A new algorithm for solving singular IVPs of Lane-Emden type

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***Abstract*—The Lane-Emden equation that describes the temperature in a self-gravitating star has been widely studied due to its applications in astrophysics and as a prototype for testing new mathematical and numerical techniques for solving nonlinear equations. In this paper a new semi-analytical algorithm for solving singular initial value problems of Lane-Emden type is presented. Solutions for polytropic indices $n = 2, 3$ and 4 are found and compared with numerical results and the Sinc-collocation approximations in the literature. Our investigations indicate that there is excellent agreement between the numerical results and the new method. Excellent agreement is also found when comparing the results of the new method against the exact analytic solution of the Lane-Emden equation for polytropic index $m = 5$ showing the accuracy and computational efficiency of the present technique.**

***Index Terms*—Lane-Emden equation, Singular IVPs, linearization method, Spectral method, series solution**

I. Introduction

The Lane-Emden differential equation of index m is one of the well studied classical equations of nonlinear mechanics. Named after the pioneering work of Lane [1] and Emden [2], the equation describes the equilibria of non-rotating polytropic fluids in a self-gravitating star. The equation has been studied extensively by physicists because of its applications in astrophysics and also because of its importance in the kinetics of combustion and the Landau-Ginzburg critical phenomena, [3], [4], [5], [6]. Some of the notable contributions in the study of isothermal gas sheets are those of Chandrasekhar [7], Eddington [8] and Spitzer [9].

For mathematicians, fascination with the Lane-Emden equation might derive partly from its nonlinearity and singular behaviour at the origin. Solving the Lane-Emden equation analytically in closed form is only possible for the polytropic indices $m = 0, 1$ and 5. The remaining solutions for other indices are generally found numerically. Initial interest might have related to the study of the properties of the known solutions of the equation (such as the classification of the solutions, bifurcations, uniqueness, etc). However, the Lane-Emden equation is numerically challenging to solve because of its singular behaviour at the origin. In recent times the Lane-Emden equation has thus been used extensively as a prototype for testing new methods for finding semi-analytic methods for solving nonlinear differential equations.

Previous studies and attempts to find analytic solutions of the Lane-Emden problem range from the approximate analytical solutions of Seidov and Sharma [10] and Sharma [11] to the power-series solutions of Nouh [12], Ramos [13] and Seidov and Kuzakhmedov [14]. There were however questions on the convergence of the power-series solutions presented in [14] and the follow-up paper by Mohan and Al-Bayaty [15] sought to find an improved power-series solution to the Lane-Emden equation that is convergent in the whole interior of a polytropic model. Power series solutions were also obtained by Shawagfeh [16] and Wazwaz [17], [18] using the Adomnian decomposition method. The series solution in [17] is given in the later study by Momoniat and Harley [19] where Lie group analysis was used to find new approximate implicit solutions to the Lane-Emden equation with a larger radius of convergence than the power series solution. This was achieved by first reducing the Lane-Emden equation to first-order and then determining a power series solution of the reduced equation. The variational Iteration Method, He [20] and the homotopy analysis method (see [21], [22], [23]) have also been used to find semi-analytical solutions of the LE equation. Other methods that have been used to find analytic solutions to the LE equation include the δ-perturbation method by Datta [24]. Padé approximants were then used to accelerate the convergence of the power series solution. The homotopy perturbation method was used by Yildrim and Özis [25] while a generalized homotopy perturbation method has been used recently by Rafiq et al. [26] to solve LE type differential equations. Numerical methods that have been used in the recent past include the Legendre Tau method by Parand and Razzaghi [27] and the sinc-collocation method by Parand and Prirkhedri [28].

In this paper we present a new successive linearization method (SLM) that promises computational efficiency, rapid convergence and better accuracy than other semi-analytical techniques currently being used, in particular, the homotopy analysis method and its variants such as the spectral homotopy analysis method [29]. So far the SLM has been successfully tested on two non-linear two-point boundary value problems arising in fluid mechanics, namely von-Karman swirling flow

[30] and general Falkner-Skan type [31]. This is the first attempt at applying the SLM to a non-linear initial value problem (IVP).

II. The Lane-Emden equation

In this paper we consider the general Lane-Emden type initial value problem with polytropic index m of the form

$$y''(x) + \frac{2}{x}y'(x) + y^m = 0, \tag{1}$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{2}$$

Equation (1) is a second-order differential equation whose closed form solutions are known only for special, integer values of the index m, namely when $m = 0, 1$ and 5. In this paper we use the successive linearization method (SLM) to find the remaining solutions for other indices, particularly for $m = 2, 3, 4$. The solution for the case $m = 5$ is also found and compared with the known exact analytical solution.

III. Successive linearisation method of solution

The successive linearisation method (SLM) is based on the assumption that the unknown functions $y(x)$ can be expanded as

$$y(x) = Y_i(x) + \sum_{n=0}^{i-1} y_n(x), \qquad i = 1, 2, 3, \ldots \tag{3}$$

where Y_i are unknown functions and y_n are successive approximations whose solutions are obtained recursively, from solving the linear part of the equation that results from substituting (3) in the governing equations (1) using $y_0(x)$ as an initial approximation. The initial approximation is chosen in such a way that it satisfies the boundary conditions (2). A suitable initial approximation for (1) is

$$y_0(x) = 1. \tag{4}$$

The linearisation technique is based on the assumption that Y_i becomes increasingly smaller as i becomes large, that is

$$\lim_{i\to\infty} Y_i = 0. \tag{5}$$

Substituting (3) in the governing equation (1), and using the Binomial theorem notation, gives

$$Y_i'' + \frac{2}{x}Y_i' + \sum_{s=0}^{m} \binom{m}{s} Y_i^{m-s}\left[\sum_{n=0}^{i-1} y_n\right]^s = -\sum_{n=0}^{i-1}\left(y_n'' + \frac{2}{x}y_n'\right) \tag{6}$$

Starting from the initial approximation (4), the subsequent solutions for y_n, $n \geq 1$ are obtained by iteratively solving the linearized form of equations (6) which are given as

$$y_i'' + \frac{2}{x}y_i' + m a_{i-1}^{m-1} y_i = \phi_{i-1}, \tag{7}$$

subject the boundary conditions

$$y_i(0) = y_i'(0) = 0, \tag{8}$$

where

$$a_{i-1} = \sum_{n=0}^{i-1} y_n, \tag{9}$$

$$\phi_{i-1} = -\left[\sum_{n=0}^{i-1} y_n'' + \frac{2}{x}\sum_{n=0}^{i-1} y_n' + \left(\sum_{n=0}^{i-1} y_n\right)^m\right]. \tag{10}$$

Once each solution for y_i $(i \geq 1)$ has been obtained, the approximate solution for $y(x)$ is obtained as

$$y(x) \approx \sum_{n=0}^{M} y_n(x) \tag{11}$$

where M is the order of SLM approximation. We remark that the coefficient parameters and the right hand side of equations (7) for $i = 1, 2, 3, \ldots$, are known (from previous iterations). Thus, equation (7) can easily be solved using analytical means (whenever possible) or any numerical methods such as finite differences, finite elements, Runge-Kutta based shooting methods or collocation methods. In this work, equation (7) is solved using the Chebyshev spectral collocation method. This method is based on approximating the unknown functions by the Chebyshev interpolating polynomials in such a way that they are collocated at the Gauss-Lobatto points defined as

$$z_j = \cos\frac{\pi j}{N}, \qquad j = 0, 1, \ldots, N. \tag{12}$$

where N is the number of collocation points used (see for example [32], [33], [34]). In order to implement the method, the physical region $[0, 1]$ is transformed into the region $[-1, 1]$ using the mapping

$$x = \frac{z+1}{2}, \quad -1 \leq z \leq 1 \tag{13}$$

The unknown functions y_i are approximated at the collocation points by

$$y_i(z) \approx \sum_{k=0}^{N} y_i(z_k) T_k(z_j), \quad j = 0, 1, \ldots, N \tag{14}$$

where T_k is the kth Chebyshev polynomial defined as

$$T_k(z) = \cos[k \cos^{-1}(z)]. \tag{15}$$

The derivatives of the variables at the collocation points are represented as

$$\frac{d^r y_i}{dx^r} = \sum_{k=0}^{N} \mathbf{D}_{kj}^r y_i(z_k), \quad j = 0, 1, \ldots, N \tag{16}$$

where r is the order of differentiation and $\mathbf{D} = 2\mathcal{D}$ with $\mathcal{D}$ being the Chebyshev spectral differentiation matrix (see [32], [34]). Substituting equations (13 - 3) in (7) leads to the matrix equation given as

$$\mathbf{A}_{i-1}\mathbf{Y}_i = \mathbf{\Phi}_{i-1}, \tag{17}$$

in which $\mathbf{A}_{i-1}$ is a $(N+1)\times(N+1)$ square matrix and $\mathbf{Y}_i$ and $\mathbf{\Phi}$ are $(N+1)\times 1$ column vectors defined by

$$\begin{aligned}\mathbf{Y}_i &= [y_i(z_0), y_i(z_1), \ldots, y_i(z_{N-1}), y_i(z_N)]^T,\\ \mathbf{\Phi}_{i-1} &= [\phi_{i-1}(z_0), \phi_{i-1}(z_1), \ldots, \phi_{i-1}(z_{N-1}), \phi_{i-1}(z_N)]^T,\\ \mathbf{A}_{i-1} &= \mathbf{D}^2 + \left[\frac{2}{x}\right]_d \mathbf{D} + m\mathbf{a}_{i-1}^{m-1}.\end{aligned}$$

In the above definitions, $[\ ]_d$ and $\mathbf{a}_{i-1}$ are diagonal matrices of size $(N+1)\times(N+1)$. After modifying the matrix system (17) to incorporate boundary conditions, the solution is obtained as

$$\mathbf{Y}_i = \mathbf{A}_{i-1}^{-1}\mathbf{\Phi}_{i-1}. \tag{18}$$

IV. RESULTS

In this section we show the accuracy and fast convergence of the SLM by comparing the current results with those in the literature. Table I gives a comparison of the SLM results against the numerical computations of Horedt [35] and the sinc-collocation results of Parand and Prirkhedri [28] for polytropic indices $m = 2, 3, 4$. The successive linearisation method gives a series solution and improvement in the accuracy and convergence of the results in general increases with the order of the solution series. In this case accuracy up to seven decimal places is obtained at the fourth order SLM results when comparing with the numerical results of Horedt [35] which are widely used as a benchmark for testing the accuracy of new methods of solution.

We note that the sinc-collocation results of Parand and Pirkhedri [28] appear to be less accurate than both our current results and those of Horedt [35]. This is confirmed in Table II where the solution $y(x)$ is presented for the case $m = 3$. In all the selected values of x, matching up to six decimal places is achieved at the sixth order of the SLM series solution and the numerical results of Horedt [35]. The sinc-collocation results of Parand and Prirkhedri [28] are only accurate up to three decimal places.

In Table III we compare the SLM results for $m = 5$ against the exact solution (19) at different values of the scaled variable x. When $m = 5$ the Lane-Emden equation (1) has an exact solution given by

$$y(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}. \tag{19}$$

The results are self-evident and show that the SLM results match with the exact solution at the fifth order of the SLM series solution. For smaller values of x exact matching is obtained at the second order but the level of accuracy decreases with x. From these findings we may conclude that the successive linearisation technique is computationally efficient, accurate and converges rapidly.

Figure 1 shows the Lane-Emden equation solution curves for $m = 2, 3$ and 4 obtained using the fourth order SLM solution series. These solution curves are accurate and match with those obtained from the numerical solution of the Lane-Emden equation.

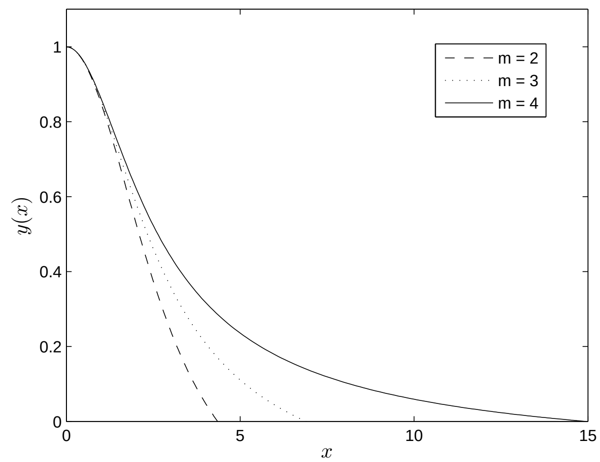

Fig. 1. Lane-Emden graphs for the 4th order successive linearisation solution for $m = 2, 3, 4$

V. CONCLUSION

We have presented a successive linearisation method for the solution of the Lane-Emden equation. We have shown that the SLM is a robust and reliable technique for obtaining semi-analytical solutions of nonlinear differential equations. The SLM is computationally efficient, accurate and converges rapidly. It is worth noting that the SLM provides a simple unified treatment of the nonlinear equation that requires no acceleration of convergence unlike the power series and the δ-perturbation methods.

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TABLE I
COMPARISON OF THE FIRST ZERO OF $y(x)$ BETWEEN THE PRESENT METHOD AND NUMERICAL VALUES OF [35] AND [28] FOR $m = 2, 3, 4$

	Present Results			Previous results	
m	3rd order	4th order	6th order	Numerical [35]	Sinc-Collocation[28]
2	4.35287461	4.35287458	4.35287458	4.35287460	4.352001691
3	6.89695528	6.89684860	6.89684860	6.89684862	6.896318068
4	14.9000000	14.9723760	14.9715463	14.9715463	14.97174475

TABLE II
APPROXIMATION OF $y(x)$ BETWEEN THE PRESENT METHOD AND NUMERICAL VALUES OF [35] AND [28] FOR $m = 3$

	Present Results			Previous results	
x	2nd order	4th order	6th order	Numerical [35]	Sinc-Collocation[28]
0.000	1.000000	1.000000	1.000000	1.000000	1.000000
0.100	0.998336	0.998336	0.998336	0.998336	0.998354
0.500	0.959868	0.959839	0.959839	0.959839	0.959951
1.000	0.856616	0.855058	0.855058	0.855058	0.855165
5.000	0.690959	0.122602	0.110820	0.110820	0.110618
6.000	0.636545	0.141308	0.043738	0.043738	0.043688
6.800	0.640623	0.259197	0.004168	0.004168	0.003875
6.896	0.644165	0.276648	0.000037	0.000037	0.000015

TABLE III
COMPARISON OF $y(x)$ BETWEEN THE PRESENT METHOD AND EXACT SOLUTION FOR $m = 5$ AT SELECTED VALUES OF x WHEN $N = 200$

x	2nd order	3rd order	4th order	5th order	Exact
0.80	0.90784130	0.90784130	0.90784130	0.90784130	0.90784130
1.20	0.82199494	0.82199494	0.82199494	0.82199494	0.82199494
1.60	0.73455329	0.73455317	0.73455317	0.73455317	0.73455316
2.00	0.65465684	0.65465367	0.65465367	0.65465367	0.65465367
2.40	0.58523585	0.58520574	0.58520574	0.58520574	0.58520574
2.80	0.52622459	0.52607297	0.52607297	0.52607297	0.52607297
4.00	0.39982330	0.39736001	0.39735971	0.39735971	0.39735971
6.00	0.29503625	0.27738204	0.27735010	0.27735010	0.27735010
8.00	0.25514452	0.21192492	0.21160369	0.21160368	0.21160368
10.00	0.24385079	0.17204783	0.17066410	0.17066404	0.17066404

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