Nonlinearly Coupled Oscillators and State Space Energy Approach

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Abstract: - The presented contribution is motivated by some fundamental questions arising in theoretical analysis of nonlinear state space energy exchange. Knowledge of principles of nonlinear dynamics and synthesis of chaotic attractors is found to be very important also for other potential applications, such as encryption by secure communication, modeling of nonlinear phenomena in power networks, modeling in biomedical engineering, etc. This provides a strong motivation for the current research on exploiting some new chaotic attractors and their implementations. In this paper, an electronic circuit was designed and built to confirm typical behavior of a class of nonlinearly coupled chaotic oscillators. It was explained theoretically, as well as demonstrated by computer simulation combined with laboratory experiments that some typical chaotic phenomena can appear as a consequence of irregular energy exchange between two coupled oscillators. In our experiments, the system consists of a 2-nd order nonlinear antidissipative subsystem nonlinearly coupled with a linear oscillator with dissipation.

Key-Words: Chaotic systems, signal energy, structure, state space energy, nonlinear system, oscillators, dissipation, simulation.

1 Introduction
In this paper, a simple nonlinear electrical circuit consisting of two coupled subsystems as depicted in the Fig.1. is investigated. The first subsystem is nonlinear antidissipative LC oscillator and the other is linear and dissipative. The negative impedance converter was used as a negative resistance element depicted in the Fig. 2., thus giving rise to undamped oscillations in the first oscillator. Both the LC circuits are coupled by a nonlinear element $N$.

![Fig. 1. The circuit diagram of nonlinearly coupled oscillators.](image1)

![Fig. 2. Measured characteristic of nonlinear negative resistance.](image2)

2 Energy based approach
For the theoretical analysis as well as for computer simulation experiments the generalized structure given by Fig.3 has been considered [1 – 6].

![Fig. 3. Generalized structure of two nonlinearly coupled systems.](image3)

For the theoretical investigation some basic results of the state space energy based approach is used [1, 7, 8]. The abstract state equations of the corresponding state space representation are given in the form:

$$\begin{align*}
\dot{x}_1 &= \alpha_1 x_1 - k_1 x_1^3 + \alpha_2 x_2 - N(x_1, x_3, p) \\
\dot{x}_2 &= -\alpha_2 x_1 \\
\dot{x}_3 &= -\alpha_3 x_3 + \alpha_4 x_4 - N(x_1, x_3, p) \\
\dot{x}_4 &= -\alpha_4 x_3
\end{align*}$$

where $N(x_1, x_3, p)$ represents a coupling nonlinearity.
The internal structure of the abstract state space system representation is illustrated in the Fig. 4.

Fig. 4. Structure of the state space representation.

The state space energy of system (1) is given by [9]:

$$E = \frac{1}{2} \sum_{i=1}^{n} x_i^2$$

(2)

where $n=4$ is the order of the system representation.

Two specific forms of the coupling nonlinearity $N$ have been chosen for comparison. The first one is the following diode-like nonlinearity

$$N = k_{N1} [x_j - x_i - p + \text{abs}(x_i - x_j - p)]$$

(3)

specified by a threshold parameter $p$. Alternatively, the dead zone-like nonlinear function, generating zero output within a specified region, was considered.

In this paper, the lower and upper limits of the dead zone are the same and can be fixed by a free parameter $p$. Dead zone nonlinear function is defined by eq. (4):

$$N = k_{N2} [(x_j - x_i - p) + \text{abs}(x_i - x_j - p) - (x_j - x_i + p) - \text{abs}(x_i - x_j + p)] = k_{N2} [2 * (x_j - x_i) + \text{abs}(x_i - x_j - p) - \text{abs}(x_i - x_j + p)]$$

(4)

The both nonlinearities are shown in the Fig. 5.

Fig. 5. The nonlinear functions: diode (top), dead zone (bottom).

As an example of the nonlinear system behavior with diode-like coupling nonlinearity a typical course of the state vector components is illustrated by computer simulation results, shown in the Fig. 6.

Fig. 6. Typical example of the time evolution of state variables.

It is worthwhile to notice that the time evolution of the first two state variables in the Fig. 6 clearly shows the irregularly repeating switching intervals of typical anti-dissipative behavior of the first subsystem, as illustrated in the upper part of the Fig. 7, [10, 11, 12].
Similarly the lower part of the Fig. 7 illustrates the course of state variables, characterizing the typical behavior of the linear dissipative subsystem. It is the matter of fact that the state space energy of the isolated dissipative subsystem would monotonically decrease until the corresponding state would reach its equilibrium state. It is easy to understand that an irregular interruption of this process is the dominating effect of the coupling nonlinearity. Similarly we could ask why the state space energy of the anti-dissipative subsystem does not go to infinity. It is quite plausible that the cubic feedback nonlinearity prevents such a situation and holds the state variables bounded in a finite region of the state space $X$.

For better understanding of the energy based considerations a typical course of the total state space energy, gained by computer simulation, is shown in the Fig. 8.

Mathematically, the rate of change of the total state space energy may be expressed by dual product of the energy gradient field and the vector field of the state space velocity as follows:

$$\frac{dE}{dt}\bigg|_{x=f(x)} = \langle \text{grad}_x E, \dot{x} \rangle \bigg|_{x=f(x)}$$

Using (2) and the state equations (1) we obtain:

$$\frac{dE}{dt}\bigg|_{x=f(x)} = x_i \left[ \alpha \dot{x}_i - k \dot{x}_i^3 + \alpha \dot{x}_j \right] - \alpha \ddot{x}_i$$

$$- \alpha \ddot{x}_i = x_i \left[ - \alpha \dot{x}_i + \alpha \dot{x}_j \right] - (\dot{x}_i + \dot{x}_j) N(x_i, x_j, p)$$

and so we get

$$\frac{dE}{dt}\bigg|_{x=f(x)} = -P_0(t) =$$

$$= \alpha \left( 1 - \frac{k}{\alpha} \right) x_i^2 - \alpha \ddot{x}_i - (\dot{x}_i + \dot{x}_j) N(x_i, x_j, p)$$

where $P_0(t)$ denotes instantaneous value of the total output dissipation power of the system including the non-linear coupling.

The negative taken value of the first term

$$P_{0,1}(t) = -\alpha \left( 1 - \frac{k}{\alpha} \right) x_i^2, \quad \alpha > 0, \quad k > 0$$

expresses the instantaneous value of the “dissipation” power of the nonlinear anti-dissipative subsystem [13 – 17].

It follows that near the equilibrium state $x_i^* = 0$ the isolated nonlinear subsystem would behave as locally unstable, or anti-dissipative, while for $x_i(t)$ far from the origin the increased state space energy will be bounded by a very strong nonlinear dissipation.

Notice that the dissipation power $P_{0,1}(t)$ vanishes not only in the zero equilibrium point $x_i(t) = x_i^* = 0$, but also in the “equilibrium states” for which it holds:

$$|x_i(t)| = \sqrt{\frac{\alpha}{k_i}}$$

The quadratic term $P_{0,2}(t) = \alpha \dot{x}_i^2$ corresponds to the isolated linear dissipation subsystem and is always positive for $x_i \in \mathbb{R}$, $a > 0$, and $x_i \neq x_i^* = 0$.

Recall that the equilibrium states of the system representation (1) are defined by the set of algebraic equations

$$0 = \alpha \dot{x}_1 - k \dot{x}_1^3 + \alpha \dot{x}_2 - N(x_1, x_j, p)$$

$$0 = -\alpha \dot{x}_2$$

$$0 = -\alpha \dot{x}_3 + \alpha \dot{x}_4 - N(x_j, x_j, p)$$

$$0 = -\alpha \dot{x}_4$$

where the diode-like nonlinearity $N$ is described by the eq. (3), and the dead zone-like nonlinearity by (4).
Using (3) it follows from (10):

\[
\begin{align*}
\alpha_1 \left( 1 - \frac{k_1}{\alpha_1} x_i^2 \right) x_i + \alpha_2 x_2 &= w(t) \\
\alpha_2 x_1 &= 0 \\
-\alpha_3 x_3 + \alpha_4 x_4 &= w(t) \\
\alpha_4 x_3 &= 0
\end{align*}
\]

(11)

where

\[
\begin{align*}
w(t) &= k_{N1} v(t) \left[ I + \text{sign}(v(t)) \right] \\
v(t) &= x_1 - x_3 - p
\end{align*}
\]

(12)

Using (12), the last term in (7) reads:

\[
\begin{align*}
P_n(t) &= (x_i + x_j) N(x_i, x_j, p) \\
&= k_{N1} (x_i^2 - x_j^2) \left[ I + \text{sign}(v(t)) \right] \\
&= -p k_{N1} (x_i + x_j) \left[ I + \text{sign}(v(t)) \right]
\end{align*}
\]

(13)

and by substitution into (7) we get the value of the total output power \( P_0(t) \) in the modified form:

\[
P_0(t) = P_{1,0}(t) + P_{2,0}(t) + P_n(t) = \\
\begin{align*}
-\alpha_1 \left[ 1 - \frac{k_{N1}}{\alpha_1} \left[ I + \text{sign}(v(t)) \right] - \frac{k_1}{\alpha_1} x_i^2 \right] x_i^2 \\
+ \alpha_3 \left[ 1 - \frac{k_{N1}}{\alpha_3} \left[ I + \text{sign}(v(t)) \right] - \frac{k_1}{\alpha_3} x_i^2 \right] x_i^2 \\
+ p k_{N1} \left[ I + \text{sign}(v(t)) \right] (x_i + x_j)
\end{align*}
\]

(14)

where

\[
\begin{align*}
1 + \text{sign}(v(t)) &= \begin{cases} 
0 & \text{if } x_i \geq x_j + p \\
2 & \text{if } x_i < x_j + p
\end{cases}
\end{align*}
\]

(15)

and thus it follows

\[
w(t) = \begin{cases} 
0 & \text{if } v(t) \leq 0 \\
2 k_{N1} v(t) & \text{if } v(t) > 0
\end{cases}
\]

(16)

Summarizing the resulting conditions for equilibrium, we get:

\[
\begin{align*}
\alpha_2 \neq 0 &\Rightarrow x_i^* = 0 \\
\alpha_4 \neq 0 &\Rightarrow x_j^* = 0 \\
\alpha_4 \neq 0 &\Rightarrow x_j^* = \frac{1}{\alpha_4} w(t) \\
\alpha_2 \neq 0 &\Rightarrow x_i^* = \frac{1}{\alpha_2} w(t)
\end{align*}
\]

(17)

and hence for \( v(t) > 0 \) we get

\[
\begin{align*}
x_i^* = 0; &\quad x_i^* = \frac{2 k_{N1}}{\alpha_2} \\
x_j^* = 0; &\quad x_j^* = \frac{2 k_{N1}}{\alpha_4}
\end{align*}
\]

(18)

and hence for \( v(t) \leq 0 \) the unique zero equilibrium state \( x^* = 0 \) follows.
In order to confirm the chaotic nature of the system trajectories by Liapunov exponents as a broadly used standard tool, the four Liapunov exponents have been computed and are displayed in the Fig.12. Because two of them are positive not only chaotic but even hyperchaotic behavior has been proven.

\[ Z(s) = \frac{sC_1R_1R_2}{R_3} \]  

(19)

The synthesis inductor and the $C_1$ is parallel resonance circuit. The OA3 is used as negative impedance converter for negative resistance realization with input resistance (until saturation of OA output):

\[ R_{inp} = -\frac{R_3}{R_6} \]  

(20)

The rest, circuit around OA4 presents bandpass filter with gain controlled by R7, R8, and Pot. R7 for slope of negative resistance and Pot. R11 for dissipation of 2'nd oscillator [18, 19, 20].

3 System realization

In order to be able to verify the theoretical results by a real-world system realization, the state equivalent electronic system implementation has been build. Its circuit structure is shown in Fig. 13. The electronic circuit consists of 4 JFET operational amplifiers (OA) in one package (TL074). The OA1 and OA2 were used for synthetic inductor. The impedance of the circuit is given by [18]:

\[ N \text{-nonlinear element.} \]

OA TL074. C1=C2=C3=C4=33 nF, R1=R2=R3=1 k, R4=10 kΩ, R5=R6=10 kΩ, R7=47kΩ, R8=680 Ω, R9=1k5, R10=M12, R11=M33
4 DIGITAL VERSION OF SYSTEM
The digital version of the system was also tessimulated (by Simulink). The block diagram is shown in Fig. 16., phase portrait in 2-D is shown in Fig. 17. State space waveforms $x_1$ and $x_3$ are shown in Fig. 18. The time evolution of the total state space energy is shown in Fig. 19 and time evolution of the state space energy of both the nonlinearly coupled subsystems – dead zone-like coupling is displayed in Fig. 20. From this Figures can be seen that analog and digital systems are similar [21, 22, 23].

5 CONCLUSION
The simulation results and experimental electronic circuit measuring gives good agreement. The theoretical results based on energy approach have been confirmed by electronic circuit construction. This circuit with different modification can be used in variety of practical applications in which effects of nonlinear energy exchange can’t be neglected.

As typical application field’s especially cardiovascular system simulation, secured communication systems and/or modern control systems of power networks can be expected in the near future.
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