Chebyshev economization in transformations of nonlinear systems with polynomial structure

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Abstract: This paper presents a new method of making an approximately equivalent extended linear systems for autonomous nonlinear systems with polynomial structure. This method is based on the method of extended model. The Chebyshev economization is used to obtain higher accuracy.

Key–Words: Chebyshev economization, Chebyshev polynomials, Telescoping power series, Nonlinear systems, Approximation, Extended systems

1 Introduction

We consider a system of autonomous dynamic equations in the normal form with a polynomial right side [1],[5],[6]. We apply a method of extending of an object by introduction of a countable set of additional phase coordinates, that transforms the nonlinear equations of perturbed motion into the linear infinite system of equations [2], [3], and then we apply Chebyshev economization method [4], [7] to obtain higher accuracy of the finite linear extended system.

2 Initial and extended systems

Suppose a nonlinear system can be defined or approximated up to desired accuracy as below:

\[
\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j + \sum_{|\nu|=2}^{m} x_\nu a_\nu, \quad x_j(t_0) = x_{i0}, \quad (1)
\]

where \(a_{ij}\) are constant coefficients, \(\nu = (\nu_1, \ldots, \nu_n)\) are vector indexes, \(x_\nu = x_{\nu_1} \ldots x_{\nu_n}\) are monomials of the degree \(|\nu| = \nu_1 + \ldots + \nu_n\), \(a_\nu\) are constant coefficients with vector indexes \(\nu\), \(m = 2n\), \(i = 1, \ldots, n\).

By definition put

\[
\tilde{D} = \{x = [x_1, \ldots, x_n] \in \mathbb{R}^n, |x_i| \leq r, i = 1, \ldots, n\},
\]

where \(\tilde{D}\) is a finite region of the state-space.

As it is known [3], [2] systems with polynomial nonlinearity of the form (1) can be transformed to the infinite set of linear equations using the object extension method.

Let us introduce the modified method. First we add a finite set of additional variables

\[
x_\nu = x_1^{\nu_1} \ldots x_n^{\nu_n}, |\nu| = 2, 3, \ldots, m, \quad m = 2n, \quad (3)
\]

Note that each homogeneous form of the power \(|\nu| = k\) includes \(N_k = \frac{(k + 1)(k + 2)\ldots(k + n - 1)}{(n - 1)!}\) monomials. As a whole we add \(N = N_2 + N_3 + \ldots + N_m\) new variables. Let us renumber these variables as follows

\[
\{x_\nu, |\nu| = 2, 3, \ldots, m\} \Rightarrow [x_{n+1}, x_{n+2}, \ldots, x_{n+N}]. \quad (4)
\]

Differentiating (3), taking into account (1) and using vector indexes \(e_i = (0\ldots010\ldots0)\), we obtain

\[
\dot{x}_\nu = \sum_i \nu_i \hat{x}_i x_{\nu-e_i} = \sum_i \nu_i \hat{x}_i x_{\nu-e_i} (\sum_j a_{ij} x_j + \sum_{\nu'} x_{\nu'} a_{\nu'}). \quad (5)
\]

By \(X^{(2,m)}\) denote the polynomials with terms of the power \(2, 3, \ldots, m\) and by \(X^{(m+1,2m-1)}\) denote the remainder polynomials. We obtain

\[
\dot{x}_\nu = \sum_{i,j} \nu_i a_{i,j} x_{\nu-e_i+e_j} + X^{(2,m)} + X^{(m+1,2m-1)}. \quad (6)
\]

Since every term of the remainder polynomials has at least one variable to the power \(k \geq 3\), we can apply a Chebyshev economization method [7], [4] as
follows:
\[ x_3^3 \approx \frac{3}{4} r^2 x_s^3, \delta(3) \leq \frac{1}{4} r^3, r \leq 1 \]
\[ x_4^4 \approx r^2 x_s^2 - \frac{1}{8} r^4, \delta(4) \leq \frac{1}{8} r^4, \]
\[ x_5^5 \approx \frac{5}{4} r^2 x_s^3 - \frac{5}{16} x_r^4, \delta(5) \leq \frac{1}{16} r^5, \]
where \( \delta \) is the economization error. Note that for some cases it is more convenient to use the alternative approximations \( x_s^4 = x_s x_r^3 \approx \frac{3}{4} r^2 x_s^2, \) although they have higher error \( \delta = \delta(3). \)

Then using this method repeatedly, we finally get the following approximation of the remainder polynomials \( X^{(m+1,2m-1)} \) by polynomials \( X^{(2,m)} \) and additional linear and constant terms

\[ X^{(m+1,2m-1)} \approx c_v + \sum_i b_v x_i + \tilde{X}^{(2,m)} \]

Substituting (8) for (6), we get the extended linear system in a matrix form

\[ \frac{dy}{dt} = yB + C, \quad B = [b_{ij}]_1^{n+N}, \quad C = [c_1, \ldots, c_{n+N}], \]
\[ y \in D, \quad D = \{ y = [x_1, \ldots, x_{n+N}] : |x_i| \leq r, i = 1, 2, \ldots, n + N \} \]

with the initial conditions

\[ x_i(t_0) = x_{i0}, \quad i = 1, \ldots, n \]
\[ x_j(t_0) = x_{j0} = x_{10}^{\nu_1} \cdots x_{10}^{\nu_n}, \quad j = n + 1, \ldots, N \]  

(11)

Note that in (9) we neglected small quantities. Therefore the solutions of the extended system (9-11) will satisfy (1) with some residual error.

\section{Example}

Consider the equation of holonomic systems with stationary nonlinear resistance:

\[ \ddot{y} + 2ny' + k^2y + ay^3 = 0, \]
\[ \hat{D} = \{ (\dot{y}, y) : |\dot{y}| \leq r, \quad y \leq r, \quad r < 1 \} \subset R^2, \]

or

\[ \dot{x}_1 = -(2nx_1 + k^2x_2) - ax_1^3, \quad \dot{x}_2 = x_1, \]
\[ \hat{D} = \{ (x_1, x_2) : |x_1| \leq r \}. \]

Then \( X = [x_1, \ldots, x_6] \) is the extended phase row vector of the system and \( X(0) = [x_{10}, \ldots, x_{60}] \) is its initial phase, where

\[ x_3 = x_1, \quad x_4 = x_2^2, \quad x_5 = x_1^2 x_2^2, \quad x_6 = x_3^2, \]
\[ x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \quad x_3(0) = x_1^3, \]
\[ x_4(0) = x_{10}^2 x_{20}, \quad x_5(0) = x_{10} x_{20}^2, \quad x_6(0) = x_{20}^3 \]

We apply the following economizations for additional monomials:

\[ x_5^3 \approx \frac{5}{4} r^2 x_s^3 - \frac{5}{16} r^4 x_s, \quad x_5 x_j \approx r^2 x_s x_j - \frac{1}{8} r^4 x_j, \]
\[ x_5^2 x_j^2 \approx \frac{3}{4} r^2 x_s^2 x_j^2, s, j = 1, 2. \]  

(13)

For example here is how we get the first additional equation using (12) and (13)

\[ \dot{x}_3 = -3x_1^2 (2nx_1 + k^2 x_2) - 3ax_1^5 = -6nx_3 - 3k^2x_4 - 3a (\frac{5}{4} r^2 x_3 - \frac{5}{16} r^4 x_1) \Rightarrow \]
\[ \dot{x}_3 = XB_3 \quad B_3 = [b, 0, -c, -3k^2, 0, 0]^T, \]
\[ b = \frac{15}{16} ar^4, \quad c = 6n + \frac{15}{4} ar^2. \]

Finally, we get the extended linear system \( \dot{X} = XB, \)
where \( B \) be as follows:

\[ B = \begin{bmatrix} -2n & 1 & b & 0 & 0 & 0 \\ -k^2 & 0 & 0 & e & 0 & 0 \\ -a & 0 & -c & 1 & 0 & 0 \\ 0 & 0 & -3k^2 & -g & 2 & 0 \\ 0 & 0 & 0 & -2k^2 & -h & 3 \\ 0 & 0 & 0 & 0 & -k^2 & 0 \end{bmatrix}, \]

where \( e = ar^4/4, g = 4n + 2ar^2, h = 2n + 3ar^2/4. \)

Now we can estimate the stability of the initial nonlinear system by the eigenvalue spectrum of the extended system, i.e. by the six roots of the characteristic equation for \( B. \) For example, for \( r = 0.8, n = 0.1, k = 1, a = 0.8 \) we get \( \lambda = [-0.156 \pm 0.988i, -1.167 \pm 0.936i, -1.041 \pm 2.736i]. \) The error and the initial conditions are defined as follows:

\[ \Delta_1 = x_1 - \dot{y}, \Delta_2 = x_2 - y, \quad y_0 = x_{10}, y_0 = x_{20} \Leftrightarrow \Delta_1(0) = \Delta_2(0) = 0. \]

Two plots are shown in Fig. 1: one - of the solution of initial nonlinear system \( y(t) \) at \( y(0) = 0.8, \dot{y}(0) = 0, t \in [0, 10] \) and another one - of the solution of the extended linear system \( x_2(t) \) with the corresponding initial conditions. As it could be seen, these two graphs almost merged into a one. That proofs the result of the paper.
Figure 1: Solutions of the nonlinear and extended linear systems

4 Conclusion

In this paper the new method of making an approximately equivalent extended linear systems for autonomous nonlinear systems with polynomial structure is presented. It is based on the method of extended model. The Chebyshev economization is used to obtain higher accuracy. The example illustrates the developed method and shows good accuracy of the approximation.

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References: