Non Axysimmetrical Stability Study of Swirling Flows Using a Projection Algorithm

DIANA ALINA BISTRIAN
Department of Electrical Engineering and Industrial Informatics
Engineering Faculty of Hunedoara, “Politehnica” University of Timisoara
Str. Revolutiei Nr.5, Hunedoara, 331128
ROMANIA
diana.bistrian@fih.upt.ro

GEORGE SAVII
Department of Mechatronics
Mechanical Engineering Faculty, “Politehnica” University of Timisoara
Mihai Viteazu Nr.1, Timisoara, 300222
ROMANIA
george.savii@mec.upt.ro

Abstract: This paper reports a numerical investigation of the hydrodynamic instability of swirling flow with application in Francis hydraulic turbine. An \( L^2 \) - projection algorithm is developed assessing both an analytical methodology and implementation using symbolic and numerical conversions. The model of the trailing vortex is used to validate the code with existing results in the literature and the results of the stability of the vortex rope are pointed out, together with the advantages of using the algorithm in flow control problems.

Key-Words: hydrodynamic stability, swirling flow, projection algorithm, spectral collocation.

1 Introduction

Swirling flows behavior has long been an intensive subject of research, especially in Francis hydropower turbine design because, at partial loads, the rope hydrodynamic instability appears in the draft tube. The amount of computational resources required to accurately simulate the vortex rope is huge, so a complementary stability analysis is a critical requirement to predict the flow dynamics. Pozrikidis [10] offer an introductory course in fluid mechanics, covering the traditional topics in a way that unifies theory, computation, computer programming and numerical simulation. An experimental investigation of the suction side boundary layer of a large scale turbine cascade has been performed by Simoni et al. in [11]. Resiga et al. [12] carried out an experimental and theoretical investigation of the flow at the outlet of a Francis turbine runner, in order to elucidate the causes of a sudden drop in the draft tube pressure recovery coefficient at a discharge near the best efficiency operating point.

The objective of this paper is to present new instruments that can provide relevant conclusions on the stability of swirling flow downstream the Francis turbine runner assessing both an analytical methodology and numerical methods for a spatial stability investigation of the bending modes [1].

The mathematical model governing the linear spatial stability of the fluid system downstream the runner, corresponding to the values of tangential wavenumber \( m = \pm 1 \), in operator form is

\[
\Lambda_j(k, F, G, H, P) = \frac{d}{dr} (rG) + rk F + mH,
\]

\[
\Lambda_j(k, F, G, H, P) = kUG - \omega G + \frac{mWG}{r} + \frac{2WH}{r} - \frac{d}{dr} P, (1)
\]

\[
\Lambda_j(k, F, G, H, P) = kUH - \omega H + \frac{mW}{r} [HW + P] + \frac{WG}{r} + \frac{d}{dr} W, (2)
\]

\[
\Lambda_j(k, F, G, H, P) = kFU - \omega F + \frac{FmW}{r} + G \frac{d}{dr} U + kP, (3)
\]

where \( F, G, H, P \) represent the complex amplitudes of the perturbations, \( k \) is the complex axial wavenumber, \( m \) is the integer tangential wavenumber, \( \omega \) represents the temporal frequency, \( U \) and \( W \) represent the axial and the tangential velocity, respectively, both depending only on the radial coordinate \( r \).

For a given real \( \omega \), the system (1)-(4) is equivalent to the complex eigenvalue problem

\[
\Lambda_j = \Lambda_\pm = \Lambda_\pm = \Lambda_\pm = 0,
\]

on the domain \( (0, r_{wall}) \) together with the boundary conditions in axis origin

\[
H \pm G = 0, F = P = 0
\]
and the wall boundary conditions
\[
\frac{2W_{\text{wall}}}{r_{\text{wall}}} H - P' = 0, \quad G = 0, \quad (6)
\]
\[
r_{\text{wall}} H \left( k U_{\text{wall}} - \omega \right) \pm H W_{\text{wall}} \pm P = 0 = 0, \quad (7)
\]
\[
r_{\text{wall}} F \left( k U_{\text{wall}} - \omega \right) \pm F W_{\text{wall}} + k H_{\text{wall}} P = 0. \quad (8)
\]

The hydrodynamic stability model developed in the forthcoming sections of this paper involves spectral differentiation operators derived by means of shifted orthogonal expansions of the perturbation field. The hydrodynamic model presented in Section 2 is developed using an \( L^2 \)-projection method with operator scheme. The sophisticated boundary conditions corresponding to the real flow case in a Francis turbine runner motivated the use of this method, suitable for non-periodic problems with complicated boundary conditions.

Dongara [6] used the Chebyshev tau method to examine in detail a variety of eigenvalue problems arising in hydrodynamic stability studies, particularly those of Orr-Sommerfeld type. The orthogonality of Chebyshev functions was first used by Bourne [4] to rewrite the differential equations as a generalized eigenvalue problem, assembling a very efficient projection based technique.

The classical approaches imply a transformation of the physical domain onto the standard interval for the definition of the Chebyshev polynomials. For the approach presented in this paper, instead of using classical Chebyshev polynomials, we used shifted Chebyshev polynomials, directly defined on the physical domain of the problem \((0, r_{\text{wall}})\), preserving the orthogonality properties detailed in Section 2.1. The numerical algorithm was developed to work automatically for any number of expansion terms, using symbolic and numeric conversions, the implementation technique being detailed in Section 2.2. In Section 3.1 the algorithm is validated upon the model of a trailing vortex and applied to the swirl flow downstream a Francis hydraulic turbine in Section 3.2. The main results of the paper are summarized in Section 4.

2 \( L^2 \)-Projection Algorithm For Bending Modes Investigation

2.1 Methodology

Following [4] the difficult eigenvalue problem (1)-(8) is transformed into a system of linear equations describing the hydrodynamic context for the cases \( m = \pm 1 \). The difference between the classical tau method and the modified version proposed here is given by the selected spaces involved in the approximation process. An appropriate solution is sought in the truncated Chebyshev series form
\[
(F, G, H, P) = \sum_{i=1}^{N} (f_i, g_i, h_i, p_i) \cdot T_i, \quad (9)
\]
with \( f_i, g_i, h_i, p_i, \ k = 1..N \) the sets of expansions coefficients to be found. The tau method is an algorithm implying in the first step expanding the residual function as a series of shifted Chebyshev polynomials.

Since the range \([0, r_{\text{wall}}]\) is more convenient to use than the standard definition interval of classical Chebyshev polynomials \([-1, 1]\) to discretize our hydrodynamic stability problem, where \( r_{\text{wall}} \) represents the radial distance to the wall of the draft cone, we map the independent variable \( r \in [0, r_{\text{wall}}]\) to the variable \( \xi \in [-1, 1] \) by the linear transformation
\[
\xi = 2rr_{\text{wall}}^{-1} - 1 \Leftrightarrow r = r_{\text{wall}}(\xi + 1)2^{-1}. \quad (10)
\]

The shifted Chebyshev polynomials of the first kind \( T_n^*(r) \) of degree \( n-1 \) in \( r \) on \([0, r_{\text{wall}}]\) are given by
\[
T_n^*(r) = T_n(\xi) = T_n(2rr_{\text{wall}}^{-1} - 1). \quad (11)
\]

The shifted Chebyshev class is orthogonal in the Hilbert space \( L^2_w(0, r_{\text{wall}}) \), weighted by
\[
w(r) = \left( 1 - (2rr_{\text{wall}}^{-1} - 1) \right)^{-1/2}. \quad (12)
\]

They have the next orthogonality properties
\[
\left(T_n^*, T_m^*\right)_w = 0, n \neq m, n, m = 1..N, \quad (13)
\]
with respect to the inner product
\[
\left(f, g\right)_w = \int_0^{r_{\text{wall}}} w(r) f(r) g(r) dr.
\]

We obtain a set of \( 4(N-2) \) linear equations. The eight remaining equations are provided by the boundary conditions applied as side constraints.

Introducing the notations
\[
I_{\text{gldd}}^V = \left\{ r^d (U^d)^{j\ell} T_{i_1} T_{i_2} \right\}_w, \quad I_{\text{gldd}}^W = \left\{ r^d (W^d)^{j\ell} T_{i_1} T_{i_2} \right\}_w, \quad (14)
\]
with \( d \) the derivation order, the first truncated \( 4(N-2) \) equations of the hydrodynamic model are
\[
k \sum_{j=1}^{N} \left( f_{j, \text{odd}}^V \right) \left( g_{j, \text{odd}} + \frac{2}{r_{\max}} I_{\text{gldd}}^V \right) + g_{\text{max}} c + g_{\text{max}} + \frac{2}{r_{\max}} I_{\text{gldd}}^V +
\]
\[
+ \sum_{j=1}^{N} g_{j, \text{odd}} \left( \frac{2(j-1)}{r_{\max}} \right) + \sum_{j=1}^{N} \frac{2}{r_{\max}} I_{\text{gldd}}^V + \]

ISSN: 1792-4251 104 ISBN: 978-960-474-201-1
\[+ \sum_{j=0}^{N} g_k \frac{2(j-1)}{r_{\text{max}}} \left[ \sum_{i=0}^{2} \left( 2T_{i,100}^{(1)} + T_{i,100}^{(2)} \right) \right] + m h r_{\text{max}} c = 0,\]  
\[k \sum_{j=1}^{N} g_j T_{i,010}^{(1)} - \omega g r_{\text{max}} c + m \sum_{j=1}^{N} g_j T_{i,110}^{(1)} + 2 \sum_{j=1}^{N} h_j T_{i,110}^{(1)} - \sum_{j=1}^{N} \sum_{r=1, j, \text{odd}}^{N} 2 \left( A_j + A_i \right) = 0,\]

where the number \( c \) is defined as
\[ c = \begin{cases} \pi/2, & j = 1 \\ \pi/4, & j = 2 \ldots N - 2 \end{cases} \quad \text{and} \quad A \in M_{N \times 2} \quad \text{is square}.
\]

The eigenvalue problem is written as a system of equations with the matricial formulation
\[kM \tilde{s} = M \tilde{s}, \quad \tilde{s} = (f, g, h, p), \quad \text{with} \quad \tilde{s} = \left( \ast_1, \ldots, \ast_N \right),
\]
\[\ast = f, g, h, p.\]  
The method has the obvious advantage that the highest degree of the Chebyshev polynomials multiplying the residual in the method inner-product is only \( N - 2 \).

### 2.2 Implementation Technique Using Symbolic And Numeric Conversions

The recurrence relation for \( T_n \) has the form
\[T_n^* (r) = 2 \left( 2r_{\text{wall}}^{-1} \right) T_{n-1}(r) - T_{n-2}(r), n = 3, 4, \ldots, \]
in which the initial conditions are \( T_1^* (r) = 1, T_2^* (r) = \frac{2r}{r_{\text{wall}}} - 1 \). The use of a recurrence relation significantly increases the elapsed time to generate the shifted Chebyshev polynomials. To improve the performance of the numerical algorithm, we introduce in our code the equivalent polynomial relation
\[T_n^* (r) = \frac{1}{2} \left[ \left( r + \sqrt{r^2 - 1} \right)^{n-1} + \left( r - \sqrt{r^2 - 1} \right)^{n-1} \right],\]
\[\tilde{r} = \frac{2r}{r_{\text{wall}}} - 1,\]  
to automatically generate the shifted Chebyshev polynomials \( \{ T_n^* (\xi) \}_{n \in \mathbb{Z}} \) on \( [0, r_{\text{wall}}] \).

A modified Chebyshev-Gauss grid \( \Xi = \{ \xi_j \}_{j \in \mathbb{Z}, j < N} \) in \([-1, 1]\) was constructed
\[\xi_j = \cos \left( \pi + \frac{j \pi}{N-1} \right), \quad j = 0 \ldots N - 1,\]

mapped into the physical range of our problem by the simple linear transformation \( r = r_{\text{wall}} \left( \xi + 1 \right)/2 \), \( i = 1 \ldots N \). The collocation nodes clustered near the boundaries diminishing the negative effects of the Runge phenomenon [5]. Another aspect is that the convergence of the interpolation function on the clustered grid towards unknown solution is extremely fast. The numerical algorithm was developed to work automatically for any number of expansion terms, using the routines of a high level.
language such Matlab. We list in Table 1 the code sequences for generating symbolically the products $I_{ijkl}^{14}$ from (14) and in Table 2 the function for numerically conversion and evaluation of the inner products.

Table 1. The function that generates symbolically the products $r^k(U^j)^{(d)}T_rT_r^jw(r)$.

<table>
<thead>
<tr>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tmn=CODEproductsU(M1,N1,rmax,U0,U1,U2,R1,R2,k,l,d)</td>
</tr>
<tr>
<td>syms x</td>
</tr>
<tr>
<td>Tx=policevs(N1,rmax);</td>
</tr>
<tr>
<td>Ty=policevs(M1,rmax);</td>
</tr>
<tr>
<td>W=1./sqrt(1-(2.*x/rmax-1).^2);</td>
</tr>
<tr>
<td>U=U0+U1.*exp(-(x.^2)./(R1^2)) + ...</td>
</tr>
<tr>
<td>U2.*exp(-(x.^2)./(R2^2));</td>
</tr>
<tr>
<td>Uder=(-2.*U1.*x.*exp(-(x.^2)./(R1^2)))/(R1^2) + ...</td>
</tr>
<tr>
<td>(-2.*U2.*x.*exp(-(x.^2)./(R2^2)))/(R2^2);</td>
</tr>
<tr>
<td>if d==0 AXI=U</td>
</tr>
<tr>
<td>elseif d==1 AXI=Uder</td>
</tr>
<tr>
<td>end</td>
</tr>
<tr>
<td>Tmn=(x.^k).*(AXI.^l).*Tx.*Ty.*W;</td>
</tr>
</tbody>
</table>

Table 2. The function for numerically conversion and evaluation of the inner products $(r^k(U^j)^{(d)}T_rT_r^j)_{w}$.

<table>
<thead>
<tr>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>valueint=INNERPR(fun,lim1,lim2)</td>
</tr>
<tr>
<td>tt=sym2str(fun);</td>
</tr>
<tr>
<td>f=strcat('@(x)', tt);</td>
</tr>
<tr>
<td>fn=eval(f);</td>
</tr>
<tr>
<td>valueint=quad(fn,lim1,lim2);</td>
</tr>
</tbody>
</table>

Table 3. The main program sequence that generates the $N \times 4N$ system matrix.

<table>
<thead>
<tr>
<th>INDICES=</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0 0;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 1 0;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 1 0;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 0 1;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>k=INDICES(:,1); l=INDICES(:,2); d=INDICES(:,3);</td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTVALS=zeros(N,4*N); for subind=1:4 % block position</td>
<td></td>
<td></td>
</tr>
<tr>
<td>for i=1:N</td>
<td></td>
<td></td>
</tr>
<tr>
<td>for j=1:N</td>
<td></td>
<td></td>
</tr>
<tr>
<td>k1=k(subind); l1=l(subind); d1=d(subind);</td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTVALS(i,j+N*(subind-...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1)=INNERPR(CODEproductsU(i,j,rwall,...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>U0,U1,U2,R1,R2,k1,l1,d1,0,rwall);</td>
<td></td>
<td></td>
</tr>
<tr>
<td>end, end, end</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The main program sequence that returns the $N \times 4N$ matrix whose $N \times N$ blocks represent the collocated values of the inner products $(rT_r,T_r^j)_w$, $(UT_r^j,T_r^j)_w$, $(rUT_r^j,T_r^j)_w$ and $(U'T_r^j,T_r^j)_w$ is listed in Table 3.

3.1 Code Validation

The basic flow under consideration for the validation of the proposed methods is the Batchelor vortex case or the q-vortex [2], that trails on the tip of each delta wing of the airplanes. The properties of the Batchelor vortex were pointed out in [9] using a shooting method. In order to compare our results with the ones from [9] numerical evaluations of the axial wavenumber $k$ and the critical distance were obtained for various values of the spectral parameter $N$. In Table 4 these values are presented in comparison with the ones from Olendradru et al. [9] and the ones obtained using a spectral boundary adapted technique in a previous investigation [3].

Table 4. Comparative results of the most amplified spatial wave: eigenvalue with largest imaginary part $\lambda_{cr}$ and critical distance of the most amplified perturbation $r_c$.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\lambda_{cr}$</th>
<th>$r_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^2$ Projection method</td>
<td>(0.46375,−0.27935)</td>
<td>1.166</td>
</tr>
<tr>
<td>Spectral boundary adapted [3]</td>
<td>(0.50842,−0.14243)</td>
<td>0.959</td>
</tr>
<tr>
<td>Shooting method [9]</td>
<td>(0.506,−0.139)</td>
<td>1.005</td>
</tr>
</tbody>
</table>

Following Tadmor [13], when differencing analytic functions using Chebyshev pseudospectral methods, the error committed is expected to decay to zero at an exponential rate. The convergence behavior of the algorithms with respect to the number of expansion terms is shown in Table 5.

Table 5 The convergence behavior of the algorithm.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Axial wavenumber $k_{cr}$</th>
<th>Critical distance $r_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.64887 − 3.74334i</td>
<td>0.00302</td>
</tr>
<tr>
<td>120</td>
<td>0.50854 − 0.14216i</td>
<td>0.95451</td>
</tr>
<tr>
<td>150</td>
<td>0.50842 − 0.14243i</td>
<td>0.95923</td>
</tr>
<tr>
<td>180</td>
<td>0.50847 − 0.14232i</td>
<td>0.95994</td>
</tr>
</tbody>
</table>

Clearly the numerical computation costs were less expansive in the projection method approach.
since the number of terms in the approximations was significantly reduced. In fact, in comparison with the boundary adapted collocation method this number was more than twenty times reduced. As a result, with a reduced by far computational time, we can obtain accurate results in an acceptable agreement with existing ones.

3.2 Error analysis and stability results
The start of the stability analysis technique presented in this section is the analytical representation of the velocity field of the vortex rope derived by Resiga et al. [12]. Although the projection method is a very efficient technique, the inclusion of the boundary conditions as equations in the system of the generalized eigenvalue problem have been observed to be one cause of spurious eigenvalues. The spurious eigenvalues, which are not always easy to identify, may lead one to a false conclusion regarding the stability of the fluid system, thus the elimination of them is of great importance. These are values returned by the algorithm which do not satisfy the eigenvalue problem. The spurious eigenvalues problems have been the attention of much study recently. Gardner et al. [7] and McFadden et al. [8] describe the tau methods to avoid spurious eigenvalues and in Dongara [6] the occurrence of the spurious eigenvalues is assessed in application to the Benard convection problem.

We implement in our numerical procedure a code sequence that identifies if an eigenvalue of the spectra is spurious or not. First the algorithm provides the entire spectra, then calculates the residual vector of the eigenvalue problem

$$\text{IDENTIFIER} = k \cdot M \cdot \vec{s} - M \cdot \vec{s}, s = (f, g, h, p)^T$$ (27)

with $f = (f_1, f_2, ..., f_N)^T$, $g = (g_1, g_2, ..., g_N)^T$, $h = (h_1, h_2, ..., h_N)^T$, $p = (p_1, p_2, ..., p_N)^T$, for each eigenvalue of the spectra.

A true value of $k$ must satisfy the eigenvalue problem. We evaluate the $L^2$ norm of the vector with respect to a given tolerance $\varepsilon$. If the condition

$$\text{norm (IDENTIFIER)} > \varepsilon$$ (28)

holds, the eigenvalue $k$ is declared spurious and discarded from the spectra.

Figure 1 presents the residual of the eigenvalue problem, solved using the QZ algorithm and the corresponding histogram. Performing a spatial stability analysis of the vortex rope, we observed the behavior of the growth rate $-k_i$ as functions of real frequency.

Figure 2 describes the behavior of the radial perturbation computed for several discharge coefficients. It is shown that the sensitivity is located near the center, for a small radius.

Figure 3 presents the results obtained by projection method for bending modes $m = 1$ and $m = -1$, for several values of the discharge coefficient $\text{PHI}$.
4 Conclusion

This paper reports a numerical spectral approach based on shifted Chebyshev polynomials for numerically solving linear stability problems with sophisticated boundary conditions. Numerical results showed that the use of this method improved the computational time and the obtained critical values of the eigenparameter involved are fairly accurate.

The numerical results have been compared with some other existing spatial investigations [3] and [9]. The collocation method proved to be more accurate, however the projection method was less expensive with respect to the numerical implementation costs, i.e. numerical results were obtained for a much smaller number of terms in the discretization. The results are very useful not only from their numerical values point of view, but also for their physical interpretation in fluid dynamics in flow control problems.

References:


