

A Fuzzy Differential Approach to strong Allee Effect based on the Fuzzy Extension Principle

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Abstract: The Allee effect is related to those aspects of dynamical of populations connected with a decreasing in individual fitness when the population size diminishes to very low levels. In this work we propose a fuzzy approach to Allee effect that permits to deal with uncertainty. This fuzzy proposal is based on the extension principle and shows a different behaviour of the equilibrium points with regard to the crisp model.

Key-Words: Allee Effect, Fuzzy Number, Level Sets, Extension Principle, Fuzzy Differential Equation

1 Introduction

In single population dynamics the rate of change of the number of individuals $x(t)$ is often expressed as a function of the per capita rate of change $\varphi(x)$ such as

$$\frac{dx}{dt} = x\varphi(x) \quad (1)$$

The Malthus equation considers a constant per capita growth and is the simplest way to model the growth of a specie. This equation has only one equilibrium point at $x = 0$ and the population tends to infinity if $k > 0$ (0 is unstable) or vanishes if $k < 0$ (0 is stable). This point of view implies an exponential growth of the population.

The logistic equation

$$\frac{dx}{dt} = g(x) = rx\left(1 - \frac{x}{k}\right) \quad (2)$$

solves the question of the existence of a positive equilibrium point because $x = k$ is an asymptotically stable point and the size of the population tends to k . Figure 1 describes this very well known equation.

One feature of Equation (2) often criticized in ecological dynamics is that for small values the rate growth is always positive. Several equations have been proposed to deal with this problem. We will focus our attention to the strong Allee effect what implies that bellow a positive number the population goes to extinction. A differential equation that governs the population growth taking into account these aspects is

$$\frac{dx}{dt} = f(x) = rx\left(1 - \frac{x}{k}\right)(x - a) \quad (3)$$

Equation 3 has three equilibrium points, two are stable (0 and k) and one is unstable (a). Figure 1 shows the behaviour of function f and its differences with the logistic equation.

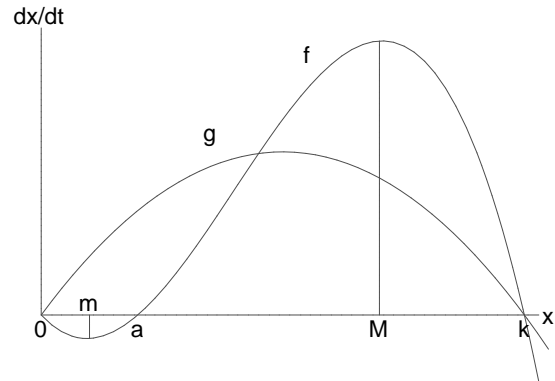


Figure 1: Logistic (g) and Allee (f) functions

The per capita growth of the logistics equation is a linear decreasing function instead of the Allee effect is increasing for small values although negative. This remark is reflected in Figure 2 where v and u are the per capita growth of the logistic and Allee effect respectively.

The Allee effect has strong evidence in ecological systems ([7]) and is associated to a wide range of systems focusing in conspecific interactions, rarity and animal sociality ([2, 3, 19, 20]). Conspecific interaction can help to understand aggregative behaviour ([18]), Allee effect can explain some proprieties about the distribution of individuals between patches ([11]), it is obviously related to extinction ([16]) and influ-

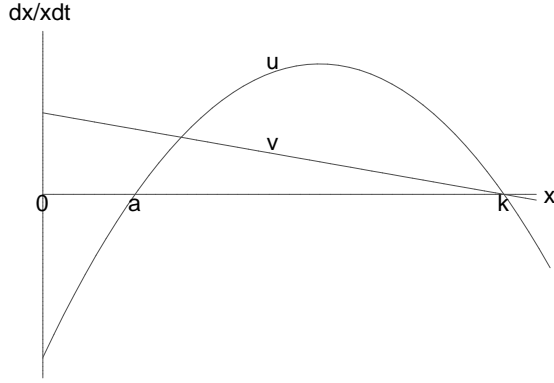


Figure 2: Per capita growth for logistic (v) and Allee (u) equations

ence any system of maximum sustainable yields as some fisheries ([13]). Bidimensional models also benefit with the introduction of the Allee effect ([1]).

2 Fuzzy analysis based on the extension principle

Environmental and economical systems are usually subject to uncertainty which affects the mathematical structure of them. References dealing with theoretical or applied ([17]) fuzzy differential equations show a great number of subjects of interest and points of view. J.J. Buckley et al, P Diamond and J.J. Nieto deal with first order differential equations ([5, 9, 10, 15]), Buckley et al and D.N. Georgiou et al with high order differential equations ([6, 12]) and for a more applied focus related to this work we cite ([4, 8, 14]).

In our context we must consider that the population is given by a fuzzy real number $\tilde{x}(t)$ for every t identifying the symbol of its membership function with itself in order to simplify the notation, so satisfying

- Normality: $\exists s_0 \in \mathbf{R}$ such that $\tilde{x}(t)(s_0) = 1$
- $\tilde{x}(t)$ is convex
- $\tilde{x}(t)$ is upper-continuous
- The support of $\tilde{x}(t)$, $[\tilde{x}(t)] = \{s : \tilde{x}(t)(s) > 0\}$ is a compact set

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a real function. Given a fuzzy number $\tilde{x} \subseteq \mathbf{R}$ then the fuzzy extension \tilde{f} of f following the Zadeh's extension principle ([21]) is defined as

$$\tilde{f}(\tilde{x})(s') = \begin{cases} \sup_{s \in f^{-1}(s')} \{\tilde{x}(s)\} & \text{if } f^{-1}(s') \neq \emptyset \\ 0 & \text{if } f^{-1}(s') = \emptyset \end{cases} \quad (4)$$

and the level sets (or α -levels with $\alpha \in [0, 1]$) of the fuzzy number $\tilde{f}(\tilde{x}(t)) = \tilde{y}(t)$ take the known form $\tilde{y}(\alpha, t) = [\underline{y}(\alpha, t), \overline{y}(\alpha, t)]$ with

$$\begin{aligned} \underline{y}(\alpha, t) &= \min\{f(s) | s \in [\underline{x}(\alpha, t), \overline{x}(\alpha, t)]\} \\ \overline{y}(\alpha, t) &= \max\{f(s) | s \in [\underline{x}(\alpha, t), \overline{x}(\alpha, t)]\} \end{aligned} \quad (5)$$

In order to simplify the notation we note $\underline{x}(\alpha, t) = x_1$ and $\overline{x}(\alpha, t) = x_2$.

It is simple to show that for the Malthus fuzzy equation and $k > 0$ the trajectories tend to infinite. On the other way, if $k < 0$ the trajectories tend to the line $x_1 = 0$.

For the fuzzy logistic equation J. J. Nieto et al divided the study of the trajectories in three regions showing a complex behaviour. As general rule the trajectories leave one region and enter into another region, but can also tend to infinite. This behaviour depends on the initial conditions ([14]).

In the case of the Allee effect the autonomous equation has three equilibrium points and as a consequence increases the number of regions. Moreover, in most cases it is not possible to get an explicit expression of the trajectories and only an implicit expression is possible. All that makes more complex the study of the fuzzy Allee effect.

The real function $f(x)$ attains a minimum at m (we note $f(m) = p$) between 0 and a and a maximum at M (we note $f(M) = q$) between a and k following the expressions obtained from $f'(x) = 0$.

Applying the Zadeh's extension principle given by (4) six cases arises depending on x_1 and x_2 belong to the zones $Z_1 = [0, m]$, $Z_2 = (m, M]$ or $Z_3 = (M, +\infty)$. This zones lead to the study of six regions

$$R_1 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \in Z_1 \wedge x_2 \in Z_1\}$$

$$R_2 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \in Z_1 \wedge x_2 \in Z_2\}$$

$$R_3 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \in Z_1 \wedge x_2 \in Z_3\}$$

$$R_4 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \in Z_2 \wedge x_2 \in Z_2\}$$

$$R_5 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \in Z_2 \wedge x_2 \in Z_3\}$$

$$R_6 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \in Z_3 \wedge x_2 \in Z_3\}$$

The fuzzy first order differential equation is transformed to a bidimensional system of differential equations which variables are x_1 and x_2 . In the next section we analyze the trajectories of this system.

3 Solution of the Fuzzy Allee equation

From the definition of the system it is clear that the solutions depend on which region the initial conditions belong. When it is possible we get the solutions, if not, we obtain the trajectories and explain their behaviour. We consider as initial conditions $x_1(t_0) = x_{10}$ and $x_2(t_0) = x_{20}$.

3.1 Solution of $\tilde{x}' = \tilde{f}(\tilde{x})$ in region R_1

In this case $f(x_1) \geq f(x_2)$ so

$$\begin{cases} [x'_1, x'_2] = [f(x_2), f(x_1)] = \\ \begin{cases} x'_1 = \frac{r}{k}x_2(k - x_2)(x_2 - a) \\ x'_2 = \frac{r}{k}x_1(k - x_1)(x_1 - a) \end{cases} \end{cases} \quad (6)$$

We get a non linear system of differential equations. Dividing both equations and integrating we obtain the implicit solution of the trajectories:

$$\begin{aligned} \frac{x_1^4}{4} - (k + a)\frac{x_1^3}{3} + ak\frac{x_1^2}{2} &= \\ = \frac{x_2^4}{4} - (k + a)\frac{x_2^3}{3} + ak\frac{x_2^2}{2} + C \end{aligned} \quad (7)$$

We obtain the value of C imposing the initial conditions

$$\begin{aligned} C &= \frac{x_{10}^4}{4} - (k + a)\frac{x_{10}^3}{3} + ak\frac{x_{10}^2}{2} - \frac{x_{20}^4}{4} + \\ &+ (k + a)\frac{x_{20}^3}{3} - ak\frac{x_{20}^2}{2} \end{aligned}$$

There are different possibilities for the behaviour of the trajectories that we summarize as follows

- Behaviour starting at Region R_1
 - If $x_{10} = x_{20}$ then the trajectories tend to the point $(0, 0)$.
 - If $x_{10} = 0$ and $x_{20} > 0$ then the trajectories leave the region R_1 .
 - Otherwise the trajectories tend to points belonging to the ordinates axe.

3.2 Solution of $\tilde{x}' = \tilde{f}(\tilde{x})$ in region R_2

$$[x'_1, x'_2] = [m, \max\{f(x_1), f(x_2)\}]$$

We need to distinguish two cases.

Case A: $f(x_1) \geq f(x_2)$

$$\begin{cases} x'_1 = p \\ x'_2 = \frac{r}{k}x_1(k - x_1)(x_1 - a) \end{cases} \quad (8)$$

Solving the trajectories we get

$$\frac{x_1^4}{4} - (k + a)\frac{x_1^3}{3} + ak\frac{x_1^2}{2} = \frac{kp}{r}x_2 + C$$

Applying the initial conditions we obtain

$$C = \frac{x_{10}^4}{4} - (k + a)\frac{x_{10}^3}{3} + ak\frac{x_{10}^2}{2} - \frac{kp}{r}x_{20}$$

- Behaviour starting at Region R_2 (Case A)

- If $x_{10} = 0$ then the trajectories leave the region.
- If $x_{20} = a$ and $x_{10} > 0$ then the trajectories tend to the point $(0, a)$.
- Otherwise some trajectories tend to the ordinates axe and others enter the region R_1 .

Case B: $f(x_1) < f(x_2)$

$$\begin{cases} x'_1 = p \\ x'_2 = \frac{r}{k}x_2(k - x_2)(x_2 - a) \end{cases} \quad (9)$$

which is an uncoupled system of differential equations. Solving it

$$\begin{cases} x_1 = pt + C_1 \\ \frac{(x_2 - a)^k}{x_2^{k-a}(k - x_2)^a} = C_2 e^{ar(k-a)t} \end{cases} \quad (10)$$

Imposing the initial conditions we obtain

$$\begin{aligned} C_1 &= x_{10} - pt_0 \\ C_2 &= \left(\frac{x_{20} - a}{x_{20}}\right)^k \left(\frac{x_{20}}{k - x_{20}}\right)^a e^{-ar(k-a)t_0} \end{aligned}$$

Finally, the trajectory is

$$x_1 = \ln \left(\frac{(x_2 - a)^k x_{20}^{k-a} (k - x_{20})^a}{x_2^{k-a} (x_{20} - a)^k (k - x_2)^a} \right) + x_{10}$$

- Behaviour starting at Region R_2 (Case B)

- If $x_{10} = 0$ then the trajectories leave the region R_2 .
- If $x_{20} = M$ then the trajectories leave the region R_2 and enter into the region R_3 .
- Otherwise the trajectories keep on the region and some of them tend head for the ordinates axe or to the line $x_2 = M$.

3.3 Solution of $\tilde{x}' = \tilde{f}(\tilde{x})$ in region R_3

$$[x'_1, x'_2] = [\min\{f(x_1), f(x_2)\}, q]$$

We need also to distinguish two cases.

Case A: $f(x_1) < f(x_2)$

$$\begin{cases} x'_1 = pt + C_1 \\ x'_2 = qt + C_1 \end{cases} \quad (11)$$

and imposing the initial conditions we obtain

$$C_1 = x_{10} - pt_0 \text{ and } C_2 = x_{20} - qt_0$$

The trajectory is a line with negative slope

$$x_2 = \frac{q}{p}x_1 + \frac{px_{20} - (1+q)pt_0 - x_{10}}{p}$$

Case B: $f(x_1) \geq f(x_2)$

$$\begin{cases} x'_1 = \frac{r}{k}x_2(k - x_2)(x_2 - a) \\ x'_2 = q \end{cases} \quad (12)$$

Calculating the trajectories we get

$$\frac{x_2^4}{4} - (k+a)\frac{x_2^3}{3} + ak\frac{x_2^2}{2} = \frac{kq}{r}x_1 + C$$

Applying the initial conditions we obtain

$$C = \frac{x_{20}^4}{4} - (k+a)\frac{x_{20}^3}{3} + ak\frac{x_{20}^2}{2} - \frac{kq}{r}x_{10}$$

- Behaviour starting at Region R_3

- If $x_{10} = 0$ then the trajectories leave the region R_3 .
- Otherwise the trajectories tend to the line $x_1 = 0$.

3.4 Solution of $\tilde{x}' = \tilde{f}(\tilde{x})$ in region R_4

$$\begin{cases} [x'_1, x'_2] = [f(x_1), f(x_2)] \\ x'_1 = \frac{r}{k}x_1(k - x_1)(x_1 - a) \\ x'_2 = \frac{r}{k}x_2(k - x_2)(x_2 - a) \end{cases} \quad (13)$$

which is a uncoupled system of differential equations. Solving it

$$\frac{(x_i - a)^k}{x_i^{k-a}(k - x_i)^a} = C_i e^{ar(k-a)t} \quad i = 1, 2$$

Imposing the initial conditions for $i = 1, 2$

$$C_i = \left(\frac{x_{i0} - a}{x_{i0}}\right)^k \left(\frac{x_{i0}}{k - x_{i0}}\right)^a e^{-ar(k-a)t_0}$$

- Behaviour starting at Region R_4

- If $x_{10} = x_{20} < a$ then the trajectories tend to the point (m, m) .
- If $x_{10} = x_{20} > a$ then the trajectories tend to the point (M, M) .
- If $x_{10} = a$ and $x_{20} > a$ then the trajectories tend to the point (a, M) .
- If $x_{10} < a$ and $x_{20} \leq a$ then the trajectories keep on the region. Some of them head on to the line $x_1 = m$ and the others to the line $x_1 = x_2$.
- If $x_{10} < a$ and $x_{20} \geq a$ then the trajectories keep on the region. Some of them head on to the line $x_1 = m$ and the others to the line $x_2 = M$.
- If $x_{10} > a$ and $x_{20} > a$ then the trajectories keep on the region. Some of them head on to the line $x_1 = x_2$ and the others to the line $x_2 = M$.

3.5 Solution of $\tilde{x}' = \tilde{f}(\tilde{x})$ in region R_5

$$[x'_1, x'_2] = [\min\{f(x_1), f(x_2)\}, q]$$

We need to distinguish two cases.

Case A: $f(x_1) < f(x_2)$

$$\begin{cases} x'_1 = \frac{r}{k}x_1(k - x_1)(x_1 - a) \\ x'_2 = q \end{cases} \quad (14)$$

Calculating the trajectories we get

$$\frac{(x_1 - a)^k}{x_1^{k-a}(k - x_1)^a} = C e^{qkr^{-1}x_2}$$

Applying the initial conditions we obtain

$$C = \left(\frac{x_{10} - a}{x_{10}}\right)^k \left(\frac{x_{10}}{k - x_{10}}\right)^a e^{-qkr^{-1}x_{20}}$$

- Behaviour starting at Region R_5 (Case A)

- If $x_{10} < a$ then the trajectories tend to the line $x_1 = m$.
- If $x_{10} > a$ then the trajectories tend to the line $x_1 = M$.

Case B: $f(x_1) \geq f(x_2)$

$$\begin{cases} x'_1 = \frac{r}{k}x_2(k - x_2)(x_2 - a) \\ x'_2 = q \end{cases} \quad (15)$$

which is an uncoupled system of differential equations. Solving it

$$\frac{x_2^4}{4} - (k + a)\frac{x_2^3}{3} + ak\frac{x_2^2}{2} = \frac{kq}{r}x_1 + C$$

Imposing the initial conditions we obtain

$$C = \frac{x_{20}^4}{4} - (k + a)\frac{x_{20}^3}{3} + ak\frac{x_{20}^2}{2} - \frac{kq}{r}x_{10}$$

- Behaviour starting at Region R_5 (Case B)

- If $x_{20} < k$ then the trajectories tend to the line $x_1 = m$.
- If $x_{20} > k$ then the trajectories tend to the line $x_1 = M$.

3.6 Solution of $\tilde{x}' = \tilde{f}(\tilde{x})$ in region R_6

This case is analytically identical to the case 3.1 but the behaviour of the trajectories is different.

- Behaviour starting at Region R_6

- If $x_{10} = x_{20} < k$ then the trajectories tend to the point (k, k) .
- If $x_{10} = x_{20} > k$ then the trajectories tend to the point (k, k) .
- If $x_{10} \neq x_{20}$, $x_{10} < k$ and $x_{20} < k$ then the trajectories tend to the line $x_1 = x_2$ or to the line $x_2 = k$.
- If $x_{10} \leq k$ and $x_{20} > k$ then the trajectories tend to the line $x_1 = M$.
- If $x_{10} > k$ and $x_{20} > k$ then the trajectories tend to the line $x_1 = k$ or to the line $x_1 = x_2$.

4 Qualitative study of the Fuzzy Allee equation

The qualitative study of the fuzzy Malthus equation for $k > 0$ shows the same behaviour that the crisp because there is an unstable equilibrium point. On the other way, if $k < 0$ there is an unstable point unlike the crisp approach in which case the equilibrium point $x = 0$ is stable.

The qualitative study of the fuzzy logistic equation made by J. J. Nieto shows that there are two equilibrium points that, surprising, are unstable ([14]). We

will follow a similar procedure for the study of the fuzzy Alle effect.

The crisp differential equation has three equilibrium points: $e_1 = 0$, $e_2 = a$ and $e_3 = k$. Looking at the Figure 1 it is obvious that e_1 and e_3 are stable (asymptotically) and e_2 is unstable.

In the fuzzy context the differential equation is converted to a system of two crisp differential equations depending on where belong x_1 and x_2 . Solving the homogenous system we find three equilibrium points $E_1 = (0, 0)$, $E_2 = (a, a)$ and $E_3 = (k, k)$ which belong to the region R_1 , R_4 and R_6 respectively. We note $F(x_1, x_2) = (x'_1, x'_2)$.

4.1 Study of the point E_1

As this point belongs to the zone R_1 we get $F(x_1, x_2) = (f(x_2), f(x_1))$ and

$$[x'_1, x'_2] = [f(x_2), f(x_1)] =$$

$$= \left[\frac{r}{k}x_2(k - x_2)(x_2 - a), \frac{r}{k}x_1(k - x_1)(x_1 - a) \right]$$

Linearizing the system around the equilibrium point we obtain

$$J(F(0, 0)) = \begin{pmatrix} 0 & -ar \\ -ar & 0 \end{pmatrix}$$

Solving the secular equation we find its eigenvalues $\lambda_1 = -ar$ and $\lambda_2 = ar$, hence $(0, 0)$ is a saddle point.

4.2 Study of the point E_2

As this point belongs to the zone R_4 we get $F(x_1, x_2) = (f(x_1), f(x_2))$ and

$$[x'_1, x'_2] = [f(x_1), f(x_2)] =$$

$$= \left[\frac{r}{k}x_1(k - x_1)(x_1 - a), \frac{r}{k}x_2(k - x_2)(x_2 - a) \right]$$

Linearizing the system around the equilibrium point we obtain

$$J(F(a, a)) = \begin{pmatrix} ar(k - a)k^{-1} & 0 \\ 0 & ar(k - a)k^{-1} \end{pmatrix}$$

Solving the secular equation we get its eigenvalues $\lambda_1 = \lambda_2 = ar(k - a)k^{-1} > 0$ hence (a, a) is an unstable node.

4.3 Study of the point E_3

As this point belongs to the zone R_6 we get $F(x_1, x_2) = (f(x_2), f(x_1))$ and

$$\begin{aligned} [x'_1, x'_2] &= [f(x_2), f(x_1)] = \\ &= \left[\frac{r}{k} x_2 (k - x_2) (x_2 - a), \frac{r}{k} x_1 (k - x_1) (x_1 - a) \right] \end{aligned}$$

Linearizing the system around E_3 we obtain

$$J(F(k, k)) = \begin{pmatrix} 0 & -r(a - k) \\ -r(a - k) & 0 \end{pmatrix}$$

Similarly at the previous cases we find the eigenvalues $\lambda_1 = -r(a - k) > 0$ and $\lambda_2 = r(a - k) < 0$, hence (k, k) is a saddle point.

5 Conclusion

The study of the strong fuzzy Allee effect under uncertainty from the point of view of fuzzy extension principle shows that fuzziness changes the behaviour of the set of solutions because all equilibrium points are unstable unlike the crisp case.

References:

- [1] P. Aguirre, E. González-Olivares, E. Sáez, Three limit cycles in a Leslie-Gower predator-prey model with additive Allee effect, *SIAM J. Appl. Math.*, 69(5), 2009, pp. 1244–1262.
- [2] W.C. Allee, *The Social Life of Animals*, William Heinemann 1938, pp. 27–33.
- [3] P. Amarasekare, Allee effects in metapopulation dynamics, *Am. Nat.*, 152, 1998, pp. 298–302.
- [4] L.C. Barros, R.C. Bassanezi, P.A. Tonelli, Fuzzy modelling in population dynamics, *Ecological Modelling*, 128, 2000, pp. 27–33.
- [5] J.J. Buckley, T. Feuring, Fuzzy differential equations, *Fuzzy Sets and Systems*, 110, 2000, pp. 43–54.
- [6] J.J. Buckley, T. Feuring, Fuzzy initial value problem for N th-order linear differential equations, *Fuzzy Sets and Systems*, 121, 2001, pp. 247–255.
- [7] F. Courchamp, T. Clutton-Brock, B. Grenfell, Inverse density dependence and the Allee effect, *Trends Ecol. Evol.*, 14, 1999, pp. 405–410.
- [8] E.Y. Deeba, A. de Korvin, E.L. Koh, On a fuzzy logistic difference equation, *Differential Equations Dynamical Systems*, 4, 1996, pp. 149–156.
- [9] P. Diamond, Brief note on the variation of constants formula for fuzzy differential equations, *Fuzzy Sets and Systems*, 129, 2002, pp. 65–71.
- [10] P. Diamond, P. Kloeden, *Metric Spaces of Fuzzy Sets. Theory and Applications*, World Scientific Publishing Co. 1994,
- [11] S.D. Fretwell, *Populations in a Seasonal Environment*, Princeton University Press 1972, pp. 27–33.
- [12] D.N. Georgiou, J.J. Nieto, R. Rodríguez López, Initial value problems for higher-order fuzzy differential equations, *Nonlinear Analysis*, 63, 2005, pp. 587–600.
- [13] R. Lande, S. Engen, B.E. Saether, Optimal harvesting, economic discounting and extinction risk in fluctuating populations, *Nature*, 372, 1994, pp. 88–90.
- [14] J.J. Nieto, R. Rodríguez López, Analysis of a logistic differential model with uncertainty, *Int. J. Dynamical Systems and Differential Equations*, 1(3), 2008, pp. 164–177.
- [15] J.J. Nieto, R. Rodríguez López, D. Franco, Linear first-order fuzzy differential equations, *Internat. J. Uncertain, Knowledge-Based Systems*, 14(6), 2006, pp. 687–709.
- [16] J.C. Poggiale, From behavioural to population level: growth and competition, *Math. Comput. Model.*, 27, 1998, pp. 41–49.
- [17] I.G. Ratiu, C. Carstea, L.P. Doru Plesea, M. Boscoianu, Fuzzy modeling economics, *Proceedings of the 9th WSEAS International conference on simulation, modelling and optimization*, 2009, pp. 51–56.
- [18] J.M. Reed, A.P. Dobson, Behavioural constraints and conservation biology: conspecific attraction and recruitment, *Trends Ecol. Evol.*, 8, 1993, pp. 253–256.
- [19] P.A. Stephens, W.J. Sutherland, Consequences of the Allee effect for behaviour, ecology and conservation, *TREE*, 14(10), 1999, pp. 401–405.
- [20] W.J. Sutherland, The importance of behavioural studies in conservation biology, *Anim. Behav.*, 56, 1998, pp. 801–809.
- [21] L.A. Zadeh, Fuzzy Sets, *Information and Control*, 8, 1965, pp. 338–353.