Complete blow-up for a degenerate semilinear parabolic problem with a localized nonlinear term

P. Sawangtong, B. Novaprateep and W. Jumpen

***Abstract*—**We here establish the local existence and uniqueness of a continuous solution under certain conditions of a degenerate semilinear parabolic problem with a localized nonlinear term: let T be any positive real number and x_0 be a fixed number in the interval $(0,1)$,

$$u_t - \frac{1}{k(x)}\left(p(x)u_x\right)_x = f(u(x_0,t)) \text{ for } (x,t)\in(0,1)\times(0,T),$$
$$u(0,t)=0=u(1,t) \text{ for } t\in(0,T),$$
$$u(x,0)=u_0(x) \text{ for } x\in[0,1],$$

where k, p, f and u_0 are given functions. Moreover, the sufficient condition to blow-up in finite time and the blow-up set of a such solution u are shown.

***Keywords*—**Blow-up in finite time, Blow-up set, Complete blow-up, Localized nonlinear terms, Semilinear parabolic problems

I. Introduction

Without loss of generality and for simplicity, we take the interval of x to [0,1]. Let $I=(0,1)$, $Q_T = I\times(0,T)$, $\bar{I}$ and $\bar{Q}_T$ be the closure of I and Q_T, respectively. We here study the following degenerate semilinear parabolic problem with a localized nonlinear term:

$$\left.\begin{aligned} u_t - \frac{1}{k(x)}\left(p(x)u_x\right)_x &= f(u(x_0,t)) \text{ for } (x,t)\in Q_T,\\ u(0,t)=0&=u(1,t) \text{ for } t\in(0,T),\\ u(x,0)&=u_0(x) \text{ for } x\in\bar{I},\end{aligned}\right\} \quad (1.1)$$

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where u_t denotes partial differentiation of u with respect to t and k, p, f and u_0 are given functions. The purpose of this paper is to prove that before blow-up occurs, there exists a $T_1(>0)$ such that problem (1.1) has a unique nonnegative continuous solution u on the time interval $[0,T_1]$ for any $x\in\bar{I}$. In addition to prove the existence and uniqueness of solution, the sufficient condition to blow up in finite and the blow-up set of such a solution u are given. A solution u of problem (1.1) is said to blows up at $x=b$ in finite time t_b if there exists a sequence (x_n,t_n) with $t_n<t_b$ such that $(x_n,t_n)\to(b,t_b)$ as $n\to\infty$ and $\lim_{n\to\infty}u(x_n,t_n)=\infty$. The set of all blow-up points of solution u is called the blow-up set. In order to obtain our results, throughout this paper, we need following assumptions.

(A) $p\in C^1(\bar{I})$, $p(0)=0$, p is positive on $(0,1]$.

(B) $k\in C(\bar{I})$, $k(0)=0$, k is positive on $(0,1]$.

(C) $f\in C^2[0,\infty)$ is convex with $f(0)=0$ and $f(s)>0$ for $s>0$.

(D) $u_0\in C^2(\bar{I})$, $u_0(0)=0=u_0(1)$, u_0 is nonnegative on I, $u_0(x_0)>0$, and u_0 satisfies

$$\frac{1}{k(x)}\frac{d}{dx}\left(p(x)\frac{du_0(x)}{dx}\right)+f(u_0(x_0))\geq\varsigma u_0(x) \text{ in } I \quad (1.2)$$

for some positive constant ς. By separation of variables, we obtain the corresponding singular eigenvalue problem to (1.1) defined by

$$\left.\begin{aligned}\frac{d}{dx}\left(p(x)\frac{d\varphi(x)}{dx}\right)+\lambda k(x)\varphi(x)=0 \text{ on } I,\\ \varphi(0)=0=\varphi(1).\end{aligned}\right\} \quad (1.3)$$

We note that conditions (A) and (B) implies that the point $x=0$ is a singular point of problem (1.3). By proposition 2.1 [7], condition (C) yields that f is increasing and locally Lipschitz on $[0,\infty)$.

We rewrite equation (1.3) in a new form:

$$\left.\begin{aligned}x^2\varphi''(x)+x\left[x\frac{p'(x)}{p(x)}\right]\varphi'(x)+x^2\left[\lambda\frac{k(x)}{p(x)}\right]\varphi(x)=0 \text{ on } I,\\ \varphi(0)=0=\varphi(1).\end{aligned}\right\} \quad (1.4)$$

We have to add some conditions on functions p and k to make the point $x=0$ to be regular singular point, that is,

(E) The limit of $\dfrac{xp'(x)}{p(x)}$ and $\dfrac{x^2k(x)}{p(x)}$ are finite as $x\to 0$ and $\dfrac{xp'(x)}{p(x)}$ and $\dfrac{x^2k(x)}{p(x)}$ are analytic at $x=0$.

We note that theorem 5.7.1 [1] yields that eigenfunctions φ_n and eigenvalues λ_n of a corresponding singular eigenvalue problem (1.4) exist. Completeness of eigenfunctions φ_n of problem (1.4) follows from next assumption.

(E) $\int_0^1\int_0^1 H(x,\xi)^2 k(x)k(\xi)d\xi dx$ is finite where H is the corresponding Green's function to problem (1.4).

Previously there are mathematicians who studied blow-up problems of parabolic type with a localized nonlinear term. In 1992, J. M. Chadam, A. Peirce and H. M. Yin [2] investigated the blow-up behaviour of solutions to heat equation with a localized reaction term: let Ω be a bounded domain in $\mathbb{R}^n$ and x_0 a fixed point in Ω,

$$\left.\begin{aligned} u_t-\nabla^2u &= f(u(x_0,t)) \text{ for } (x,t)\in\Omega\times(0,T),\\ u(x,t) &= 0 \text{ for } (x,t)\in\partial\Omega\times(0,T),\\ u(x,0) &= u_0(x) \text{ for } x\in\overline{\Omega}, \end{aligned}\right\}$$

(1.5) where f and u_0 are given functions and $\partial\Omega$ and $\overline{\Omega}$ denote boundary and closure of $\Omega,$ respectively. They showed that under some conditions the solution u of problem (1.5) exhibits global blow-up and the blow-up set is $\overline{\Omega}$. In 2000, C.Y. Chan and J. Yang [5] studied the degenerate semilinear parabolic problem with a localized nonlinear term: let q be a nonnegative constant:

$$\left.\begin{aligned} x^qu_t-u_{xx} &= f(u(x_0,t)) \text{ for } (x,t)\in Q_T,\\ u(0,t) &= 0 = u(1,t) \text{ for } t\in(0,T),\\ u(x,0) &= u_0(x) \text{ for } x\in\overline{I}, \end{aligned}\right\} \qquad (1.6)$$

where f and u_0 are given functions. They proved that under certain hypotheses a nonnegative classical solution u of problem (1.6) blows up at all points $x\in\overline{I}$ in finite time. Moreover they gave a sufficient condition for solution a u of problem (1.6) to blow-up in finite time.

II. Local Existenec and Uniqueness

This section deal with the local existence and uniqueness of a nonnegative continuous solution u of problem (1.1). Referred to [8], we have well-know properties of eigenvalues λ_n and eigenfunctions φ_n of problem (1.4) as the following lemma.

Lemma 2.1.

2.1.1. $\int_0^1 k(x)\varphi_n(x)\varphi_m(x)dx=\begin{cases}1 & \text{for } m=m,\\ 0 & \text{for } m\neq n.\end{cases}$

2.1.2. All eigenvalues are real and positive.

2.1.3. Eigenfunctions are complete with the weight function k.

2.1.4. $\lambda_1<\lambda_2<\lambda_3<\ldots$ and $\lim_{n\to\infty}\lambda_n=\infty.$

2.1.5. $\int_0^1 p(x)\varphi_n'(x)\varphi_m'(x)dx=\begin{cases}\lambda_n & \text{for } n=m,\\ 0 & \text{for n} \neq \text{m}.\end{cases}$

2.1.6. For any $n\in\mathbb{N},\ \varphi_n\in C^\infty(\overline{I}).$

Let us construct Green's function $G(x,t,\xi,\tau)$ corresponding to problem (1.1). It is determined by the following system: for $x,\xi\in I$ and $t,\tau\in(0,T),$

$$\left.\begin{aligned} G_t-\frac{1}{k(x)}\left(p(x)G_x\right)_x &= \delta(x-\xi)\delta(t-\tau),\\ G(0,t,\xi,\tau) &= 0 = G(1,t,\xi,\tau),\\ G(x,t,\xi,\tau) &= 0 \text{ for } t>\tau, \end{aligned}\right\} \qquad (2.1)$$

where δ is the Dirac delta function. By the eigenfunction expansion, the corresponding Green's function G to problem (1.1) is defined by

$$G(x,t,\xi,\tau)=\sum_{n=1}^{\infty}\varphi_n(\xi)\varphi_n(x)e^{-\lambda_n(t-\tau)} \text{ for } x,\xi\in I \text{ and } t>\tau.$$

By using Green's second identity, we get the integral equation equivalent to problem (1.1) given by

$$u(x,t)=\int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi + \int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau. \qquad (2.2)$$

The following lemma is due to properties of G.

Lemma 2.2. Let $\lambda_n=O(n^s)$ for some $s>1$ as $n\to\infty.$

2.2.1. G is continuous for $x,\xi\in I$ and $0\le\tau<t<T.$

2.2.2. G is positive for $x,\xi\in I$ and $0\le\tau<t<T.$

2.2.3. $\lim_{t\to\tau^+}k(x)G(x,t,\xi,\tau)=\delta(x-\xi)$

2.2.4. For any $(x,t,\tau)\in I\times(0,T)\times(0,T),$

$\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi\le C_0$ for some $C_0>0.$

Proof. By modifying proof of lemma 4.a and 4.c [5], we obtain the proof of 2.2.1 and 2.2.2, respectively. For proof of 2.2.3, let us consider the following problem:

$$\begin{aligned} w_t-\frac{1}{k(x)}\left(p(x)w_x\right)_x &= 0 \text{ for } x,\xi\in I \text{ and } 0<\tau<t<T,\\ w(0,t,\xi,\tau) &= 0 = w(1,t,\xi,\tau) \text{ for } 0<\tau<t<T,\\ \lim_{t\to\tau^+}k(x)w(x,t,\xi,\tau) &= \delta(x-\xi). \end{aligned}$$

By equation (2.2), we have that for any $t>\tau,$

$$w(x,t,\xi,\tau)=\int_0^1 k(\zeta)G(x,t,\zeta,\tau)\frac{1}{k(\zeta)}\delta(\zeta-\xi)d\zeta=G(x,t,\xi,\zeta).$$

Hence, we obtain the proof of 2.2.3. We next prove 2.2.4.

Case 1. For any $t<\tau.$

Definition for G yields that $\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi = 0.$

Case 2. $t=\tau$.

It follows lemma 2.2.3 and a property of Dirac delta function δ that $\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi = \int_0^1 \delta(x-\xi)d\xi = 1.$

Case 3. For any $t>\tau$.

Let us consider the series

$$\sum_{n=1}^{\infty}\int_0^1 k(\xi)\varphi_n(\xi)\varphi_n(x)e^{-\lambda_n(t-\tau)}d\xi.$$

Since

$$\left|\int_0^1 k(\xi)\varphi_n(\xi)\varphi_n(x)e^{-\lambda_n(t-\tau)}d\xi\right| \le \left(\max_{x\in\bar{I}}\varphi_n(x)\right)^2 e^{-\lambda_n(t-\tau)}$$

and the series $\sum_{n=1}^{\infty} e^{-\lambda_n(t-\tau)}$ converges,

$\sum_{n=1}^{\infty}\int_0^1 k(\xi)\varphi_n(\xi)\varphi_n(x)e^{-\lambda_n(t-\tau)}d\xi$ converges uniformly for any $(x,t,\tau)\in I\times(0,T)\times(0,T)$. Hence we get the proof of 2.2.4.

Therefore, the proof of lemma 2.2 is complete.

Next theorem says to local existence of a solution u of the equivalent integral equation (2.2).

Theorem 2.1. There exists a T_1 with $0<T_1<T$ such that the equivalent integral equation (2.2) has a unique continuous solution u for any $(x,t)\in\bar{Q}_{T_1}$.

Proof. We will use the fixed point theorem to prove existence of a solution u of the equivalent integral equation (2.2). Let $M=\max_{x\in\bar{I}}|u_0(x)|+1$. Locally Lipschitz property of f implies that there exists a positive constant $L(M)$ depending on M such that $|f(x)-f(y)|\le L(M)|x-y|$ for any $x,y\in\mathbb{R}$ with $|x|\le M$ and $|y|\le M$. We then choose

$$T_1 < \min\left\{\frac{1}{C_0 f(M)}, \frac{1}{L(M)C_0}\right\}. \tag{2.3}$$

Define a set E by

$$E=\left\{u\in C(\bar{Q}_{T_1}) \text{ such that } \max_{(x,t)\in\bar{Q}_{T_1}}|u(x,t)|\le M\right\}.$$

Then, E is a Banach space equipped with the norm $|u|_E = \max_{(x,t)\in\bar{Q}_T}|u(x,t)|$. Let

$$\Lambda u(x,t) = \int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi + \int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau. \tag{2.4}$$

for any $u\in E$. We next show that the operator Λ defined by (2.4) maps E into itself and that Λ is contractive. Let $u,v\in E$. We then have that

$$|\Lambda u(x,t)| \le \left|\int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi\right| + \left|\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau\right|.$$

(2.5)

Let us consider the following auxiliary problem:

$$\left.\begin{aligned} u_t-\frac{1}{k(x)}\left(p(x)u_x\right)_x &= 0 \text{ for } (x,t)\in Q_{T_1},\\ u(0,t)=0=u(1,t) &\text{ for } t\in(0,T_1),\\ u(x,0)=u_0(x) &\text{ for } x\in\bar{I}. \end{aligned}\right\} \tag{2.6}$$

It follows from (2.2) that a solution u of problem (2.6) is given by

$$u(x,t)=\int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi \text{ for } (x,t)\in\bar{Q}_{T_1}.$$

Moreover, maximum principle for parabolic type implies that $0\le u(x,t)\le\max_{x\in\bar{I}}|u_0(x)|$ for any $(x,t)\in\bar{Q}_{T_1}$. Thus, we obtain that $\int_0^1 k(\xi)G(x,t,\xi,0)d\xi\le 1$. From (2.5) and lemma 2.2.4.,

$$\begin{aligned} |\Lambda u(x,t)| &\le \max_{x\in\bar{I}}|u_0(x)| + f(M)\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi d\tau.\\ &\le \max_{x\in\bar{I}}|u_0(x)| + f(M)C_0T_1. \end{aligned}$$

By definition of T_1, $\Lambda u\in E$ for any $u\in E$. Since

$$\begin{aligned} &|\Lambda u(x,t)-\Lambda v(x,t)|\\ &\le\left|\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)f(u(x_0,\tau))-f(v(x_0,\tau))d\xi d\tau.\right|\\ &\le\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)\left|f(u(x_0,\tau))-f(v(x_0,\tau))\right|d\xi d\tau\\ &\le L(M)\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)d\xi d\tau|u-v|_E\\ &\le C_0T_1L(M)|u-v|_E, \end{aligned} \tag{2.7}$$

definition of T_1 and (2.7) yield that Λ is contractive. The fixed point theorem then implies that there exists a unique u in E satisfying the integral equation (2.2). Therefore, the proof is complete.

Lemma 2.3. Let v be a classical solution of the following problem:

$$\left.\begin{aligned} v_t-\frac{1}{k(x)}\left(p(x)v_x\right)_x &\ge B(x,t)v(x_0,t) \text{ for } (x,t)\in Q_T,\\ v(0,t)=0=v(1,t) &\text{ for } t\in(0,T),\\ v(x,0)=u_0(x)\ge 0 &\text{ for } x\in\bar{I}, \end{aligned}\right\} \tag{2.8}$$

where $B(x,t)$ is a nonnegative and bounded function on $\bar{Q}_T$. Then $v(x,t)\ge 0$ for any $(x,t)\in\bar{Q}_T$.

Proof. In order to prove this lemma we have to add a nonnegative continuous function $z(x,t)$ on $\overline{Q}_T$ to right-hand side of equation (2.8) and then we have that

$$\left.\begin{aligned} v_t - \frac{1}{k(x)}\left(p(x)v_x\right)_x = B(x,t)v(x_0,t) + z(x,t) \text{ on } Q_T,\\ v(0,t) = 0 = v(1,t) \text{ for } t\in(0,T),\\ v(x,0) = u_0(x) \geq 0 \text{ for } x\in\overline{I}. \end{aligned}\right\} \quad (2.9)$$

From equation (2.2), we obtain that for $(x,t)\in\overline{Q}_T$,

$$\begin{aligned} v(x,t) &= \int_0^1 k(\xi)G(x,t,\xi,0)u_0(\xi)d\xi \\ &+\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)B(\xi,\tau)v(x_0,\tau)d\xi d\tau \\ &+\int_0^t\int_0^1 k(\xi)G(x,t,\xi,\tau)z(\xi,\tau)d\xi d\tau. \end{aligned} \quad (2.10)$$

From (2.10), we have

$$\begin{aligned} v(x_0,t) &= \int_0^1 k(\xi)G(x_0,t,\xi,0)u_0(\xi)d\xi \\ &+\int_0^t\int_0^1 k(\xi)G(x_0,t,\xi,\tau)B(\xi,\tau)v(x_0,\tau)d\xi d\tau \\ &+\int_0^t\int_0^1 k(\xi)G(x_0,t,\xi,\tau)z(\xi,\tau)d\xi d\tau. \end{aligned}$$

Let
$$\begin{aligned} h_0(t) &= \int_0^1 k(\xi)G(x_0,t,\xi,0)u_0(\xi)d\xi \\ &+\int_0^t\int_0^1 k(\xi)G(x_0,t,\xi,\tau)z(\xi,\tau)d\xi d\tau. \end{aligned}$$

Since functions k, z, G and u_0 are nonnegative, h_0 is nonnegative. Let $u(x_0,t)=h(t)$ for $t\in[0,T]$. Define an operator Φ mapping from $C[0,T]$ to $C[0,T]$ by

$$\Phi h(t) = \int_0^t\int_0^1 k(\xi)G(x_0,t,\xi,\tau)B(\xi,\tau)h(\tau)d\xi d\tau.$$

By corollary 5.2.1. [6], there exists a $T_2(<T)$ such that

$$v(x_0,t) = h(t) = \sum_{m=0}^{\infty}\Phi^{(m)}h_0(t) \quad (2.11)$$

where $\Phi^{(0)}h_0(t)=h_0(t)$ and $\Phi^{(m+1)}h_0(t)=\Phi\left[\Phi^{(m)}h_0(t)\right]$ for $m\in\mathbb{N}$. Mathematical induction yields that $\Phi^{(m)}h_0(t)\geq 0$ for $m\in\mathbb{N}$. Thus, from equation (2.11), we obtain that $v(x_0,t)\geq 0$ for any $t\in[0,T_2]$. It follows from equation (2.10) that $v(x,t)\geq 0$ on $\overline{Q}_{T_2}$. Finally, we can repeat the previous procedure to obtain the desired result for $(x,t)\in\overline{Q}_T$.

Next lemma gives additional properties of a solution u of problem (1.1).

Lemma 2.4. Let u be a continuous solution of problem (1.1). Then $u(x,t)\geq u_0(x)$ and $u_t(x,t)\geq 0$ for any $(x,t)\in\overline{Q}_{T_1}$.

Proof. Let $z(x,t)=u(x,t)-u_0(x)$ on $\overline{Q}_{T_1}$. Let us consider that for any $(x,t)\in Q_{T_1}$,

$$z_t - \frac{1}{k(x)}\left(p(x)z_x\right)_x = f(u(x_0,t)) + \frac{1}{k(x)}\left(p(x)\frac{du_0(x)}{dx}\right).$$

Equation (1.2) yields $\frac{1}{k(x)}\left(p(x)\frac{du_0(x)}{dx}\right)\geq -f(u_0(x_0))$ on I and then we obtain that for any $(x,t)\in Q_{T_1}$,

$$z_t - \frac{1}{k(x)}\left(p(x)z_x\right)_x = f(u(x_0,t)) - f(u_0(x_0)) \geq f'(\eta_1)z(x_0,t)$$

where η_1 is between $u(x_0,t)$ and $u_0(x_0)$. Moreover, for any $(x,t)\in\{0,1\}\times(0,T)\cup\overline{I}\times\{0\}$, $z(x,t)=0$. Lemma 2.3 implies that $z\geq 0$ on $\overline{Q}_{T_1}$ or $u\geq u_0$ on $\overline{Q}_{T_1}$. Let h be any positive constant less that T and $w(x,t)=u(x,t+h)-u(x,t)$ on $\overline{Q}_{T_1}$. Then we have that on Q_{T_1},

$$\begin{aligned} w_t - \frac{1}{k(x)}\left(p(x)w_x\right)_x &= f(u(x_0,t+h)) - f(u(x_0,t)) \\ &= f'(\eta_2)w(x_0,t), \end{aligned}$$

for η_2 between $u(x_0,t+h)$ and $u(x_0,t)$. Furthermore, $w=0$ on $\{0,1\}\times(0,T_1)$ and $w\geq 0$ on $\overline{I}\times\{0\}$. It then follows from lemma 2.3 that $w\geq 0$ on $\overline{Q}_{T_1}$. This shows that $u_t\geq 0$ on $\overline{Q}_{T_1}$.

We note that before blow-up occurs, there exists a positive constant M such that $\left|u(x,t)\leq M\right|$ for all $(x,t)\in\overline{Q}_{T_1}$. Locally Lipschitz continuity of f yields that there exists a positive constant $L(M)$ depending on M such that $\left|f(u(x_0,t))\right|\leq L(M)\left|u(x_0,t)\right|$ for any $t\in[0,T_1]$.

Lemma 2.5. If $f'(u_0(x_0))\geq L(M)$, then $u_t(x,t)\geq L(M)u(x,t)$ on $\overline{Q}_{T_1}$.

Proof. Let $z(x,t)=u_t(x,t)-L(M)u(x,t)$ on $\overline{Q}_{T_1}$. We then have that for $(x,t)\in Q_{T_1}$,

$$z_t - \frac{1}{k(x)}\left(p(x)z_x\right)_x = f'(u(x_0,t))u_t(x_0,t) - L(M)f(u(x_0,t)).$$

Locally Lipschitz continuity of f implies that for $(x,t)\in Q_{T_1}$,

$$\begin{aligned} z_t - \frac{1}{k(x)}\left(p(x)z_x\right)_x &\geq f'(u(x_0,t))u_t(x_0,t) - L^2(M)u(x_0,t) \\ &\geq f'(u_0(x_0))u_t(x_0,t) - L^2(M)u(x_0,t) \\ &\geq L(M)z(x_0,t). \end{aligned}$$

From lemma 2.4, $z(0,t)=u_t(0,t)\geq 0$ and $z(1,t)=u_t(1,t)\geq 0$ for $t\in(0,T_1)$. If we set $\varsigma=L(M)$, then equation (1.2) implies that for any $x\in I$,

$$z(x,0) = \lim_{t\to 0}u_t(x,t) - L(M)u_0(x)$$

$$=\frac{1}{k(x)}\left(p(x)\frac{du_0(x)}{dx}\right)+f(u(x_0))-L(M)u_0(x)\geq 0.$$

Therefore, by lemma 2.3, the proof is complete.

Lemma 2.6. If $u_0(x_0)\geq u_0(x)$ for any $x\in\overline{I}$, then $u(x_0,t)\geq u(x,t)$ on $\overline{Q}_{T_1}$.

Proof. Let $z(x,t)=u(x_0,t)-u(x,t)$ on $\overline{Q}_{T_1}$. We then have that on Q_{T_1}, lemma 2.5 yields that

$$z_t-\frac{1}{k(x)}\left(p(x)z_x\right)_x=u_t(x_0,t)-f(u(x_0,t))$$
$$=u_t(x_0,t)-L(M)u(x_0,t)\geq 0.$$

Since $z(0,t)=u(x_0,t)\geq u_0(x)\geq 0,\ z(1,t)=u(x_0,t)\geq u_0(x)\geq 0$ for $t\in(0,T_1)$, and $z(x,0)=u_0(x_0)-u_0(x)\geq 0$ for any $x\in\overline{I}$, by lemma 2.3, the proof of this lemma is complete.

Theorem 2.2. Let $T_{\max}$ be the supremum of all T_1 such that the continuous solution u of an equivalent integral equation (2.2) exists. If $T_{\max}$ is finite, then $u(x_0,t)$ is unbounded as t tends to $T_{\max}$.

Proof. Suppose that $u(x_0,T_{\max})$ is finite. Let $N=u(x_0,T_{\max})+1$. By theorem 2.1 and a fact that u is nondecreasing in t, there exists a finite time $\tilde{T}(>T_{\max})$ depending on N such that the equivalent integral equation (2.2) has a unique continuous solution on the time interval $[0,\tilde{T}]$ for any $x\in\overline{I}$. By the definition of $T_{\max}$, we get a contradiction.

A proof similar to that of theorem 3 of Chan and Tian [3] gives the following result.

Theorem 2.3 Such a continuous solution u of the equivalent integral equation (2.2) is a classical solution.

III. A SUFFICIENT CONDITION TO BLOW-UP IN FINITE TIME

Let φ_1 be the first eigenfunction of a singular eigenvalue problem (1.3) and let λ_1 be its corresponding eigenvalue. Without loss of generality we assume

$$\int_0^1 k(x)\varphi_1(x)dx=1. \tag{3.1}$$

We then define a function H by $H(t)=\int_0^1 k(x)\varphi_1(x)u(x,t)dx.$

Theorem 3.1. Assume that

3.1.1. u_0 attains its maximum at point x_0.

3.1.2. $f(\xi)\geq b\xi^p$ with $b>0$ and $p>1$.

3.1.3. $H(0)>\left(\frac{\lambda_1}{b}\right)^{\frac{1}{p-1}}$.

Then a solution u of problem (1.1) blows up in finite time.

Proof. Multiplying equation (1.1) by $k(x)\varphi_1(x)$ and integrating equation (1.1) with respect to x over its domain yield

$$\frac{dH(t)}{dt}=-\lambda_1H(t)+\int_0^1 k(x)f(u(x_0,t))\varphi_1(x)dx.$$

By lemma 2.6 and assumption 3.1.2, we have

$$\frac{dH(t)}{dt}\geq-\lambda_1H(t)+\int_0^1 k(x)f(u(x,t))\varphi_1(x)dx$$
$$\geq-\lambda_1H(t)+b\int_0^1 k(x)u^p(x,t)\varphi_1(x)dx.$$

(3.2)

Holder inequality implies that

$$\int_0^1 k(x)\varphi_1(x)u(x,t)dx$$
$$\leq\left(\int_0^1 k(x)\varphi_1(x)dx\right)^{\frac{p-1}{p}}\left(\int_0^1 k(x)\varphi_1(x)u^p(x,t)dx\right)^{\frac{1}{p}}.$$

From (3.1), we get

$$\int_0^1 k(x)\varphi_1(x)u^p(x,t)dx\geq\left(\int_0^1 k(x)\varphi_1(x)u(x,t)dx\right)^p=H^p(t). \tag{3.3}$$

Form equation (3.2) and (3.3), we obtain

$$H'(t)\geq-\lambda_1H(t)+bH^p(t)$$

or

$$H^{p-1}(t)\geq\frac{1}{\frac{b}{\lambda_1}+\left[H^{1-p}(0)-\frac{b}{\lambda_1}\right]e^{-\lambda_1(1-p)t}}.$$

It then follows from assumption 3.1.3 that there exists a $\hat{T}(>0)$ such that H tends to infinity as t converges to $\hat{T}$. By the definition of H, we find that

$$H(t)\leq\left(\int_0^1 k(x)\varphi_1(x)dx\right)u(x_0,t)=u(x_0,t).$$

Therefore, a solution u of problem (1.1) blows up at point x_0 as t tends to $\hat{T}$.

IV. THE BLOW-UP SET

Theorem 4.1. The blow-up set of a solution u of problem (1.1) is I.

Proof. From (2.2), we have that for $t\in(0,T_{\max})$,

$$u(x_0,t)=\int_0^1 k(\xi)G(x_0,t,\xi,0)u_0(\xi)d\xi$$
$$+\int_0^t\int_0^1 k(\xi)G(x_0,t,\xi,\tau)f(u(x_0,\tau))d\xi d\tau$$
$$\leq\max_{x\in\overline{I}}|u_0(x)|+C_0\int_0^t f(u(x_0,\tau))d\tau. \tag{4.1}$$

By theorem 2.2, we obtain that as t tends to $T_{\max}$,

$$\int_0^{T_{\max}} f(u(x_0,\tau))d\tau=\infty. \tag{4.2}$$

On the other hand, by positivity of k, G, and u_0, we get that for any $(x,t) \in Q_{T_{\max}}$,

$$u(x,t) \geq \int_0^t \int_0^1 k(\xi) G(x,t,\xi,\tau) f(u(x_0,\tau)) d\xi d\tau.$$

Since there exists a positive constant C_1 such that

$$\int_0^1 k(\xi) G(x,t,\xi,\tau) d\xi \geq C_1,$$

we obtain that

$$u(x,t) \geq C_1 \int_0^t f(u(x_0,\tau)) d\tau \quad \text{for all } (x,t) \in Q_{T_{\max}}.$$

Hence, the solution u tends to infinity for all $x \in I$ as t approaches to $T_{\max}$. Therefore the proof of this theorem is complete.

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