Non-delay Parameter Depending Stability of a Time-Delay System

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Abstract: - A characteristic equation of a time-delay system contains a quasipolynomial rather then a polynomial. Solutions of the equation, system poles, have as the same meaning as for delay-free systems, thus they decide about system stability. This contribution studies the influence of a non-delay real parameter in a selected characteristic quasipolynomial to time-delay system stability. The analysis is based on the argument principle which also holds for the selected quasipolynomial. Upper and lower stability bounds for the parameter are found through proven lemmas, propositions and theorems.

Key-Words: - Stability, time-delay systems, characteristic quasipolynomial, argument principle, Mikhaylov criterion.

1 Introduction

It is well known fact that a large number of real-life processes, e.g. in a wide spectrum of natural sciences [1]-[4] or in pure informatics [5], is affected by delays which can have various forms. Linear dynamic systems with distributed, lumped, input-output or even internal delays can be represented by Laplace transfer function as a ratio of so-called *quasipolynomials* [6] in one complex variable [7]-[9], instead of polynomials which are usual in system and control theory. Delay can significantly deteriorate the quality of feedback control performance, namely stability and periodicity.

Control theory has been dealing with problem of delay effect on the feedback system since its nascence – indeed, the well known Smith predictor has been known for longer than five decades [10]. Linear time delay systems in technological and other processes have been usually assumed to contain delay elements in input-output relations only, which results in shifted arguments on the right-hand side of differential equations. However, this conception is somewhat restrictive in effort to fit the real plant dynamics since in many cases inner feedbacks are of distributed or delayed nature, which yield delay elements on the left-hand side of a differential equation. Internal delays also appear in the feedback system when control plants with input-output transport delays, the dynamics of which is characterized using the Laplace transform by the *characteristic quasipolynomial*. This quasipolynomial decides about the control system asymptotic stability because of the fact that its zeros are system poles with the same meaning as for polynomials; however, the number of poles is infinite.

A large number of conference and journal papers were dedicated to stability analysis of systems with delay elements on the left-hand side of a differential equation, e.g. in [7]-[8], [11]-[13]. In this paper, we address the stability analysis of a selected quasipolynomial. In contrast to some other papers, the presented contribution investigates the stability with respect to the single non-delay coefficient and not with respect to the delay. Presented derivations and calculations are based on the fact that the argument principle (i.e. the Mikhaylov criterion) holds for a class of quasipolynomials represented by the studied one as well [7]-[9]. The information about lower and upper bounds on the selectable real parameter can serve engineers to decide quickly about system stability or to set a selectable controller parameter which appears in the characteristic quasipolynomial of a closed loop. Notice that the investigated quasipolynomial was analyzed already e.g. in [13]-[14]; however, these authors utilized different approaches.

2 Argument Principle and Studied Quasipolynomial

For a general *retarded quasipolynomial*

$$m(s) = s^n + \sum_{i=0}^{n-1}\sum_{j=1}^{h_i} m_{ij} s^i \exp\left(-s\vartheta_{ij}\right) \qquad (1)$$

it holds that the number N_U of unstable roots (i.e. those with non-negative real parts) is given by

$$N_U = \frac{n}{2} - \frac{1}{\pi}\Delta \underset{s=\omega j,\omega\in[0,\infty]}{\arg} m(s) \qquad (2)$$

see [7]. It means that the well known argument principle, or the Mikhaylov stability criterion, holds for stable quasipolynomials ($N_U = 0$)

$$\underset{s=\omega\mathrm{j},\omega\in[0,\infty]}{\Delta \arg} m(s) = \frac{n\pi}{2} \tag{3}$$

see also [15].

This observation is a powerful tool for retarded quasipolynomials stability analysis and thus for delayed feedback systems. Note that statement (3) does not say anything about the Mikhaylov curve for *unstable* quasipolynomials. Although formula (2) gives the answer about the overall argument change, calculation of N_U by analytic means, in a general case, is almost impossible. However, one can draw the Mikhaylov curve using software tools and thus to observe its behavior, or to estimate the curve analytically if it is possible. Information about the overall argument shift in unstable case is extremely important e.g. when using the Nyquist criterion for stabilization of internally delayed systems. For example, a first order (n = 1) unstable retarded quasipolynomial can behave in the frequency domain e.g. like an unstable polynomial of the first ($\Delta \arg m(s) = -\pi/2$) or that of the third order with one ($\Delta \arg m(s) = \pi/2$) or three unstable roots ($\Delta \arg m(s) = -3\pi/2$).

The aim of this paper is to analyze the following quasipolynomial with respect to the setting of a free parameter $q \neq 0 \in \circ$

$$m(s) = s + a\exp(-\vartheta s) + kq \tag{4}$$

where $a \neq 0 \in \circ$; $k, \vartheta > 0 \in \circ$. The goal is to find the interval for q so that quasipolynomial (4) is asymptotically stable, whereas all the other parameters are fixed, using the criterion (3). The approach is based on the requirement that the appropriate Mikhaylov curve for $\omega \in [0, \infty]$ must have the overall argument change equal to $\pi/2$, see Fig. 1.

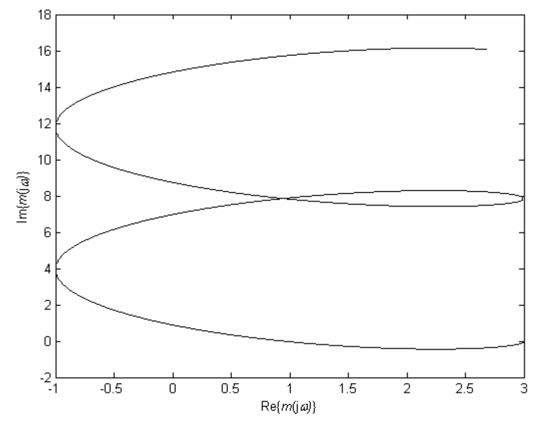

Fig. 1 – The Mikhaylov curve of a stable quasipolynomial (4)

3 Quasipolynomial Properties

Let us study stability properties of quasipolynomial (4) in the form of proven lemmas, propositions and theorems with respect to q. This task is equivalent to the appropriate setting a proportional stabilizing feedback controller q when control a plant with internal delay ϑ.

Lemma 1. For ω = 0, the imaginary part of the Mikhaylov curve of quasipolynomial (4) equals zero and it approaches infinity for $\omega \to \infty$.

Proof. Decompose $m(\mathrm{j}\omega)$ into real and imaginary parts as

$$\mathrm{Re}\{m(\mathrm{j}\omega)\} = a\cos(\vartheta\omega) + kq \tag{5}$$

$$\mathrm{Im}\{m(\mathrm{j}\omega)\} = \omega - a\sin(\vartheta\omega) \tag{6}$$

Obviously

$$\mathrm{Im}\{m(\mathrm{j}\omega)\}\big|_{\omega=0} = 0, \lim_{\omega\to\infty} \mathrm{Im}\{m(\mathrm{j}\omega)\} = \infty. \quad \square$$

Lemma 2. If (4) is stable, the following inequality holds

$$q > \frac{-a}{k} \tag{7}$$

and thus the Mikhaylov curve starts on the positive real axis.

Proof. If (4) is stable, the overall argument shift equals to $\pi/2$ according to (3). Moreover, Lemma 1 states that the imaginary part goes to infinity. These two requirements imply that for stable quasipolynomial is

$$\mathrm{Re}\{m(\mathrm{j}\omega)\}\big|_{\omega=0} > 0 \tag{8}$$

By application of (8) onto (5) yields the condition (7). □

Lemma 2 represents the necessary stability condition and the lower bound for q. The curve can either pass through the first or the fourth quadrant for an infinitesimally small $\omega = \Delta > 0$, which is clarified in the following simple lemma.

Lemma 3. A point on the Mikhaylov curve of (4) lies in the first quadrant for an infinitesimally small $\omega = \Delta > 0$ iff

$$a\vartheta \leq 1 \tag{9}$$

This point lies in the fourth quadrant iff

$$a\vartheta > 1 \tag{10}$$

Proof. (Necessity.) If the point on the curve goes to the first quadrant for an infinitesimally small $\omega = \Delta > 0$, then the change of function $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ in $\omega = 0$ is positive or this function is increasing in $\omega = \Delta$. It is known fact that this is satisfied if either

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\bigg|_{\omega=0} > 0 \tag{11}$$

or there exists *even* $n \in {}^2$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0}=\ldots=\left.\frac{\mathrm{d}^{n-1}}{\mathrm{d}\omega^{n-1}}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0}=0,$$
$$\left.\frac{\mathrm{d}^{n}}{\mathrm{d}\omega^{n}}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0}>0 \qquad (12)$$

(i.e. there is a local minimum of $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ in $\omega=0$)

or there is *odd* $n \geq 3 \in^2$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0}=\ldots=\left.\frac{\mathrm{d}^{n-1}}{\mathrm{d}\omega^{n-1}}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0}=0,$$
$$\left.\frac{\mathrm{d}^{n}}{\mathrm{d}\omega^{n}}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0}\neq 0,\ \left.\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=\Delta}>0 \qquad (13)$$

(i.e. there is a point of inflexion of $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ in $\omega=0$; however, the function is increasing in $\omega=\Delta$).

Analyze the previous three conditions. First, relation (11) w.r.t. (6) reads

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0}=1-a\vartheta\cos(\vartheta\omega)|_{\omega=0} = 1-a\vartheta>0 \qquad (14)$$

which is satisfied for $a\vartheta<1$.

Second, condition (12) can be taken into account if

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\omega=0}=0 \Leftrightarrow a\vartheta=1 \qquad (15)$$

hence

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\substack{\omega=0\\ a\vartheta=1}}=0,\ \left.\frac{\mathrm{d}^3}{\mathrm{d}\omega^3}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\substack{\omega=0\\ a\vartheta=1}}>0 \qquad (16)$$

where the least non-zero *n*th derivation is odd, and thus (12) can not be satisfied for $a\vartheta=1$; however, we can test (13). Indeed

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\{m(\mathrm{j}\omega)\}\right|_{\substack{\omega=\Delta\\ a\vartheta=1}}>0 \qquad (17)$$

and thus function $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ in $\omega=\Delta$ is increasing.

Similarly, one can easily verify that if the Mikhaylov plot pass through the fourth quadrant first, then function $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ decreases in $\omega=0$ when (10) holds .

(Sufficiency.) If conditions (9) or (10) are considered, particular derivations of $\mathrm{Im}\{m(\mathrm{j}\omega)\}$ can be calculated, which guarantee, according to (11) – (13), whether there is a tendency of the Mikhaylov curve to go to the first or the fourth quadrant, respectively. □

Lemma 4. If the lower bound (7) holds and *a*, *k*, *q* are bounded, then $\mathrm{Re}\{m(\mathrm{j}\omega)\}$ is bounded for all $\omega > 0$.

Proof. Assume that $a>0$. Then

$$\begin{aligned}-2a<-a+kq\leq \mathrm{Re}\{m(\mathrm{j}\omega)\}\\ =a\cos(\vartheta\omega)+kq\leq a+kq\end{aligned} \qquad (18)$$

On the other hand, if $a < 0$

$$0<a+kq\leq \mathrm{Re}\{m(\mathrm{j}\omega)\}\leq -a+kq<2kq \qquad (19)$$

where the left-hand sides of (18) and (19) and the right-hand one of (19) employ condition (7). The case when *a* = 0 can be discarded due to definition (4) of the quasipolynomial. □

The requirement of bounded parameters is natural with regard to the their physical meaning as process quantities or controller gains.

Lemma 5. If (7) holds, there it exists an intersection of the Mikhaylov plot with the imaginary axis for some $\omega > 0$ iff

$$a>0 \text{ and } |kq|\leq a \qquad (20)$$

Proof. (Necessity.) Show a contradiction, hence if $a<0$ and (7) holds, then $0<a+kq\leq \mathrm{Re}\{m(\mathrm{j}\omega)\}$ according to Lemma 4 and thus there is no intersection with the imaginary axis.

(Sufficiency.) Consider $a>0$. If $|kq|\leq a$, there must exists $\omega > 0$ such that $a\cos(\vartheta\omega)=kq$, hence, $\mathrm{Re}\{m(\mathrm{j}\omega)\}=0$. □

Searching of the stability upper bound will be made in two branches, so that conditions (9) and (10) are solved separately. The following theorem presents the necessary and sufficient stability condition for the former case.

Theorem 1. If (9) holds, then asymptoticaly quasipolynomial (4) is stable iff condition (7) is satisfied.

Proof. (Necessity.) See Lemma 2.

(Sufficiency.) Lemma 2 indicates that if (7) is satisfied, the Mikhailov curve starts for $\omega=0$ on the positive real axis. According to Lemma 1 the imaginary part of the curve goes to infinity and Lemma 4 states that for bounded parameters, the curve is bounded in the real axis. Now for the stability it is sufficient to certify that for $a\vartheta\leq 1$ the Mikhailov plot does not leave either the first and the fourth quadrant, or the first and the second quadrant, since then the overall phase shift is $\pi/2$.

Indeed, Lemma 4 and Lemma 5 state that if $a<0$, there is no intersection with the imaginary axis and thus the plot lies in the first and the fourth quadrant. Otherwise, if $0<a\leq 1/\vartheta$, an intersection with the imaginary axis can exist because of Lemma 5. Thus, it ought to be verified that there is no intersection with the real axis. Consider two cases:

1) If $\sin(\vartheta\omega)\geq 0,\ \omega>0$, then

$$\mathrm{Im}\{m(\mathrm{j}\omega)\} = \omega - a\sin(\vartheta\omega) \geq \omega - \frac{\sin(\vartheta\omega)}{\vartheta} = \omega\left(1 - \frac{\sin(\vartheta\omega)}{\vartheta\omega}\right) > 0 \quad (21)$$

2) If $\sin(\vartheta\omega) < 0,\ \omega > 0$, we induce a contradiction. Hence, assume that there exists $\omega > 0$ such that $\sin(\vartheta\omega) < 0$ and $\mathrm{Im}\{m(\mathrm{j}\omega)\} = 0$. Then

$$a = \frac{\omega}{\sin(\vartheta\omega)} \quad (22)$$

which yields $\sin(\vartheta\omega) > 0$ and thus we have a contradiction. □

Now consider the second case, i.e. $a\vartheta > 1$. The following result reinforces condition (7).

Definition 1. Let (7) holds. The *crossover frequency* ω_C is defined as

$$\omega_C := \min\{\omega : \omega > 0, \mathrm{Im}\{m(\mathrm{j}\omega)\} = 0\} \quad (23)$$

for some $a \neq 0,\ \vartheta > 0$. In other words, it represents the least solution of (22).

Theorem 2. If (10) holds, then quasipolynomial (4) is asymptotically stable iff

$$q > \frac{-a\cos(\vartheta\omega_C)}{k} \quad (24)$$

Proof. (Necessity.) Lemma 1 and Lemma 2 state that the Mikhaylov curve for stable quasipolynomial (4) starts on the positive real axis. Condition (10) guaranties that the initial change of the curve in the imaginary axis is negative, see Lemma 3. Thus, the curve has to pass through the fourth followed by the first quadrant. In other words, the first crossing with the real axis on the frequency $\omega_C > 0$ has to satisfy

$$\begin{aligned} \mathrm{Im}\{m(\mathrm{j}\omega_C)\} &= \omega_C - a\sin(\vartheta\omega_C) = 0 \\ \mathrm{Re}\{m(\mathrm{j}\omega_C)\} &= a\cos(\vartheta\omega_C) + kq > 0 \end{aligned} \quad (25)$$

which gives (24) directly.

(Sufficiency.) If (10) holds, then $a > 0$ and

$$q > \frac{-a\cos(\vartheta\omega_C)}{k} \geq \frac{-a}{k} \quad (26)$$

and thus the Mikhaylov curve for quasipolynomial (4) starts on the positive real axis according to Lemma 2 and the initial change of the curve in the imaginary axis is negative, see Lemma 3. Condition (24) then agrees with the fact that the curve crosses positive real axis first, as it is obvious from (6). Since the curve is bounded in the real part and the imaginary part goes to infinity (see Lemma 1 and Lemma 4), the overall phase shift is $\pi/2$ and thus the quasipolynomial is stable. □

4 Example

Let us demonstrate the utilization of the presented quasipolynomial stability properties in a control problem. Consider a simple feedback loop with an unstable delayed plant giving rise to the transfer function

$$G(s) = \frac{1}{s + 2\exp(-0.8s)} \quad (27)$$

The plant is controller by a proportional controller (gain) q. The characteristic equation of the feedback loop is

$$m(s) = s + 2\exp(-0.8s) + q \quad (28)$$

hence, in the comparison with (4), $a = 2,\ \vartheta = 0.8$. Since (10) holds, the necessary and sufficient asymptotic stability condition is given by (24). The corresponding crossover frequency can be found as $\omega_C \doteq 2$ [rad·s^{-1}] and thus the closed loop is stable iff $q > 5.71 \cdot 10^{-2}$. Choose e.g. $q = 1$, then the corresponding Mikhaylov plot of the characteristic quasipolynomial is displayed in Fig. 1. The stabilized closed loop step response can be seen in Fig. 2.

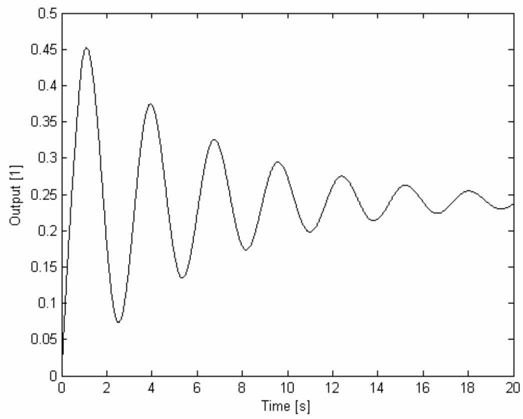

Fig. 2 – The closed loop step response for plant (27) and controller q = 1

4 Conclusion

In this contribution, we have addressed the stability analysis of a selected first order quasipolynomial. The aim has been to find acceptable upper and lower limits for a non-delay parameter. The analysis has been based on the argument principle, i.e. the Mikhaylov stability criterion, in order to keep the desired shape of the Mikhaylov curve. The task can be also comprehend as a suitable setting of a proportional stabilizing controller when controlling a delayed (anisochronic) system, because of the fact that the characteristic closed loop quasipolynomial has the form of the studied one. A simulation example figures the Mikhaylov plot of a stable characteristic quasipolynomial together with control stabilized responses. The analytic tools utilized in this contribution can be employed when studying other quasipolynomials as well.

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