$(.2*10^{-5})$					-0.002	0.041	6.68	0.985
(9)&	-0.3872	1.3803	49.6334	-1.5728	-	-	-	-				
(12)	(0.0042)	(0.0141)	(0.2271)	(0.0195)					-10^{-4}	0.048	7.89	0.976
(10)&	1.4009	0.9315	0.4527	-0.1059	0.00009	-0.1334	-	-				
(12)	(0.0091)	(0.0020)	(0.0029)	(0.0101)	(0.00005)	(0.0043)			0.026	0.048	7.78	0.982
(11)&	2.0204	0.9283	0.9745	-1.1928	26.6647	-381.4061	-0.7426	13.8017	0.006	0.041	6.73	0.985
(12)	(0.0763)	(0.0104)	(0.0937)	(0.0571)	(5.4078)	(114.667)	(0.0239)	(0.7875)				
						Taper models						
(9)	-0.3872	1.3803	49.6334	-1.5728	-	-	-	-	0.092	1.079	6.10	0.981
	(0.0042)	(0.0141)	(0.2271)	(0.0195)								
(10)	1.4009	0.9315	0.4527	-0.1059	0.00009	-0.1334	-	-	0.077	1.218	6.87	0.978
	(0.0091)	(0.0020)	(0.0029)	(0.0101)	(0.00005)	(0.0043)						
(11)	2.0204	0.9283	0.9745	-1.1928	26.6647	-381.4061	-0.7426	13.8017	0.011	0.929	5.24	0.985
	(0.0763)	(0.0104)	(0.0937)	(0.0571)	(5.4078)	(114.667)	(0.0239)	(0.7875)				

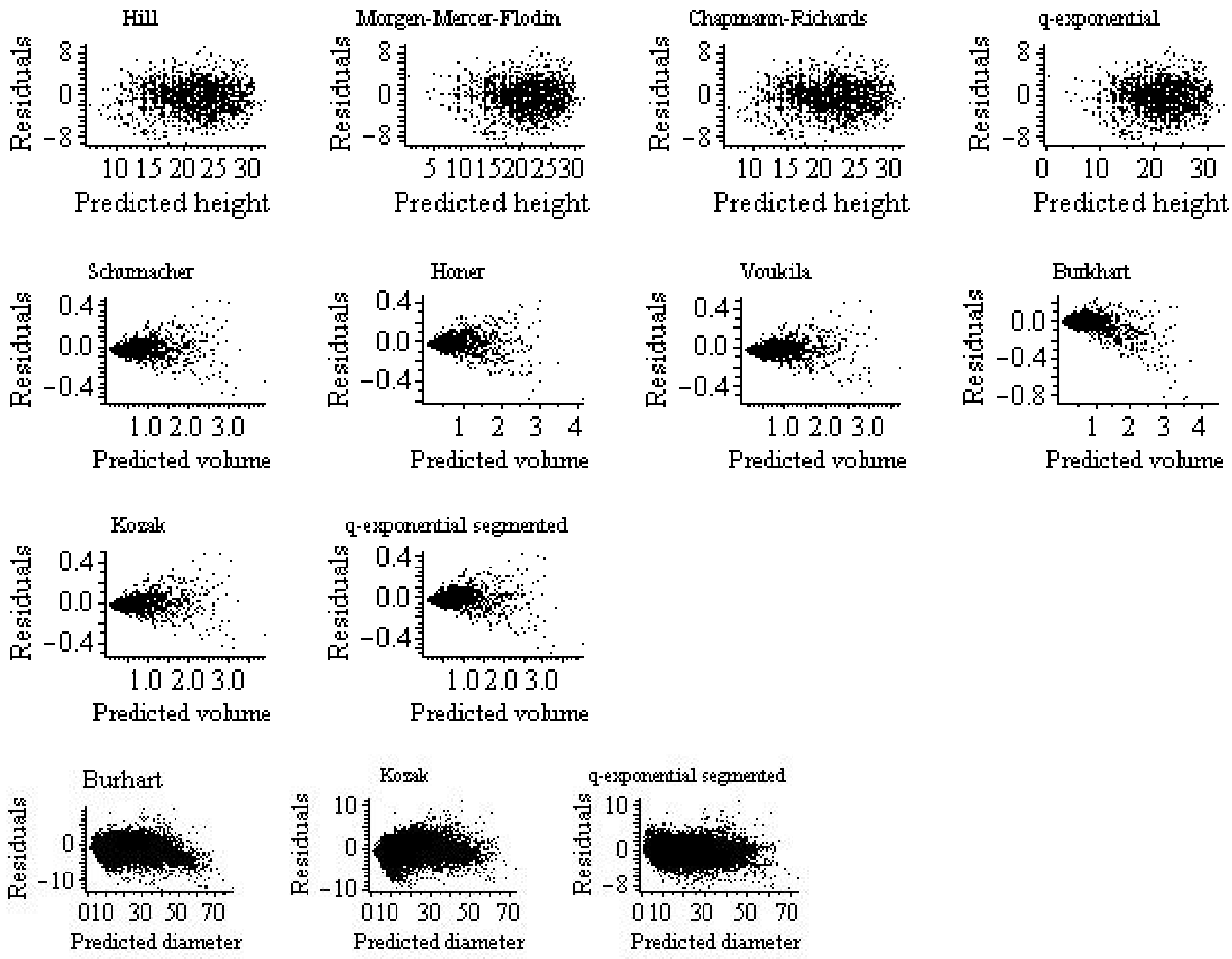

Figure 1. Plot of residuals versus predicted height, predicted stem volume, and predicted diameter outside bark

4 Conclusions

The q-exponential function was modified to model the height, volume and stem profile. We presented a study on the comparison of performance between various height, volume, stem profile models previously in the context of Scots pine data. Fit characteristics of the q-exponential models were better than those of other models.

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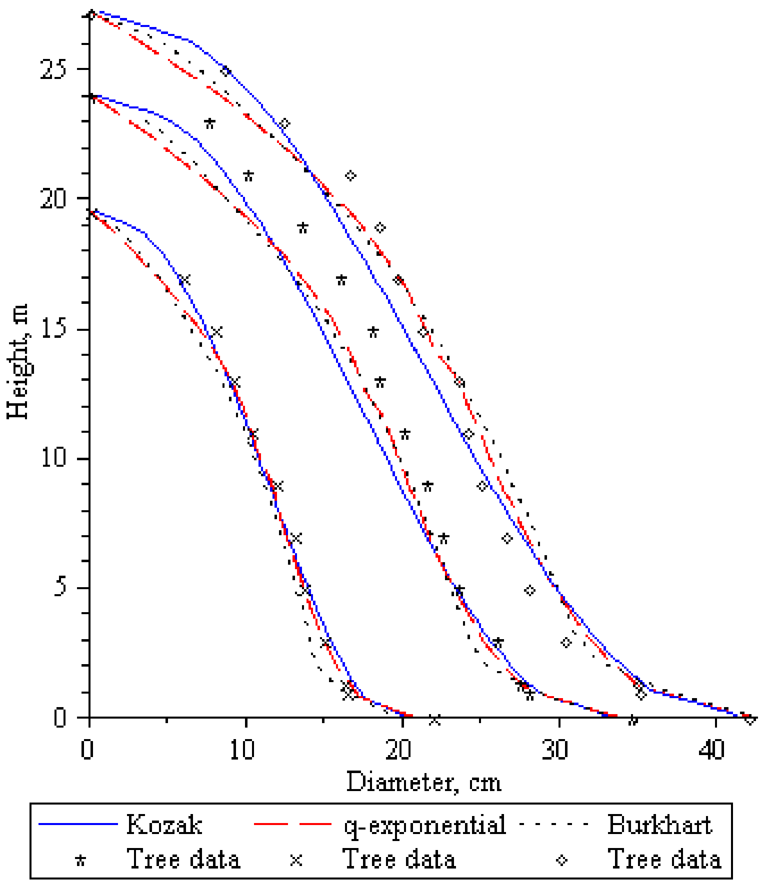

Figure 2. Tree profiles for three randomly selected pine trees generated using Burkhart (9), Kozak (10) and q-exponential segmented (11) taper models