Argument Principle Based Stability Conditions of a Retarded Quasipolynomial with Two Delays

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Abstract: - In the presented paper, we address a problem of the appropriate setting of a parameter in a selected quasipolynomial with two delay elements in order to ensure that all its zeros are located in the open left-half complex plane. The quasipolynomial can represent the dynamics of a system with internal delays and thus it can decide about system stability. In contrast to many other analyses, a non-delay real parameter is being to set. The argument principle (Mikhaylov criterion) is utilized for this purpose. Stability bounds for the parameter are found through proven lemmas, propositions and theorems.

Key-Words: - Stability, time-delay systems, characteristic quasipolynomial, argument principle, Mikhaylov criterion.

1 Introduction

The existence of delays in dynamics is a generic feature of many systems and processes. In day-to-day life, it is apparent especially on heating systems [1]-[3]. The class of delayed (anisochronic) systems can be modeled in various ways [4]. Considering continuous and linear (or linearized) models, one can naturally utilize the Laplace transform yielding the transfer function as a ratio of so-called *quasipolynomials* [5] in one complex variable s [6]-[7], instead of polynomials which are usual in system and control theory. Roots of the denominator (i.e. poles) decide about the asymptotic stability as in the case of polynomials; however, the number of poles is infinite.

In the recent years, a deep interest in problems of stability and stabilization of delayed systems can be observed, e.g. in [8-11] where the task was solved using various analytical and numerical tools. The decision about asymptotic stability of plants or control systems with internal delays can be done via studying of the corresponding characteristic quasipolynomial. A powerful tool here is the fact the conventional argument principle (i.e. the Mikhaylov criterion) holds for characteristic quasipolynomials of delayed systems of *retarded* type. Note that *neutral* delayed systems require rather modified Mikhaylov criterion [12].

In this contribution, we investigate the stability of the selected retarded quasipolynomial with two independent delays. The aim is to find lower and upper bounds for a real selectable non-delay parameter so that all its zeros are located in the open left-half complex plane, which implies the asymptotic stability of the quasipolynomial. Presented derivations and calculations are based on the argument principle (i.e. the Mikhaylov criterion) and the desired shape of the Mikhaylov plot. In contrast to the presented paper, other authors usually have studied the stability w.r.t. the delay, not w.r.t. a non-delay parameter. The result can serve engineers in setting the unknown controller parameter in the characteristic quasipolynomial of a delayed system properly or to decide about system stability.

2 Stability Criterion for Retarded Quasipolynomials

The principle of argument is a well known tool for stability analysis of delayless systems and models based on the knowledge of the characteristic polynomial. The same principal holds for retarded quasipolynomials of the form

$$m(s)=s^n+\sum_{i=0}^{n-1}\sum_{j=1}^{h_i}m_{ij}s^i\exp\left(-s\vartheta_{ij}\right) \tag{1}$$

Thus the (characteristic) quasipolynomial has all its zeros in the open left-half complex plane (i.e. the corresponding system is asymptotically stable) iff

$$\underset{s=\omega j,\omega\in[0,\infty]}{\Delta\arg}\, m(s)=\frac{n\pi}{2} \tag{2}$$

see e.g. [6]-[7].

Hence, one can take the desired number of quadrants in the complex plane which the Mikhaylov plot has to pass, and to specify quasipolynomial parameters in order to obtain the desired shape of the curve. The following stability analysis is done in the same way.

3 Quasipolynomial Stability Analysis

The main goal of this paper is to found bounds for the parameter $q \neq 0 \in ○$ such that the quasipolynomial with two independent delays

$$m(s) = s + a\exp(-\vartheta s) + kq\exp(-\tau s) \tag{3}$$

is stable, where $a \neq 0 \in ○$; $k, \vartheta, \tau > 0 \in ○$.

According to criterion (2), the Mikhaylov curve of (3) for $\omega \in [0, \infty]$ must have the overall argument change equal to $\pi/2$. Quasipolynomial stability investigation via lemmas, propositions and theorems follows. Due to a rather high complexity of such type of quasipolynomials, some statements will not be proved.

Lemma 1. For ω = 0, the imaginary part of the Mikhaylov curve of quasipolynomial (3) equals zero and it approaches infinity for $\omega \to \infty$.

Proof. Decompose $m(\mathrm{j}\omega)$ into real and imaginary parts as follows

$$\mathrm{Re}\{m(\mathrm{j}\omega)\} = a\cos(\vartheta\omega) + kq\cos(\tau\omega) \tag{4}$$

$$\mathrm{Im}\{m(\mathrm{j}\omega)\} = \omega - a\sin(\vartheta\omega) - kq\sin(\tau\omega) \tag{5}$$

Obviously

$$\mathrm{Im}\{m(\mathrm{j}\omega)\}\big|_{\omega=0} = 0$$

$$\lim_{\omega\to\infty} \mathrm{Im}\{m(\mathrm{j}\omega)\} = \infty \qquad \square$$

Lemma 2. If (3) is stable, the following inequality holds

$$q > \frac{-a}{k} \tag{6}$$

and thus the Mikhaylov curve starts on positive real axis.

Proof. If (3) is stable, the overall argument change equals to $\pi/2$ according to (2). Moreover, Lemma 1 states that the imaginary part goes to infinity. These two requirements imply that for stable quasipolynomial is

$$\mathrm{Re}\{m(\mathrm{j}\omega)\}\big|_{\omega=0} > 0 \tag{7}$$

By application of (7) onto (4) the condition (6) is obtained. □

Lemma 3. A point on the Mikhaylov curve of (3) lies in the first quadrant for an infinitesimally small $\omega = \Delta > 0$ iff

$$a\vartheta + kq\tau \le 1 \tag{8}$$

This point lies in the fourth quadrant iff

$$a\vartheta + kq\tau > 1 \tag{9}$$

Proof. (Necessity.) If the point goes to the first quadrant for an infinitesimally small $\omega = \Delta > 0$, then the change of function $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = 0$ is positive or this function is increasing in $\omega = \Delta$. It is known fact that this is satisfied if either

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} > 0 \tag{10}$$

or there exists *even* $n \in {}^2$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = \ldots = \frac{\mathrm{d}^{n-1}}{\mathrm{d}\omega^{n-1}}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = 0,$$

$$\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} > 0 \tag{11}$$

(i.e. there is a local minimum of $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = 0$), or there is *odd* $n \ge 3 \in {}^2$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = \ldots = \frac{\mathrm{d}^{n-1}}{\mathrm{d}\omega^{n-1}}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = 0,$$

$$\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} \neq 0,\ \frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=\Delta} > 0 \tag{12}$$

(i.e. there is a point of inflexion of $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = 0$; however, the function is increasing in $\omega = \Delta$).

Analyze now the previous three conditions. First, relation (10) w.r.t. (5) reads

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0}\\ &= 1 - a\vartheta\cos(\vartheta\omega) - kq\tau\cos(\tau\omega)\big|_{\omega=0}\\ &= 1 - a\vartheta - kq\tau > 0\end{aligned} \tag{13}$$

which gives $a\vartheta + kq\tau < 1$.

Second, condition (21) can be taken into account if

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = 0 \Leftrightarrow a\vartheta + kq\tau = 1 \tag{14}$$

The second derivation is

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0}\\ &= a\vartheta^2\sin(\vartheta\omega) + kq\tau^2\sin(\tau\omega)\big|_{\omega=0} = 0\end{aligned} \tag{15}$$

Generally, any *even* n-th derivation reads

$$\begin{aligned}&\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\mathrm{Im}\{m(\mathrm{j}\omega)\}\\ &= (-1)^{\frac{n}{2}-1}\left(a\vartheta^n\sin(\vartheta\omega) + kq\tau^n\sin(\tau\omega)\right)\\ &\Rightarrow \frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} = 0\end{aligned} \tag{16}$$

This implies that condition (11) can not be satisfied.

Third, assume that there exists a non-zero *odd* n-th, $n \ge 3$, derivation in $\omega = 0$

$$\begin{aligned}&\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0}\\ &= (-1)^{\frac{n-3}{2}}\left(a\vartheta^n\cos(\vartheta\omega) + kq\tau^n\cos(\tau\omega)\right)\big|_{\omega=0}\\ &\overset{a\vartheta+kq\tau=1}{=} (-1)^{\frac{n-3}{2}}\left(a\vartheta^n\cos(\vartheta\omega) - (a\vartheta-1)\tau^{n-1}\cos(\tau\omega)\right)\big|_{\omega=0}\end{aligned}$$

$$= (-1)^{\frac{n-3}{2}} \left(a\vartheta^n - (a\vartheta - 1)\tau^{n-1} \right) \tag{17}$$

Test the latter condition in (12), obviously

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \operatorname{Im}\{m(\mathrm{j}\omega)\}\Big|_{\substack{\omega=\Delta \\ a\vartheta+kq\tau=1}} = a\vartheta(\cos(\vartheta\Delta) - \cos(\tau\Delta)) + \cos(\tau\Delta) > 0 \tag{18}$$

since

$$\lim_{\Delta\to 0^+} \frac{\cos(\vartheta\omega)}{\cos(\tau\omega)} = 1 \tag{19}$$

Analogously to in (10)-(12), one can easily verify that if the Mikhaylov plot passes through the fourth quadrant first, then function $\operatorname{Im}\{m(\mathrm{j}\omega)\}$ decreases in $\omega = 0$ and (9) holds.

(Sufficiency.) Consider condition (8) and verify that it satisfies (10) or (12), respectively. In the same way, formula (9) gives rise to

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \operatorname{Im}\{m(\mathrm{j}\omega)\}\Big|_{\omega=0} < 0 \tag{20}$$

which induces the initial tendency of the Mikhaylov plot to go to the fourth quadrant. □

Lemma 4. If *a*, *k*, *q* are bounded, then $\operatorname{Re}\{m(\mathrm{j}\omega)\}$ is bounded for all $\omega > 0$.

Proof. Assume the following four various condition.

1) If $a > 0$ and $kq > 0$, then

$$\begin{aligned} -a - kq &\le \operatorname{Re}\{m(\mathrm{j}\omega)\} \\ &= a\cos(\vartheta\omega) + kq\cos(\tau\omega) \le a + kq \end{aligned} \tag{21}$$

2) If $a > 0$ and $kq < 0$, then

$$-a + kq \le \operatorname{Re}\{m(\mathrm{j}\omega)\} \le a - kq \tag{22}$$

3) If $a < 0$ and $kq > 0$, then

$$a - kq \le \operatorname{Re}\{m(\mathrm{j}\omega)\} \le -a + kq \tag{23}$$

4) If $a < 0$ and $kq < 0$, then

$$a + kq \le \operatorname{Re}\{m(\mathrm{j}\omega)\} \le -a - kq \tag{24}$$

It is possible to summarize and unify results (21) – (24) as

$$-(|a| + |kq|) \le \operatorname{Re}\{m(\mathrm{j}\omega)\} \le |a| + |kq| \tag{25}$$

□

Proposition 1. If (6) and (8) are satisfied together, then

$$a(\vartheta - \tau) \le 1 \tag{26}$$

Proof. Obviously,

$$a(\vartheta - \tau) \overset{kq>-a}{<} a\vartheta + kq\tau \le 1 \tag{27}$$

□

The preceding proposition also expresses that for a stable quasipolynomial (3) when the corresponding Mikhaylov plot passes the first quadrant as first, the condition (26) holds.

Proposition 2. If the following inequality holds

$$a(\vartheta - \tau) > 1 \tag{28}$$

then the corresponding Mikhaylov plot of a stable quasipolynomial (3) passes the fourth quadrant as first.

Proof. Lemma 2 states that (6) reads for stable quasipolynomial (3). Then

$$1 < a(\vartheta - \tau) \overset{kq>-a}{<} a\vartheta + kq\tau \tag{29}$$

which induces that the Mikhaylov plot goes to the fourth quadrant as first, due to Lemma 3.

Proposition 3. There always exists an intersection of the Mikhaylov curve of (3) with the imaginary axis.

Proof. The intersection exists if $\operatorname{Re}\{m(\mathrm{j}\omega)\} = 0$, i.e.

$$a\cos(\vartheta\omega) = -kq\cos(\tau\omega) \tag{30}$$

for some $\omega > 0$. Obviously, since $\vartheta > 0$, $\tau > 0$, there is $\omega > 0$ satisfying relation (30). □

The upper stability bound will now be found via some observations and a theorem. Due to high complexity of formulas (4) and (5) caused by goniometric functions, some numerical unproven observations compensate for exact analytic statements.

Definition 1. Let (6) holds. A *crossover frequency* ω_0 is an element of the set

$$\Omega_0 := \{\omega : \omega > 0, \operatorname{Re}\{m(\mathrm{j}\omega)\} = 0, \operatorname{Im}\{m(\mathrm{j}\omega)\} = 0\} \tag{31}$$

for some *crossover gain* q_0 and $a \ne 0, k, \tau, \vartheta > 0$. ■

A crossover frequency, hence, has to satisfy simultaneously these two identities

$$\begin{aligned} &a\cos(\vartheta\omega_0) + kq_0\cos(\tau\omega_0) = 0 \\ &\omega_0 - a\sin(\vartheta\omega_0) - kq_0\sin(\tau\omega_0) = 0 \end{aligned} \tag{32}$$

Relations (32) can also be expressed by transcendental equation

$$\omega_0\cos(\tau\omega_0) = a(\sin((\vartheta - \tau)\omega_0)) \tag{33}$$

Note that equation (33) is in the form suitable for utilization of numerical methods, i.e. some ratios of goniometric functions are not desirable for this purpose. The crossover gain q_0 can be calculated from (32) as

$$q_0 = \frac{\omega_0 - a\sin(\vartheta\omega_0)}{k\sin(\tau\omega_0)} \tag{34}$$

Definition 2. Let (6) holds. The *critical frequency* ω_C is defined as

$$\omega_C := \min\left\{ \omega : \omega \in \Omega_0, \underset{s=\omega\mathrm{j},\omega\in[0,\omega_C]}{\Delta\arg m(s)} = 0, \underset{s=\omega\mathrm{j},\omega\in[\omega_C,\infty]}{\Delta\arg m(s)} = \frac{\pi}{2} \right\} \tag{35}$$

for the corresponding *critical gain* q_C given by (34), where ω_C is placed instead of ω_0, and $a \ne 0, k, \tau, \vartheta > 0$. ■

Obviously, the critical frequency is the least crossover frequency for which the argument change is zero for $\omega \in [0, \omega_C]$ and consequently it equals $\pi/2$ for $\omega \in [\omega_C, \infty]$. The quasipolynomial is then on the stability border for q_C, which has to satisfy the necessary stability condition (6). There can hence exist some crossover frequencies less then the critical one which do not mean the stability border.

Observation 1. Let $q = q_C$, then the Mikhaylov plot of (3) circumscribes curves in the clockwise direction around the center of the rotation (like a "whirligig"). Moreover, if (8) holds, then the Mikhaylov plot of (5) initially moves to the first quadrant (as proved in Lemma 3) followed by the fourth quadrant for some frequencies $\omega > 0$. It means that although relation (8) quarantines that the plot tends to move to the first quadrant for $\omega = 0$, it immediately passes over the positive real axis to the fourth quadrant anyway.

Remark 1. In [13] and [14] is proved a lemma which states that the spectrum of a general retarded quasipolynomial, represented e.g. by (3), is continuous with respect to continuous changes of all its parameters. This fact implies that the Mikhaylov plot of an appropriate quasipolynomial is continuous in both axes with respect to these parameters' changes, and viceversa.

Theorem 1. If $\sin(\tau\omega_C) > 0$, then quasipolynomial (3) is stable iff

$$\frac{-a}{k} < q < \frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \tag{36}$$

Contrariwise, if $\sin(\tau\omega_C) < 0$, then quasipolynomial (5) is stable iff

$$q > \frac{\omega_C - a\sin(\vartheta\omega_C)}{k\sin(\tau\omega_C)} \geq \frac{-a}{k} \tag{37}$$

where ω_C is the critical frequency.

Proof. (Necessity.) The Mikhaylov curve of stable quasipolynomial (3) starts on the positive real axis, and thus the left-hand side of (36) and the right-hand one of (37) hold, as proved in Lemma 2. Lemma 3 states the condition (8) guaranties that the initial change of the Mikhaylov curve in the imaginary axis is positive. i.e. the curve tends to move to the first quadrant for $\omega = 0$; however, according to Observation 1, it immediately moves to the fourth quadrant. If (9) is satisfied, the curve passes through the fourth quadrant already for an infinitesimally small ω. The critical (marginal) case is characterized by ω_C and q_C where the curve crosses the origin of the complex plane and a small change of q would cause the quasipolynomial stability, i.e. the overall phase change would be $\pi/2$, see Remark 1. The limit stable case thus obviously means that either when $\mathrm{Im}\{m(\mathrm{j}\omega_C)\} = 0$, the real part must satisfy $\mathrm{Re}\{m(\mathrm{j}\omega_C)\} > 0$ for some q, or $\mathrm{Re}\{m(\mathrm{j}\omega_C)\} = 0$ and $\mathrm{Im}\{m(\mathrm{j}\omega_C)\} > 0$. However, the former condition has one important inconvenience described in the following paragraph.

When $\vartheta = \tau$, the critical case $\mathrm{Re}\{m(\mathrm{j}\omega_C)\} = 0$, q = q_C, reads

$$(a + kq_C)\cos(\vartheta\omega_C) = 0 \tag{38}$$

Since for $\omega_C \neq 0$ and a Mikhaylov plot starting on the positive real axis, $a < -kq_C$, i.e. $\cos(\vartheta\omega_C) = 0$ (Lemma 2), then it is not possible to satisfy $\mathrm{Re}\{m(\mathrm{j}\omega_C)\} > 0$ for any q.

Therefore take the later limit stable stability condition and apply simple calculations on (4) and (5) using (33) when $\sin(\tau\omega_C) > 0$, which yields the upper bound in (36).

Otherwise, if $\sin(\tau\omega_C) < 0$, the calculations result in the left-hand side inequality in (37). Evidently, values of q less then the necessary stability condition (6) can be discarded.

A case when $\sin(\tau\omega_C) = 0$ would mean that q reaches infinity which is not physically possible.

(Sufficiency.) Consider inequality (36) first. The lower bound means that the Mikhaylov curve initiates on the positive real axis, see Lemma 2. Lemma 3 verifies that the curve reaches infinity in the imaginary axis for $\omega \to \infty$, and Lemma 4 states that it is bounded in the real axis. Moreover, if (8) holds the Mikhaylov curve tends to move to the first quadrant and, consequently, to the fourth quadrant for $\omega = 0$; otherwise, it moves to the fourth quadrant for $\omega = \Delta$ when (9) is satisfied. For the quasipolynomial stability, expressed by the overall phase shift $\pi/2$, it is now sufficient to show that the curve does not encircle the origin of the complex plane in the clockwise direction.

Let the critical stability case be expressed by ω_C and q_C and apply the upper bound in (36) on (4) and (5) together with $\sin(\tau\omega_C) > 0$. Hence, the following conditions are satisfied simultaneously for a particular q: $q < q_C$, $\mathrm{Re}\{m(\mathrm{j}\omega_C)\} = 0$, $\mathrm{Im}\{m(\mathrm{j}\omega_C)\} > 0$. It means that the imaginary axis is crossed in the positive semi-axis first on the critical frequency and thus, with respect to Remark 1, the origin is encircled in the anti-clockwise direction with the overall phase shift $\pi/2$.

As second, the right-hand side of (37) expresses the necessary stability condition (6) which guaranties that the Mikhaylov curve starts on the positive real axis. Assume now that the left-hand side in the inequality holds. Similarly as in the previous paragraph, it is sufficient to prove that the curve encircles the origin in the complex plane in the anti-clockwise direction. Indeed, if $\sin(\tau\omega_C) < 0$, one can verify that the inequality agrees with the statement that $\mathrm{Re}\{m(\mathrm{j}\omega_C)\} = 0$ and $\mathrm{Im}\{m(\mathrm{j}\omega_C)\} > 0$ which gives rise to the stability of quasipolynomial (3). □

4 Simulation Example

Consider quasipolynomial of the form (3) with $\tau = 1.1$, $\vartheta = 1$, k = 1, a = -5. One can observe that $\omega_C = 0.953$ which gives $q_C = 5.803$, according to (34), and $\sin(\tau\omega_C) > 0$. Hence, Theorem 1 yields the stabilizing interval $5 < q < 5.803$. Let $q = 5.4$, then the corresponding Mikhaylov plot is displayed in Fig. 1.

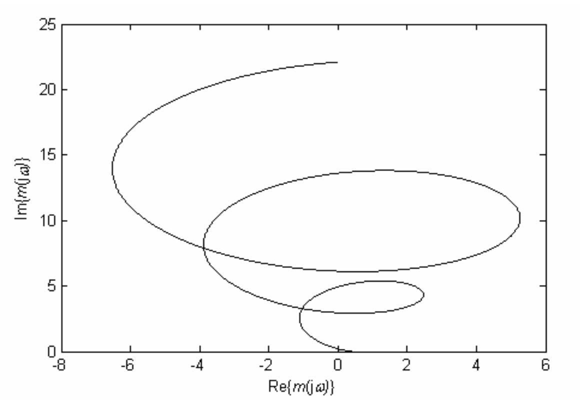

Fig. 1. The Mikhaylov plot of (3) for q = 5.4, $\omega \in [0,15]$

Obviously, the overall phase shift for $\omega \to \infty$ would be $\pi/2$, thus according to the principle of argument, all the quasipolynomial zeros are located in the open left-half complex plane.

4 Conclusion

The presented contribution has introduced some stability properties of a selected retarded quasipolynomial with two fixed independent delays. The aim has been to derive acceptable upper and lower bounds for a non-delay real parameter so that all quasipolynomial zeros are located in the open left-half complex plane. The analysis has been based on the argument principle, i.e. the Mikhaylov stability criterion, in order to keep the desired shape of the Mikhaylov curve. Almost all presented lemmas and theorems have been proved, except a hardly provable observation. The results can be instrumental in a suitable setting of the closed loop characteristic quasipolynomial for delayed systems or in the stability analyzing of these systems. A simulation example demonstrates the stabilization of a particular quasipolynomial and figures its Mikhaylov plot.

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