Robust Static Output Feedback Stabilization of an Exothermic Chemical Reactor with Input Constraints

MONIKA BAKOŠOVÁ, ANNA VASIČKANINOVÁ, MÁRIA KARŠAIOVÁ
Slovak University of Technology in Bratislava, Faculty of Chemical and Food Technology
Institute of Information Engineering, Automation and Mathematics
Radlinského 9, 812 37 Bratislava
SLOVAKIA
monika.bakosova@stuba.sk

Abstract: Robust static feedback stabilization of linear systems with uncertainties and bounded control inputs is studied. Necessary and sufficient conditions of positive invariance of polyhedral domains with respect to motion of uncertain linear systems are formulated together with conditions ensuring the asymptotic stability of uncertain linear systems with bounded control inputs. The problem of robust static feedback controller calculation is solved by the inversion method. The approach has several advantages. It enables to include control input boundaries into the controller design. It can be used for both, robust static state feedback and robust static output feedback controller design. The designed controllers can be P or PI ones. The designed approach is verified by simulation results obtained for an exothermic continuous stirred tank reactor with two uncertain parameters. The reactor is open-loop unstable. The possibility to stabilize the reactor using static output feedback P or PI controllers in the presence of boundaries on control inputs is confirmed.

Key–Words: constrained input, uncertainty, robust control, static output feedback, continuous stirred tank reactor

1 Introduction

Technological processes usually work with constraints on input variables. The most usual constraints are of saturation type and it means that there are limitations on magnitude of control signals and/or magnitude of their changes. So, it is necessary to solve the problem of controller design taking these limitation into account. There are several approaches proposed in the literature to solve the problem of controller design with the control input boundaries, e.g. the positive invariance concept [1, 2], the polynomial concept [3], predictive control techniques [4] and other approaches [5]. From these approaches, the positive invariance approach is selected in this paper, because it gives simple methods to design static output feedback controllers with control input constraints in the continuous-time case. The concept of positive invariance is involved in many control problems as constraints, robustness, pole assignment and optimization. Overview of positively invariant sets is given e.g. in [6].

Operation of technological processes is corrupted by different uncertainties. Some of them arise from varying or not exactly known parameters, as e.g. densities, specific heat capacities, reaction rate constants, reaction enthalpies, heat transfer coefficients, etc. The other reason is using linearization techniques for modelling or control system design. The uncertainty is caused by the difference between the real behaviour of the system and the behaviour of the plant model. Various techniques have been proposed in the literature combining the techniques of robust and constrained control [7, 8].

In this paper, a solution of robust constrained control problem for linear continuous-time systems is described. Robustness conditions for static feedback controllers with constrained control inputs are formulated and the algorithm for obtaining robustly stabilizing P or PI controllers is given. The second part of the paper is devoted to the application of the robust constrained control design to an exothermic continuous stirred tank reactor (CSTR). CSTRs are ones of the most important plants in process industry and exothermic reactors are very interesting systems from the control viewpoint because of their potential safety problems and the possibility of exotic behaviour such as multiple steady states, see e.g. [9]. During the last years, several approaches have been proposed to solve the stabilization problem of chemical reactors with uncertainties, see e.g. [10], [11], [12], [13], [15] and others. This paper proposes a solution of this problem based on linearization technique and subsequent design of robust static output feedback controllers in the presence of boundaries on control inputs.
tion results confirm possibility to stabilize the CSTR with uncertainties using static output feedback P or PI controllers in the presence of boundaries on control inputs.

2 Robust static output feedback controller design for linear uncertain system

2.1 Problem statement

Consider a linear uncertain continuous-time system represented by the following state-space model

\[
\dot{x}(t) = A(q_A(t)) x(t) + B(q_B(t)) u(t) \\
x_0 = x(t_0) \\
y(t) = C x(t)
\]

(1)

where \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \Omega \subseteq \mathbb{R}^m \) is the control input and \( y(t) \in \mathcal{R}^p \) is the controlled output. Vectors \( q_A(t) \in Q_A \subseteq \mathbb{R}^{p_A} \), \( q_B(t) \in Q_B \subseteq \mathbb{R}^{p_B} \) represent uncertain parameters and \( Q_A, Q_B \) are compact convex sets that include origin in their interiors. The control input is restricted by saturation to the following polyhedral set

\[
\Omega = \{ u(t) \in \mathbb{R}^m : -u_{\min} \leq u \leq u_{\max}, \\
u_{\min}, u_{\max} \in \text{Int}(\mathbb{R}^m) \}
\]

(2)

Consider further that the controlled system (1) has the affine linear structure with \( A, B \) in the form

\[
A(q(t)) = A_0 + \sum_{i=1}^{p_A} A_i q_{Ai}(t) \]  \quad \text{(3)}

\[
B(q(t)) = B_0 + \sum_{j=1}^{p_B} B_j q_{Bj}(t) \]  \quad \text{(4)}

where \( q_{Ai}(t), q_{Bj}(t) \) represent \( i \)-th and \( j \)-th components of vectors \( q_A(t), q_B(t) \), respectively.

Convexity and compactness of the sets \( Q_A, Q_B \) imply that there exist \( \mu_A \) vertices \( q_A^k, k = 1, \ldots, \mu_A \) and \( \mu_B \) vertices \( q_B^l, l = 1, \ldots, \mu_B \) such that

\[
q_A = \sum_{k=1}^{\mu_A} \alpha_k q_A^k, \quad \sum_{k=1}^{\mu_A} \alpha_k = 1, \quad \alpha_k \in [0,1] \]  \quad \text{(5)}

\[
q_B = \sum_{l=1}^{\mu_B} \beta_l q_B^l, \quad \sum_{l=1}^{\mu_B} \beta_l = 1, \quad \beta_l \in [0,1] \]  \quad \text{(6)}

Assume further that the pair \((A(q_A(t)), B(q_B(t)))\) is controllable for every \( q_A \in Q_A \) and \( q_B \in Q_B \).

The nominal system is given by

\[
\dot{x}(t) = A_0 x(t) + B_0 u(t), \quad x_0 = x(t_0) \]  \quad \text{(7)}

\[
y(t) = C x(t) \]

The robust static output feedback control problem is to find an output feedback control law

\[
u(t) = F y(t), \quad F \in \mathbb{R}^{m \times r}, \quad \text{rank}(F) = m \]  \quad \text{(8)}

or an state feedback control law

\[
u(t) = K x(t), \quad K \in \mathbb{R}^{m \times n}, \quad \text{rank}(K) = n \]  \quad \text{(9)}

which stabilize the uncertain system (1) with respect to the control constraints (2) such that the closed-loop system is robustly stable in the presence of parametric uncertainty. The nominal system (7) is stabilized as well.

Stabilization of (7) means that the nominal closed loop

\[
\dot{x}(t) = (A_0 + B_0 FC)x(t) = (A_0 + B_0 K)x(t) = A_{CL0} x(t) \]  \quad \text{(10)}

is stable, e.a. the eigenvalues of \( A_{CL0} \) have negative real parts.

Application of the control law (2) divides the state space into two regions. The first region is the set of the linear closed-loop behaviour

\[
D_1 = \{ x(t) \in \mathbb{R}^n : -u_{\min} \leq FC x(t) \leq u_{\max} \}
\]

(11)

or

\[
\{ x(t) \in \mathbb{R}^n : -u_{\min} \leq K x(t) \leq u_{\max}, \quad \text{Int}(\mathbb{R}^n) \}
\]

The uncertain closed loop is given in this region

\[
\dot{x}(t) = (A(q_A(t)) + B(q_B(t)) FC)x(t) = (A(q_A(t)) + B(q_B(t)) K)x(t) = A_{CL}(q(t)) x(t) \]  \quad \text{(12)}

where \( q(t) \in Q = (Q_A \times Q_B) \) and \( q_k^l, k = 1, \ldots, \mu_A, l = 1, \ldots, \mu_B, \) are vertices of \( Q \).

The second region is the set of the nonlinear closed-loop behaviour

\[
D_2 = \{ x(t) \in \mathbb{R}^n : u = \text{sat}(FC x(t)) \} \]

(13)

and the uncertain closed loop is given

\[
\dot{x}(t) = A(q_A(t)) x(t) + B(q_B(t)) \text{sat}(FC x(t)) = A(q_A(t)) x(t) + B(q_B(t)) \text{sat}(K x(t)) \]  \quad \text{(14)}


2.2 Robust P stabilization with constrained input

The cornerstone for deriving robust constrained P controllers $F$ satisfying (8) is the positive invariance of domain $D_1$ (11) with respect to motion of the closed-loop system (12) [2].

Definitions and theorems given in this section are taken from [2] and the proofs of them can be found also there.

**Definition 1** A set $S \subset \mathbb{R}^n$ is positively invariant with respect to motion of the system (1) if for every $x(t_0) \in S$, the motion $x(t, x(t_0)) \in S$ for all $q(t) \in Q$ and all $t > t_0$.

**Theorem 2** The domain $D_1$ (11) is positively invariant with respect to the system (10) if and only if there exist a matrix $H \in \mathbb{R}^{m \times n}$, such that

$$KA_0 + KB_0 K = HK$$

where $K = FC$ and

$$\bar{H} = \begin{bmatrix} H_1 & H_2 \\ H_2 & H_1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{\text{max}} \\ u_{\text{min}} \end{bmatrix}$$

$$H_1 = \begin{cases} h_{ij} & \text{for } i = j \\ h_{ij}^+ & \text{for } i \neq j \end{cases}, \quad h_{ij}^+ = \sup(h_{ij}, 0)$$

$$H_2 = \begin{cases} 0 & \text{for } i = j \\ h_{ij}^- & \text{for } i \neq j \end{cases}, \quad h_{ij}^- = \sup(-h_{ij}, 0)$$

Consider the polyhedral domain

$$D_p = \{x(t) \in \mathbb{R}^n : -p_2 \leq x \leq p_1, p_1, p_2 \in \text{Int}(\mathbb{R}_+^n)\}$$

and the autonomous uncertain system

$$\dot{x}(t) = A(q(t))x(t), \quad x_0 = x(t_0)$$

**Lemma 3** The set $D_p$, (20) is positively invariant with respect to motion of system (21) if and only if it is positively invariant with respect to motion of system (21) at vertices $q^{kl}$, $k = 1, \ldots, \mu_A, l = 1, \ldots, \mu_B$ of the set $Q$.

**Theorem 4** The subset $D_1$ (11) is positively invariant with respect to motion of the uncertain system (12) with constrained control (2) if and only if there exist matrices $H(q^{kl})$ for $k = 1, \ldots, \mu_A, l = 1, \ldots, \mu_B$ such that

$$KA_{CL}(q^{kl}) = H(q^{kl})K$$

$$\bar{H}(q^{kl})U \leq 0$$

The conditions stated in Lemma 3 and Theorem 4 are necessary and sufficient [2].

The additional conditions to guarantee the asymptotic stability are formulated in Proposition 5.

**Proposition 5** $K$ stabilizes asymptotic the uncertain linear system (12) with constrained control (2) if there exist matrices $H(q^{kl})$, $k = 1, \ldots, \mu_A, l = 1, \ldots, \mu_B$ such that

$$KA_{CL}(q^{kl}) = H(q^{kl})K$$

$$\bar{H}(q^{kl})U \leq 0$$

**2.3 Robust PI stabilization with constrained input**

For the system (1), it is necessary to find a static output feedback

$$u(t) = F_1 y(t) + F_2 \int_0^t y(\tau) d\tau$$

with $F_1, F_2 \in \mathbb{R}^{m \times r}$.

Define a new state $z(t) = [z_1(t), z_2(t)]^T$, where $z_1(t) = x(t)$ and $z_2(t) = \int_0^t y(\tau) d\tau$. The dynamics of the newly defined system can be described as follows

$$\dot{z}_1(t) = \dot{x}(t) = A(q_A(t))z_1(t) + B(q_B(t))u(t)$$

$$z_1(t_0) = x(t_0) = x_0$$

$$\dot{z}_2(t) = y(t) = Cz_1(t)$$

Consider the polyhedral domain

$$D_p = \{x(t) \in \mathbb{R}^n : -p_2 \leq x \leq p_1, p_1, p_2 \in \text{Int}(\mathbb{R}_+^n)\}$$

and the autonomous uncertain system

$$\dot{x}(t) = A(q(t))x(t), \quad x_0 = x(t_0)$$

**2.4 Design procedure**

The algebraic equation (15) representing the condition for positive invariance of (10) plays a fundamental role in the design procedure for robust constrained controller. Two procedures can be used for solution of this equation, the direct and the inverse ones. The inverse procedure is used in this approach [14].
The spectrum of $A_0$ in 15 contains necessarily (at least) a symmetric set of $n - m$ non-null eigenvalues when $\text{rank}(K) = m$. Assume

$$
\sigma(A_0) \supset \Lambda_2 = \{ \lambda_j \in C : \lambda_j \neq 0, j = m + 1, \ldots, n \} \quad (32)
$$

$$
A_0 \zeta_j = \lambda_j \zeta_j, j = m + 1, \ldots, n \quad (33)
$$

where $\zeta_{m+1}, \ldots, \zeta_n$ are eigenvectors belonging to $\lambda_j$ and they are distinct. Give also a diagonalizable matrix $K \in R^{m \times m}$ satisfying the following conditions

$$
\sigma(H) = \Lambda_1 = \{ \lambda_i \in C : \lambda_i \neq 0, i = 1, \ldots, m \} \quad (34)
$$

$$
H \theta_i = \lambda_i \theta_i, i = 1, \ldots, m \quad (35)
$$

where $\theta_1, \ldots, \theta_m$ are eigenvectors belonging to $\lambda_i$ and they are linearly independent.

Assume further

$$
\Lambda_1 \cap \sigma(A_0) = 0 \quad (36)
$$

The assigned spectrum to the matrix $A_{CL0} = A_0 + B_0K$ is given by

$$
\Lambda = \Lambda_1 \cup \Lambda_2 \quad (37)
$$

with

$$
A_{CL0} \xi_k = \lambda_k \xi_k, k = 1, \ldots, n \quad (38)
$$

The necessary and sufficient condition of the existence of matrix $K$, which is the solution of (15), is given in the following theorem [14].

**Theorem 6** For a given matrix $H \in R^{m \times m}$ satisfying (34) and (36), there exists a real matrix $K \in R^{m \times n}$ of full rank as the unique solution of (15) if and only if $\xi_1, \ldots, \xi_n$ are linearly independent and

$$
K = [\theta_1, \ldots, \theta_m, 0_{m+1}, \ldots, 0_n] [\xi_1, \ldots, \xi_n]^{-1} \quad (39)
$$

In Theorem 6, $\xi_i$ are eigenvectors associated to the eigenvalues of $A_{CL0}$ and they are calculated as follows

$$
\xi_i = (\lambda_i I - A_0)^{-1} B_0 \theta_i, i = 1, \ldots, m \quad (40)
$$

To use the result of Theorem 6 for calculation $K$, the matrix $H$ satisfying following condition must be given:

$$
\begin{cases}
\sigma(H) \cap \sigma(A_0) = 0 \\
B \theta_i \neq 0, i = 1, \ldots, m \\
\theta_i, i = 1, \ldots, m, \text{ are linearly independent}
\end{cases} \quad (41)
$$

**Algorithm** The input data for the problem of robust constrained controller design are system (1) with known matrices $A_i, i = 0, \ldots, p_A, B_j, j = 0, \ldots, p_B$, the constraints on the control input $u_{\text{min}}, u_{\text{max}}$ and vertices $q^{k_i} k = 1, \ldots, \mu_A, l = 1, \ldots, \mu_B$ of the set of uncertainties $Q$.

**Step 1:** Choose a matrix $H$ according to (41) and satisfying condition (16).

**Step 2:** Use the inverse procedure and calculate $K$ from (39).

**Step 3:** Use the direct procedure and compute $H(q^{k_i})$ from (24) and verify condition (25). If condition (25) holds then go to the next step. Otherwise return to Step 1 and change the matrix $H$.

**Step 4:** Compute $A_{CL}(q^{k_i})$ for $k = 1, \ldots, \mu_A, l = 1, \ldots, \mu_B$ and verify condition (26). If condition (26) holds then calculate $F$ from $K = FC$ and use $F$ as a robust static output feedback controller that robustly stabilizes uncertain closed-loop system with constrained control input. Otherwise return to Step 1 and change the matrix $H$.

The algorithm can be used for both, P or PI robust controller design.

### 3 Application to the CSTR

The approach described in the theoretical section was applied to design of robust static output feedback controller that asymptotically stabilizes the reactor with uncertainties and bounded control input. The reactor is used for production of propylene glycol from propylene oxide and is detaily described in [15]. Two uncertain parameters are considered in this reactor, e.a. pre-exponential factor $k_\infty$ in the reaction rate constant and reaction enthalpy ($-\Delta_r H$). Their values can lie in following intervals: $k_\infty \in [2.4067 \times 10^{11}, 3.2467 \times 10^{11}] \text{ min}^{-1}$, $(-\Delta_r H) \in [-5.64 \times 10^6, -5.28 \times 10^6] \text{ kJ kmol}^{-1}$. It is considered further that the reactor has one controlled output - the temperature of the reacting mixture $T_r$ [K] and two control inputs - the volumetric flow rates of the reacting mixture $q_B$ [m$^3$ min$^{-1}$] and the cooling medium $q_C$ [m$^3$ min$^{-1}$].

For the controller design, the linearized model of the reactor is used. The matrices in the description of the uncertain linear model (1) of the reactor are
The controller

\[ K = \begin{bmatrix} 112.6628 & 0.2940 & 0 \\ 3509.2445 & 4.4608 & 99.3475 \end{bmatrix} \]  

which stabilizes asymptotically the nominal system with constrained control is calculated using (39). Then the robustness of this controller with respect to the parametric uncertainties is studied and the conditions (25) and (26) are verified. Both of them are also fulfilled. Therefore the controller \( K \) stabilizes asymptotically the uncertain system with constrained control (2). Then, the static output feedback controller is calculated and obtained as \( F = [0.2940 4.4608]^T \). Fig. 2 shows the stabilization of the reactor using static output feedback P controller \( F = [0.2940 4.4608]^T \). The controller is able to stabilize the nominal system as well as all vertex systems. The nominal system and four vertex systems for simulation purposes are represented by the original nonlinear model of the reactor obtained for nominal values of uncertain parameters and by the nonlinear models of the reactor obtained for all combinations of minimum and maximum values of uncertain parameters. Designed P controller robustly stabilizes the CSTR to the open-loop unstable operating point, but using of the P controller leads to the offsets in the control responses of vertex systems.

The obtained static output feedback PI controller is

\[ F = \begin{bmatrix} 0.6731 & 0.0419 \\ 37.3615 & -1.4772 \end{bmatrix} \]
Fig. 3 shows the PI stabilization of the reactor using designed robust static output feedback PI controller for the constrained case. The controller is able to stabilize the nominal systems as well as four vertex systems of the original nonlinear model into the open-loop unstable operating point without offsets.

![Figure 3: Robust static output feedback stabilization of CSTR using PI controller](image)

4 Conclusion

The paper presents approaches to the design of static feedback controllers for linear systems with uncertainty and constrained inputs. Given algorithm is simple and easy to use especially in the case of small number of vertices of uncertain parameters. Checking and assuring necessary and sufficient conditions for asymptotic stabilization of uncertain system with bounded inputs is not so easy in the case when the dimensions of the vector of uncertainty is high. The key role in the presented algorithm plays the choice of the matrix $H$. The advantage of the algorithm is that it can be used for both, the P or PI robust static output feedback controller design.

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