Reduction Method with Finite-Difference Approximation for the Model of Small Transverse Vibrations in Thin Elastic Plates

KULESHOV A.A.
Keldush Institute of Applied Mathematics
of Russian Academy of Sciences,
Miusskaya sq., 4-A, Moscow, 125047
RUSSIA
andrew_kuleshov@mail.ru

Abstract: The problem of small transverse vibrations in a thin elastic plate of variable thickness with a bending moment and a shearing force on the plate contour is considered. The numerical method for the problem solution is described. It is based on reduction of initial partial differential equation to a system of equations with first-order time derivatives. Some results of numerical simulation for the applied problem are discussed.

Key-Words: Thin elastic plate, transverse vibrations, numerical method and applications

1 Introduction
The mathematical model of transverse vibrations in thin elastic plates has applications in various fields of science and engineering. Specifically, in geocology it can be applied to the problem of floating ice vibrations caused by various moving loads. It can also be applied to the problem of transverse vibrations in oceanic lithosphere plates because thin oceanic lithosphere (as compared with thickness on the continental lithosphere) consists of relatively uniform basalt and can be considered as the thin elastic plate. In addition to the above mentioned applications there have been other applications (e.g., in structural mechanics). And yet despite the fact that there is a wide range of applications the numerical methods to solve the above problem have not been developed sufficiently. In present paper describes a new efficient numerical method to solve the above problem and discusses some results of numerical simulation for the problem of vibrations in floating ice caused by moving loads.

2 Problem Formulation
The equation of small transverse vibrations in a thin isotropic elastic plate of variable thickness $h(x,y)$ lying on the elastic (Winkler) foundation has the form [1]:

$$\rho h W_{tt} + \Delta (D \Delta W) - (1-\sigma) \left( D_{yy} W_{xx} - 2D_{xy} W_{xy} + D_{xx} W_{yy} \right) + aW = F, \quad (x,y) \in \Omega, \tag{1}$$

where $W(x,y,t)$ is the plate deflection measured along the z axis, $\rho$ is the density of the plate material, $D = Eh^3/[12(1-\sigma^2)]$ is the cylindrical stiffness of the plate, $E$ is the modulus of elasticity, $\sigma$ is Poisson's ratio of the plate material, $aW$ is the reaction of elastic foundation (reactive pressure) proportional to the plate deflection according to the Winkler model, $a = \text{const} > 0$, $F$ is the external force given on a surface of a plate.

On the curvilinear contour of a plate $\partial \Omega$ general (Kirchhoff-generalized) conditions: the bending moment $M_{\partial \Omega}$ and the vertical shearing force $N_{\partial \Omega}$ are given:

$$B_1 W = M_{\partial \Omega}, \quad (x,y) \in \partial \Omega,$$

$$B_2 W = N_{\partial \Omega}, \quad (x,y) \in \partial \Omega,$$

where

$$B_1 W = -D \Delta W - D (1-\sigma) \left[ \sin 2\theta W_{xy} - \sin^2 \theta W_{xx} - \cos^2 \theta W_{yy} \right],$$

$$B_2 W = \frac{\partial (D \Delta W)}{\partial n} - (1-\sigma) \frac{\partial}{\partial n} \left[ D \left( \sin 2\theta W_{yy} - W_{xx} \right) + \cos 2\theta W_{xy} \right] +$$

$$+ (1-\sigma) \left[ \sin \theta D_x W_{xx} + \cos \theta D_y W_{yy} - (\sin \theta D_x + \cos \theta D_y) W_{xy} \right],$$

$n$ is the outward normal to the contour $\partial \Omega$, $\mathbf{l}$ is the tangent to the contour, $\theta$ is the angle between normal to the contour and the positive OX axis.

The initial conditions:

$$W \big|_{t=0} = \varphi(x,y), \quad W_t \big|_{t=0} = \psi(x,y).$$

A finite-difference approximation of the equation (1) based on a three-level finite-difference scheme will have a multipoint stencil and its numerical implementation will be complicated. Our new approach [2] to construction of a numerical method for the problem solution based on reduction method with finite-difference approximation is described in the next section.
3 Numerical method for the problem solution

We use the well-known approach to reducing a high-order partial differential equation to a system of equations of smaller order. We reduce the original equation, which is of the second order on time, by a system of equations with first-order on time and a two-level implicit finite-difference scheme is developed for solving this system. For this purpose at first we will write down the equation (1) in an equivalent form [1]:

\[
\rho \frac{\partial^2 W}{\partial t^2} = \frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - aW + F,
\]
(2)

where \( M_x = -D(\frac{\partial^2 W}{\partial x^2} + \sigma \frac{\partial W}{\partial y}) \), \( M_y = -D(\frac{\partial^2 W}{\partial y^2} + \sigma \frac{\partial W}{\partial x}) \) are bending moments, \( M_{xy} = (1 - \sigma)D\frac{\partial^2 W}{\partial x \partial y} \) is a torsion moment. Then in the equation of plate vibrations (2), we make the substitution \( S = W \). Further we differentiate on \( t \) formulas for the bending moments \( M_x \) and \( M_y \) and make the substitution \( S = W_t \) too. As a result we receive system of the equations of the first order on time:

\[
\rho \frac{\partial S}{\partial t} = \frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - aW + F,
\]
\[
\frac{\partial M_x}{\partial t} = -D\frac{\partial^2 S}{\partial x^2} - \sigma D\frac{\partial^2 S}{\partial y^2},
\]
\[
\frac{\partial M_y}{\partial t} = -\sigma D\frac{\partial^2 S}{\partial x^2} - D\frac{\partial^2 S}{\partial y^2},
\]
\[
\frac{\partial W}{\partial t} = S,
\]
(3)

with boundary conditions:

\[ M_x \cos^2 \theta + M_y \sin^2 \theta - M_{xy} \sin 2\theta = M_{\partial \Omega}, \quad (x, y) \in \partial \Omega, \]
\[ Q_x \cos \theta + Q_y \sin \theta - \frac{\partial M_{nl}}{\partial \mathbf{n}} = N_{\partial \Omega}, \quad (x, y) \in \partial \Omega, \]
where \( Q_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}, \quad Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} \)

are vertical shearing forces,

\[ M_{nl} = M_{xy} (\cos^2 \theta - \sin^2 \theta) + (M_x - M_y) \sin \theta \cos \theta. \]

The initial conditions:

\[ W|_{t=0} = \varphi, \quad S|_{t=0} = \psi, \]
\[ M_x|_{t=0} = -D(\varphi_{xx} + \sigma \varphi_{xy}), \quad M_y|_{t=0} = -D(\varphi_{yy} + \sigma \varphi_{yx}). \]

For numerical solution of system (3) the two-layer implicit finite-difference approximation [2] on a rectangular mesh has been created. This scheme is more convenient for the numerical solution of the problem, than the three-layer scheme for the equation (1). The system of finite-difference equations is convenient to solve by splitting into two subsystems. In the first subsystem we implicitly approximated only finite-difference derivatives on \( x \), and in the second – finite-difference derivatives on \( y \). To solve each of these subsystems efficiently, we renumber the unknowns in a certain order and reduce the system of linear algebraic equations to a system with a nine-diagonal matrix, which is solved by Gaussian elimination. In this case, the coefficients are calculated in four stages. Test calculations have shown the high efficiency of the method proposed for solving the problem.

Stability of the finite-difference scheme from the input data has been proved [2,3]. The function \( F \) describing the forces acting on the plate surface can be discontinuous. Therefore, solutions of the original problem are regarded as generalized solutions in corresponding function spaces [4]. The strong convergence of the solution of the finite-difference problem to a generalized solution of the original differential problem has been proved and the rate of convergence has been estimated [3].

4 Numerical criterion of a plate destruction

In the process of transverse vibrations in a plate its material is stretched and compressed. For a thin elastic plate the stress tensor components determined by Hooke’s law and have the forms: \( \sigma_{xx} = 12z M_x / h^3 \), \( \sigma_{yy} = 12z M_y / h^3 \), \( \sigma_{xy} = -12z M_{xy} / h^3 \). The finite-difference analogues of their corresponding moments \( M_x, M_y, M_{xy} \) are computed at each time step of the computer program. Since maximal in absolute magnitude values of tensile and compression stress occur on the plate’s surfaces, maximal in absolute magnitude values of stress for the mesh at \( z = \pm h / 2 \) are computed. When such a value reaches the ultimate tensile or compression strength of the plate’s material we conclude that at this time step irreversible deformation and destruction of the plate will take place.

5 Some applications of the method

As an example of developed numerical method application let us consider the problem of vibrations of a floating ice in response to moving load. The mathematical model of the floating ice vibration problem is described by the equation for transverse vibrations of a thin elastic plate (1) lying on a liquid base, where \( a = \rho \eta \), \( \rho \) is water density, \( \rho g W \) is the buoyancy Archimedean force on the lower surface of the ice, \( F \) is the moving load on the upper surface of the ice. Notice that the floating ice vibration problem so formulated has been considered by many authors (See, e.g., [5-8]). In [5-8] the analytical methods
have been used to solving the problem: the plane pressure front advance on the ice was specified in the right-hand side of the equation by a δ-function of the form $P\delta(x-vt)$, where $P$ is the force acting upon the ice surface, the δ-function was represented in the form of a Fourier integral, and the solution of the problem for the plate’s deflection and for liquid flow potential under the ice was also represented in the form of integrals. The focus was mainly on the study of the resulting dispersion relations, dependence of the wave amplitude on the load’s velocity, and existence of critical velocities. Consequently it may be concluded that for the above applied problem no numerical methods with direct approximations of the equation for transverse vibrations in thin elastic plates were used. The developed method and computer code were used to carry out numerical simulation of the problem of propagation of floating ice vibrations caused by moving cars. The parameters used in the computations were as follows: linear dimensions of an ice plate with uniform thickness were 100 m x 40 m, the size of the mesh was 500 x 200 mesh points, the time step was chosen corresponding with the stability condition of the method used for solving the system of difference equations [1]; the values of physical parameters of the ice were as follows: modulus of elasticity $E=5.1\cdot10^8$ N/m$^2$, Poisson's ratio $\sigma=0.35$, ice tensile strength was $0.5–1.0$ MPa, ice compression strength was $2–3$ MPa. Minimal values of ice tensile and compression strength were selected as destruction criteria. The motion of one car with a mass of 2.2 tons moving at a speed of 15 m/sec on a floating 0.26 m thick ice was numerically simulated. Tensile and compression strength was computed at each time step of the computer program. No ice destruction happened in this case. Of particular interest is the case of combined motion of two vehicles or more on floating ice. Depending on the distance between those vehicles the waves propagating from the vehicles can be quenched or added increasing their amplitudes.

Numerical simulation was carried out of the combined motion of two identical cars each having a mass of 2.2 tons moving at the same speed of 15 m/sec on a floating 0.29 m thick ice, the distance between the cars being 20 m (See Fig. 1). When the wave due to the motion of the first car reached the other car there was an increase in amplitude of vibration (See Fig. 1c) and the computed maximum stress values exceeded the minimum tensile strength value of 0.5 MPa, and it means that destruction of the ice will take place.

### 6 Conclusion

The developed numerical method and computer software to solve the problem of small transverse vibrations in thin elastic plates permit the study of wave dynamics of
transverse vibration in such plates under the action of various moving loads and also enable us to make inferences regarding the potential of the plate material destruction, which has great value for applied problems.

References: