On a new approach to obtain spline Bézier curves

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Abstract: The aim of this paper is to introduce an approach for obtaining spline Bézier curves of G^1 class. Our approach generalizes F-Mill interpolation method and depends on two parameters which influence the curve properties. The method allows modifications of the shape of the curves without changing the interpolation points and the direction of the tangents in this points. We implemented the proposed interpolation scheme using MATLAB and analyzed the shape modifications depending of the parameters values.

Key-Words: spline Bézier curve, Bézier interpolation

1. Introduction

The development of Computer Aided Geometric Design (CAGD) is strongly related to parametric curves and surfaces and especially to Bézier ones. These are parametric curves and surfaces expressed in the Bernstein basis which preserves the shape of the curve or surface. There are many studies related to curve and surface design and control. Generally, we interpolate a curve or surface in given points using fixed interpolation, i.e. the interpolating curve or surface is fixed for the given interpolating data and control polygon. The modification of shape implies the modification of given control points. An important problem in CAGD is how to modify the shape of the curve without changing the given data. The rational Bézier model allows construction of free-form curves and surfaces. Rational spline with parameters (see [6]) allows the modification of the interpolating curve by changing the parameters. In [10] is introduces an approach in which the shape of the curves and surfaces is controlled by the control edges of the control polygon, not only by the control points.

In our paper we introduce a method for modifying the shape of a spline cubic Bézier curve without modifying the given points. More, the tangents in the interpolation points remain the same in the shape modification process. The shape modification is realized by means of 2 parameters. We implemented this method and analyzed the influence of the parameters on the shape modification.

The article is organized as follows. In section 2 we briefly present Bézier and spline Bézier curves. In section 3 we introduce our approach. In section 4 we analyze the influence of the parameters from our scheme to shape modification. Conclusions and further directions of study are presented in section 5.

2. Bézier curves

Bézier curves had an important role in development of Computer Aided Geometric Design ([1], [4], [5]). They were introduced independently by P. de Casteljau at Citroen and and P. Bézier at Renault.

According to [7] they are numerically the most stable among all polynomial bases used since now in CAD systems. The idea of de Casteljau was to generate curves using a control polygon. Bézier curves are parametric curves express in Bernstein basis using the control points as coefficients. The parametric equations of a Bézier curve are given in (1):

$$f(t) = (x(t), y(t), z(t)) = \sum_{i=0}^{n} b_i \cdot B_i^n(t),\ t \in [0,1] \quad (1)$$

$B_i^n(t)$ are Bernstein polynomials defined explicitly by

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i} t^i \quad (2)$$

The control polygon is defined by the control points $b_i = (x_i, y_i, z_i)$.

The most important properties of Bézier curves derive from the Bernstein polynomials properties given below.

Bernstein polynomials satisfy the recursion:

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t) \quad (3)$$

$$B_i^n(0) = \delta_{i,0} \quad (4)$$

$$B_i^n(1) = \delta_{i,n} \quad (5)$$

$\delta_{i,j}$ is the Kronecker function.

Bernstein polynomials form a partition of unity:

$$\sum_{i=0}^{n} B_i^n(t) \equiv 1 \quad (6)$$

Many details regarding Bernstein polynomials' properties can be found in [4].

Bézier curves' properties were derived first using geometric arguments and then, they were proved using algebraic arguments (the explicit form of Bernstein curves and Bernstein polynomials' properties). We enumerate briefly these properties:

1. *Convex hull property:* the Bézier curve is contained in convex hull of control points.
2. *Endpoints interpolation:* Bézier curve interpolates in the first and last control point ($f(0)=b_0$, $f(1)=b_n$).
3. *Linear precision:* if the control points are situated on a straight line, the corresponding Bézier curve reproduces this line.
4. *Affine invariance:* the control polygon obtained by a linear transformation of the control points defines a new Bézier curve which is obtained applying the same transformation to the original curve.
5. *Endpoint Hermite interpolation:* Bézier curve is tangent to the control polygon in the first and last control point.
6. *Pseudo – local control*: if we move only one of the control points, the curves is affected by this change only in the neighborhood of this point. This makes predictable the effect of this change.
7. *Symmetry*: the sequences of control points: $b_0, \ldots, b_n$ and $b_n, \ldots, b_0$ generate the same Bézier curve.

The most used Bézier curves are cubic Bézier curves, obtained for $n=3$. Many advantages of cubic Bézier curves are given in [4]. The most complex curves can not be approximated using cubic Bézier curves. A solution is to use cubic spline Bézier curves (cubic B-spline), which are piecewise cubic Bézier curves satisfying continuity and smoothness conditions in the junction points ([4[). Cubic spline interpolation was introduced in CAGD by Ferguson ([8]), but also developed inside approximation theory by many mathematicians (see [2], [11] and the references inside).

A B–spline, of degree n, with m pieces is a piecewise polynomial, which j-th component is defined by:

$$f_j(t) = \sum_{i=0}^{n} b_i^j B_i(t - j + 1),\ t \in [j-1, j], \quad (7)$$

$j \in \{1, \ldots, m\}$.

We denoted by b_i^j the control points of the j-th component.

The continuity conditions are:

$b_0^j = b_n^{j-1},\ \forall\ j \in \{2, \ldots, m\}$

A cubic B-spline is of class G^1 if there is an unique tangent in the junction points, i.e.

$b_{n-1}^j, b_n^j, b_1^{j+1}$, *are colinear* $\forall\ j \in \{1, \ldots, m-1\}$.

The cubic B-splines are obtained for $n=3$.

3. Our approach for generating cubic B-spline of G^1 class

We consider the following problem: given $n+1$ digitalized points $P_i=(x_i,y_i,z_i)$, $i \in \{0, \ldots, n\}$ from a curve find a cubic B-spline of G^1 class which interpolates this curve. More, find a form of B-spline capable to allows shape modifications of curve without changing the control points.

Many solutions of generating B-spline can be found in literature ([1], [2], [4]-[6], [9], [12]).

In this article we propose a generalization of F-Mill method. F-Mill method approximates the original curve using a cubic B-spline with n pieces. For the $n+1$ given points, the F-Mill relations provide $3n+1$ control points b_i. The G^1 condition for the B-spline curve is obtained imposing that in the junction point P_i the tangent is parallel with the segment $P_{i-1}P_{i+1}$, the point P_i is the middle of the tangent segment $b_{3i-1}b_{3i+1}$ and $\|b_{3i-1}b_{3i+1}\| = \frac{1}{3}\|P_{i-1}P_{i+1}\|$. The F-Mill relations give the control points except the second and the last but one. These two points define the direction of tangents to the original curve in the first and last digitalized point. The F-Mill relations are given in (8) – (10).

$$b_{3(i-1)} = P_{i-1} \quad (8)$$

$$b_{3i-1} = P_i - \frac{1}{6} P_{i-1}P_{i+1} \quad (9)$$

$$b_{3i+1} = P_i + \frac{1}{6} P_{i-1}P_{i+1} \quad (10)$$

We generalize the F-Mill method, introducing two parameters k_1, k_2. For the $n+1$ given points we calculate $3n+1$ control points b_i of a G^1 class cubic B – spline with n pieces, imposing the following conditions:

1. In the junction point P_i the tangent is parallel with the segment $P_{i-1}P_{i+1}$

2. $\|b_{3i-1}P_i\| = k_1 \|b_{3i-1}b_{3i+1}\|,\ 0 < k_1 < 1$

3. $\|b_{3i-1}b_{3i+1}\| = k_2 \|P_{i-1}P_{i+1}\|,\ k_2 > 0$

The control points are defines by the following relations:

$$b_{3(i-1)} = P_{i-1} \quad (11)$$

$$b_{3i-1} = P_i - k_1 \cdot k_2 \cdot P_{i-1}P_{i+1} \quad (12)$$

$$b_{3i+1} = P_i + (1-k_1) \cdot k_2 \cdot P_{i-1}P_{i+1} \quad (13)$$

Classical F-Mill method is obtained for $k_1 = \frac{1}{2}$ and $k_2 = \frac{1}{3}$.

4. Analysis of shape modifications

We implemented the scheme introduced in section 3 using MATLAB. MATLAB offers a easy manipulation of matrix structures, a high quality graphical capabilities and a powerful tools for Graphical User Interfaces design.

We consider the case of planar curves and use the matrix form of a cubic Bézier curve:

$$f(t) = b \cdot B(t) \quad (14)$$

with $b = [b_0, ..., b_3] = \begin{pmatrix} x_0 & \cdots & x_3 \\ y_0 & \cdots & y_5 \end{pmatrix}$, the matrix of control points and

$$B(t) = \left[(1-t)^3, 3(1-t)^2 t, 3(1-t)t^2,\ t^3\right]'$$

the column vector of cubic Bernstein polynomials.

The computation of the control points and the graphical representation of the spline Bézier curve and of its control polygon is realized by the function *[b] = kmill(p,a,e,c,d,k1,k2)* which has as input data the matrix p of digitalized points, the coordinates of the second and the last but one control points: *(a,e)*, *(c,d)*.

The algorithm uses the relations (11) – (13) for computing the coordinates of the control points and the relation (14) for computing the points from the cubic Bézier spline.

For an easier manipulation we designed a user graphic interface Fig. 1 is illustrated a classical F-Mill B-spline.

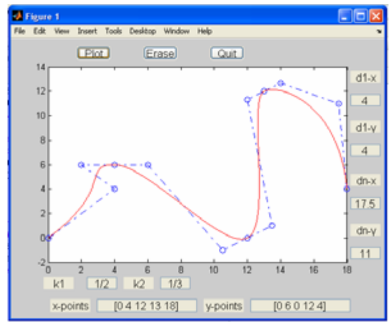

Fig. 1- Graphic interface for cubic B-spline

For a fix value of k_2=1/3, the variation of the shape for k_1=1/6 and k_1=4/5 can be view in Fig. 2a and b.

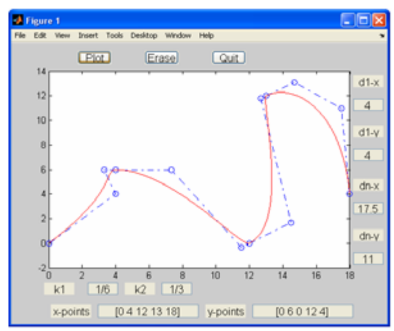

Fig. 2a- Influence of k_1 to shape variation: k_1=1/6

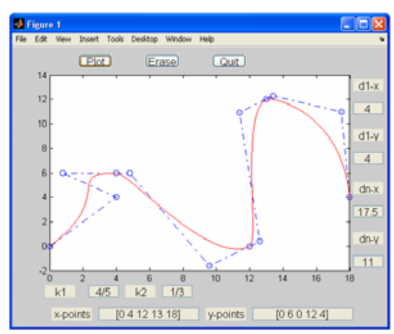

Fig. 2b- Influence of k_1 to shape variation: k_1=4/5

For a fix value of k_1=1/2, the shape variation for k_2=1/8 and k_2=4/5 can be view in Fig. 3a and 3b.

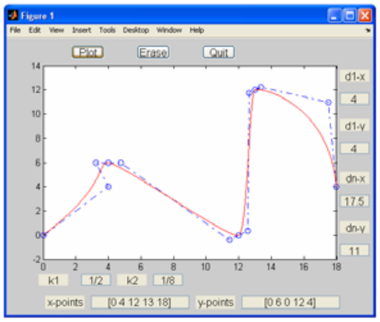

Fig. 3a- Influence of k_2 to shape variation:$k_2=1/8$

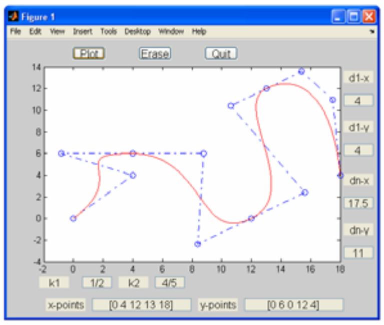

Fig. 3b- Influence of k_2 to shape variation: $k_2=4/5$

We observe that variation of k_1 and k_2 does not modify the convexity properties of Bézier curve pieces which compose the B-spline..

The results obtained suggest that k_2 influences the variation of shape more than k_1.

In the following we will give geometric reasons for this conclusion.

Let consider the control polygon $p_k = \overline{b_{3(k-1)}b_{3k-2}b_{3k-1}b_{3k}}$.

The control points are given by:

$$b_{3(k-1)} = P_{k-1}$$

$$b_{3k-2} = P_{k-1} + (1-k_1)k_2 \cdot \overline{P_{k-2}P_k}$$

$$b_{3k-1} = P_k - k_1 k_2 \cdot \overline{P_{k-1}P_{k+1}}$$

$$b_{3k} = P_k$$

Let consider k_2 fixed and $0<k_1<1$ variable. The points P_i are given points. Be cause $1-k_1>0$, the convexity of the control polygon p_k does not change. Also, the directions of the tangents on the control points $b_{3(k-1)}$ and b_{3k} do not change, therefore the convexity of the k –th B-spline piece does not change. The variation of k_1 for fixed value of k_2 produces a slip of the control points b_{3k-2}, b_{3k-1} , in opposite sense (if $b_{3k-3}b'_{3k-2} > b_{3k-3}b_{3k-2}$ then $b_{3k}b'_{3k-1} < b_{3k}b_{3k-1}$), on the control polygon edges $b_{3k-3}b_{3k-2}$ and $b_{3k-1}b_{3k}$ such that $\frac{\|b'_{3k-2}b_{3k-2}\|}{\|b'_{3k-1}b_{3k-1}\|} = \frac{\|P_{k-2}P_k\|}{\|P_{k-1}P_{k+1}\|}$

We denoted by b'_j the control point obtained for a modified value k'_1 of parameter k_1. As big the difference $|k_1-1/2|$ as sharp the curve in the points P_i is.

Let consider k_1 fixed and $k_2>0$ variable. The variation of k_2 for fixed value of k_1 produces a slip of the control points b_{3k-2}, b_{3k-1} on the control polygon edges $b_{3k-3}b_{3k-2}$ and $b_{3k-1}b_{3k}$, in the same sense (if $b_{3k-3}b'_{3k-2} > b_{3k-3}b_{3k-2}$ then $b_{3k}b'_{3k-1} > b_{3k}b_{3k-1}$), such that $\frac{\|b'_{3k-2}b_{3k-2}\|}{\|b'_{3k-1}b_{3k-1}\|} = \frac{(1-k_1)\|P_{k-2}P_k\|}{k_1\|P_{k-1}P_{k+1}\|}$. In this case the control polygons for the Bézier cubic components can change unpredictable and the shape of the B-spline can be completely different for different values of k_2 when k_2 increases, as shown in Fig. 4a ($k_2=1/3$) and Fig. 4b ($k_2=1$). Practically we observed than significant shape variations appear when $k_2 \to 1$ or $k_2 \geq 1$.

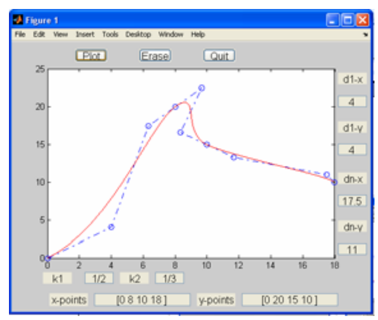

Fig. 4a – Shape for $k_2=1/3$

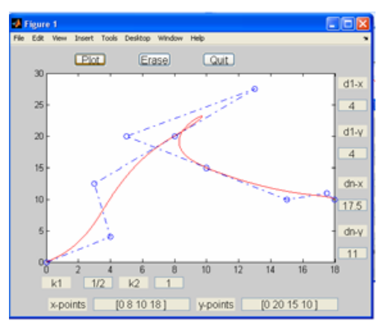

Fig. 4b – Modified shape for $k_2=1$

The influence of parameters k_1, k_2 is difficult to be express algebraic, in a general case be cause the *k*-th control polygon depends on 4 digitalized points P_{k-2}, ..., P_{k+1}.

5 Conclusion

In this article we introduce a method for obtaining a cubic Bézier spline of class G^1. Our method depends on 2 parameters which allow the shape modification without changing the given points.

Using a program which implements our method we made an analysis of the dependence of shape variation on the methods' parameters. The explanation of our results is based on geometric reasons. As a further direction of study we want to see if that can be established limits for parameters k_1 and k_2 such that other prescribed geometric condition to be respected.

We want to develop other graphic interface for interactive modification of the points b_{3i-2}, $i \in \{1,\dots,n-1\}$ for k_1 fixed or for k_2 fixed.

References:

[1] W. Boehm, G. Farin, J. Kahmann, A survey of curve and surface methods in CAGD, *Computer Aided Geometric Design*, Elsevier Press, pp. 1–60, 1984.

[2] C. de Boor, B – cubic spline interpolation, *J. Math. Phys.* , 41, pp. 212 – 218, 1962.

[3] C. de Boor, K. Hollig, M. Sabin, High accuracy geometric Hermite interpolation, *Computer Aided Geometric Design*, 4, Elsevier Press, 269–278, 1987.

[4] G. Farin, *Curves and Surfaces for Computer Aided Geometric Design. A practical guide*, Academic Press, Boston, 1996.

[5] G. Farin, J. Hoschek, M. S. Kim, *Handbook of computer aided design*, North – Holland, 2002,

[6] G. Farin, A Class of Bézier Curves, *Computer Aided Geometric Design*, 23, Elsevier Press, pp.573-581, 2008.

[7] R. Farouki, V. Rajan, On the numerical condition of polynomials in Bernstein form, *Computer Aided Geometric Design*, 4(3), pp. 192-216, 1987.

[8] J. Ferguson, Multivariable curve interpolation, *Journal ACM*, 11(2), pp. 221-228, 1964.

[9] H. Oruç, G. Phillips. q-Bernstein polynomials and Bézier curves, *Journal of Computational and Applied Mathematics*, 151, pp. 1-12, 2003.

[10] Xi-An Hana, YiChen Maa, XiLi Huangc, A novel generalization of Bézier curve and surface, *Journal of Computational and Applied Mathematics*, 217, pp. 180-193, 2008

[11] L. Schumaker, *Spline functions: Basic theory*, Wiley, 1981.

[12] D. E. Ulmet, Swept surfaces in Computer Aided Geometric Design, *Proceedings of the International Workshop New approaches, Algorithms and Advanced Computational Techniques in Approximation Theory and its Applications*, Ed. Univ. Lucian Blaga, Sibiu, pp. 19-31, 2007.